A LFT/$H_\infty$ state feedback design for Linear Parameter Varying Time Delay Systems

Corentin Briat$^1$, Olivier Sename$^1$ and Jean-François Lafay$^2$

Abstract—This paper considers $H_\infty$ control of LPV time delay systems in LFT form. We first introduce a new model transformation for systems with time varying delay which allows us to propose a systematic method to design gain-scheduled state feedback in the delay dependent framework. Further, we extend this method to elaborate a new type of controller for TDS whenever the delay is measured in real time and explicitly used in the controller as a parameter. These results are presented with single delayed systems but can be easily generalized to multiple delayed systems.

Index Terms—Time delay systems, linear parameter varying systems, robust control

I. INTRODUCTION

Since several years constant time-delay systems (TDS) have been widely studied (See [7], [13], [21], [3], [5], [30], [15] and references therein). More recently, time varying delays, appearing for example in communication networks, have suggested more and more interest (See [6], [9], [22], [27],[8] and references therein). Indeed, lags in communication channels may destabilize systems or deteriorate performances. Two types of stability criteria are often used: delay independent and delay dependent. The latter treats only delays which belongs to a compact set and is, in general, less conservative and more precise than the delay independent criteria. Indeed, in practice, the delays are bounded and thus, delay dependent stability is preferred.

In the last two decades, linear parameter varying systems (LPV systems) have been studied and several stability analysis and control synthesis methods exist (See [16], [1], [2], [17], [10], [25], [19], [11], [26], [18], [20], [14] and references therein). Three principal methods have been developed: the polytopic approach, the gridding method and the LFT approach. The first one uses the parameter values at each vertex of the polytope, but needs an online convex combination of an exponential number of controllers at each instant, which may be actually computationally expensive (the design may also require a large number of LMI constraints). The great benefits of this approach is its the synthesize simplicity and a good conservatism reduction using for example slack variables. The second one is the gridding method using infinite dimensional LMIs discretized over the parameter space. This leads to simple controllers to implement but the type and the number of basis functions, the number and the space between discretization points are not well defined. The last is the LFT approach which consists in pulling out the varying parameters, as done usually with uncertainties in robust control. According to the type of scalings (multipliers) used, this approach may leads to a small number to a exponential number of LMIs where decision matrices do not depend on parameters (which is conservative). The conservatism reduction is performed using scalings in the small-gain theorem argument. Moreover, an unique and easily implementable controller is obtained which is self smoothly scheduled by the parameters using elementary arithmetical operations. Each method has its on benefits and disadvantages. In this paper we have only interested to develop a LFT approach.

We consider in this paper LPV time-delay systems of the form:

$$\dot{x}(t) = A(\theta)x(t) + A_h(\theta)x_h(t) + B_h(\theta)w(t)$$

$$z(t) = C_1(\theta)x(t) + C_{zh}(\theta)x_h(t) + D_{zh}(\theta)u(t) + D_{zw}(\theta)w(t)$$

where $x_h(t) = x(t-h(t))$, $h$ is the time varying time delay, $x$ is the state, $u$ the control inputs, $w$ the exogenous inputs and $z$ the controlled outputs. $\theta = col(\theta_i)$ is a possibly time varying parameter vector whose components belong to the set

$$\mathcal{P} := \{\theta_i(t) \in [-1,1], \ i = 1, \ldots, N\}$$

Note that the parameters time derivative are unconstrained and since, in practice, it may lead to additional conservatism. Nevertheless, the set $\mathcal{P}$ may be extended to a larger set containing static/dynamic nonlinearities, memoryless operators... whose derivatives are difficult (even impossible) to obtain. It is proposed in [11] a method tackling parameter derivaties but is difficult to adapt in the proposed synthesis. If only the bounds are known, a robust synthesis should be done with respect to these bounds but this will not reduces conservatism. If their derivatives are known at each instant, the controller may be scheduled by the parameters and their derivatives. In practice, it is possible to compute the derivatives simply using a second order low pass filter. The parameters matrix is diagonal

$$\Theta(t) := \bigoplus_{i} \theta_i(t) I_{p_i}$$

where $\theta_i(t)$ are the time varying parameters and $p_i$ is the number of occurrences of the $i^{th}$ parameter.

LPV time-delay systems have been studied in [31], [24], [28], [29], [23] but using polytopic approach or gridding...
methods only. However, the LFT approach has not been really extended to these kinds of systems.

This paper brings a new approach to study delay-dependent stability and control of LTI/LPV time-delay systems with interval time-varying delay (ie. $0 < h_m < h(t) < h_M < +\infty$), in an unified framework of robust control theory for LPV systems.

The contributions of the paper are the following. First a new model transformation for TDS with interval time varying delays is introduced which transforms a time delay system into an uncertain LPV system. Hence a classical stability test is sufficient to conclude.

Second, we express sufficient conditions to the existence and computation feasibility of gain-scheduled state-feedback memoryless controllers, which will depend on the delay bounds and the parameters. The controllers are represented in a LFT form and are not unique. This non-uniqueness allows to the designer to add supplementary constraints to the synthesis problem such as robust pole placement, resilience... (despite of render it, in some case, unfeasible).

Third, we exploit the fact that our approach permits to transform a (LPV) time-delay system into an uncertain LPV system where the delay does not act anymore as an operator but as a bounded time-varying parameter. This allows to elaborate a new kind of controllers. The same methodology is applied to synthesize delay-scheduled controllers with approximate knowledge of the delay value. The measurement error on the delay value is here taken into account to ensure the robustness of the closed-loop system stability.

The advantage of the proposed methodology relies on the fact that we propose for the first time an unique LFT formulation to design different types of controllers for different classes of systems.

The paper is structured as follows. Section II recall useful lemmas and gain-scheduled state feedback existence theorem inspired from [1]. In section III we provide delay dependent stability and stabilization of LPV time delay systems via the use of a particular model transformation. In section IV we present a new LPV based control method for time-delay systems when delays are approximatively known and which can easily be extended to LPV time-delayed systems.

We denote by $A^T$ the transpose of the matrix $A$, $\text{Ker}(A)$ is a basis of the null space of the operator $A$, $A > 0$ ($A < 0$) means that $A$ is positive definite (negative definite), $\oplus$ is the direct sum of matrices, $\hat{x}(t)$ is the time derivative of the signal $x(t)$, $L_2([-h_{max}, +\infty])$ is the space of square integrable signals which maps $\mathbb{R}^+$ to $\mathbb{R}$ (from $[-h_{max}, +\infty]$ to $\mathbb{R}$), $\gamma$ denotes the transpose of the symmetric term in symmetric matrices.

II. BACKGROUND

Let us introduce the well-known Scaled Bounded real Lemma (SBRL):

**Proposition 2.1: Scaled Bounded Real Lemma**

Consider the system

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bw(t) + Bu(t) \\
z(t) &= Cx(t) + Dw(t) + Du(t)
\end{align*}
$$

where $x$ is the state, $w$ the exogenous inputs, $u$ the control inputs and $z$ the controlled outputs.

Let the set of scalings associated to the parameters structure $\Theta$ be defined as

$$
L_\Theta := \{ L : \Theta L = L \Theta, \ L = LT > 0 \}
$$

Note that this set is convex and so this constraint can be merged with LMI ones. The set of scalings of the repeated performance channel which has to be optimized (see figure 1).

The $L$ matrix scales the parameter dependent channels (system and controller one) which reduces the conservatism of the small gain theorem despite of render this inequality non convex. However, this apparent untractable problem can be easily exactly relaxed without inducing some conservatism using the projection lemma (See [1]). The $\gamma$ term acts on the performance channel which needs to be optimized.

Let us consider now the LPV system, represented as in Figure 1.

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bw(t) + Bu(t) \\
z(t) &= Cx(t) + Dw(t) + Du(t)
\end{align*}
$$

with matrices

$$
\begin{align*}
\bar{B} &= \begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix} \\
\bar{C} &= \begin{bmatrix} C^T_0 & C^T_1 & C^T_2 \end{bmatrix}^T \\
\bar{D} &= \begin{bmatrix} D_{00} & D_{01} & D_{02} \\
D_{10} & D_{11} & D_{12} \\
D_{20} & D_{21} & D_{22} \end{bmatrix} \\
\bar{D}_w &= \begin{bmatrix} D^T_{0u} & D^T_{1u} & D^T_{2u} \\
D^T_{0w} & D^T_{1w} & D^T_{2w} \end{bmatrix} \\
z &= \begin{bmatrix} z_0^T & z_1^T & z_2^T \end{bmatrix}^T \\
w &= \begin{bmatrix} w_0^T & w_1^T & w_2^T \end{bmatrix}^T
\end{align*}
$$

where $x$ is the state, $u$ the control input, $z_0/w_0$ the LPV channel, $z_1/w_1$ the uncertain channel and $z_2/w_2$ the performance channel which has to be optimized (see figure 1).
The aim of this paper is to design a gain-scheduled state feedback controllers (GSSF) of the form (9) (see figure 2)

\[
\begin{align*}
    u(t) & = K_{00}x(t) + K_{01}w_c(t) \\
    z_c(t) & = K_{10}x(t) + K_{11}w_c(t) \\
    w_c(t) & = \Theta(t)z_c(t)
\end{align*}
\]

where \( u \) is the control input, \( z_c \) and \( w_c \) represent the controller parameter dependent signals, such that the closed-loop system is asymptotically stable and the \( H_\infty \) norm of the channel \( w_2 \rightarrow z_2 \) is less than \( \gamma \) in delay-dependent and delay-scheduled framework.

We present here a LPV state feedback existence theorem inspired from [1].

**Theorem 2.1: LPV state-feedback existence theorem**

Consider the LPV system of the form (7)-(8) with \( C_y = I \) and \( D_{yu} = 0 \), then if there exist some positive definite matrices \( X, L_3, J_3 \) and a positive scalar \( \gamma \) such that these LMI holds

\[
    K_i^T \begin{bmatrix} L_i & I \\ I & J_i \end{bmatrix} K_i < 0, \text{ for } i = 1, 2
\]

with

\[
\begin{align*}
    \Lambda_1 & = -L_3 \oplus -I_1 \oplus -\gamma I_{w_2} & T_1 & = \tilde{D} \\
    M_1 & = L_3 \oplus J_1 \oplus \gamma^{-1} J_2 \oplus \gamma^{-1} I_{w_2} & K_1 & = I \\
    \Lambda_2 & = \begin{bmatrix} YA \oplus AY & YC \end{bmatrix} \hspace{1cm} V_2 = J_3 \oplus I_2 \oplus \gamma^{-1} I_{w_2} \\
    T_2 & = \begin{bmatrix} B^T & \tilde{D}^T \end{bmatrix} \hspace{1cm} M_2 = J_3 \oplus I_1 \oplus \gamma^{-1} I_{w_2} \\
    \mathcal{K}_2 & = \text{Ker} \begin{bmatrix} B_u^T & \tilde{D}_u^T \end{bmatrix}
\end{align*}
\]

then there exists a GSSF of the form (9) which stabilizes (7)-(8) with an achieved \( H_\infty \) performance index lower than \( \gamma \). The controller matrices (9) can be found by solving the following LMI

\[
    \Psi_{sf} + P_{sf}^T K Q_{sf} + P_{sf} K^T Q_{sf}^T < 0
\]

where \( K = \begin{bmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \end{bmatrix} \) and \( \Psi_{sf} \) is the matrix (6) with matrices

\[
\begin{align*}
    \tilde{A} & = \tilde{A} \oplus \tilde{B} \hspace{1cm} \tilde{C} = \begin{bmatrix} 0 \\ C \end{bmatrix} \hspace{1cm} \tilde{D} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \\
    L_w & = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \oplus I_1 \oplus \gamma I_{w_2} \\
    L_z^{-1} & = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \oplus I_1 \oplus I_{z_2} \\
    P_{sf} & = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \hspace{1cm} Q_{sf} = \begin{bmatrix} I & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

\[
\text{Proof:} \quad \text{A sketch of a proof and the way to find the whole matrix } L \text{ are discussed in appendix A.}
\]

Note that several controllers may satisfy the LMI (12), therefore it may be useful consider other specifications robust pole placement, stable controller, small coefficients.

III. LPV CONTROLLERS SYNTHESIS - DELAY DEPENDENT FRAMEWORK

In all what follows we will consider TDS and provide some transformations such that the TDS is rewritten under the LFT form (7)-(8), which allows to use theorem 2.1.

**A. A new model transformation**

Consider a linear time delay system

\[
\begin{align*}
    \dot{x}(t) & = Ax(t) + A_h x_h(t) + B x_1(t) \\
    z_1(t) & = C x(t) + C_h x_h(t) + D x_1(t)
\end{align*}
\]

where \( x \) is the state, \( x_h = x(t-h(t)) \), \( w_1 \) the exogenous inputs and \( z_1 \) the controlled outputs. Let \( D_h(\cdot) \) be the operator \( D_h := \nu(t) \to \frac{h_m}{h_M} \int_{t-h(t)}^t h(\eta)^{-1} \nu(\eta) d\eta \) where \( 0 < h_m \leq h(t) \leq h_M < +\infty \). One can check that \( \|D_h\|_\infty \leq 1 \) (see appendix B).

Then the system (13) may be rewritten as the following uncertain linear parameter varying where \( h(t) \) is viewed as a parameter. Indeed \( x_h = x - w_0 \) and \( z_0 = \alpha(t) \hat{x} \) then replace them into the system (13) leads to

\[
\begin{align*}
    \dot{\hat{x}} & = (A + A_h)x(t) - A_h w_0(t) + B x_1(t) \\
    z_0(t) & = (C + C_h)x(t) - C_h w_0(t) + D x_1(t) \\
    z_1(t) & = \alpha(t)(A + A_h)x(t) + \alpha(t)A_h w_0(t) + \ldots + \alpha(t)B x_1(t) \\
    w_0(t) & = D_h(z_0(t)) = x(t) - x_h(t) \\
    \alpha(t) & = h(t)\frac{h_M}{h_m}
\end{align*}
\]

Hence, we can see that the stability of the system (14) implies the stability of system (13) because this transformation induces additional dynamics (the initial value is now defined on \([-2h_M, 0] \)). We may also analyze the stability of these additional dynamics (see [8] for details). An advantage of this transformation comes from the fact that the delay bounds appear explicitly in the expression of the system and
allow us to establish a delay-dependent stability test. This transformation permits to transform the time-delay system stability analysis problem into a LPV uncertain system stability analysis. A similar result is provided in [13] for constant time-delay. That means that we could use classical tools from robust analysis and control as theorem 2.1.

Note that the operator $D_h(\cdot)$ is treated as an uncertainty which is less conservative than a delay independent criteria where the whole delay is considered as uncertainties. A similar model transformation has been performed for constant time delays in [12].

B. $L_2$-$L_2$ stability Analysis

Theorem 3.1: Consider the transformed time-delayed system (14), if there exist symmetric positive definite matrices $P$, $Q$ and a scalar $\gamma > 0$ such that the two LMIs hold

$$
\begin{bmatrix}
    A^T X + X A - X A_h X B_2 & \alpha_1 A^T Q & C^T \\
    * & -Q & -C_h^T \\
    * & * & -\gamma I_{w_1} \\
    * & * & * & -Q \\
    * & * & * & -\gamma I_{z_1}
\end{bmatrix} < 0
$$

(15)

for $i = 1, 2$ with $\alpha_1 = h_M$, $\alpha_2 = \frac{h^2_M}{T_m}$, $\bar{A} = A + A_h$ and $\bar{C} = C + C_h$, then the system (14) is asymptotically stable and hence time-delayed system (13) is asymptotically stable for all delays belonging to $[h_m, h_M]$ and the channel $w_1(t) \rightarrow z_1(t)$ has an induced $L_2$-norm lower than $\gamma$.

Proof: The first step is to inject the matrices of the uncertain LPV system (14) into the scaled bounded real lemma LMI (6), with $L_w = L + \gamma I_{w_1}$, $L_z = L + \gamma I_{z_1}$.

Then we perform a congruence transformation with the non-singular matrix $I \oplus I \oplus I \oplus L \oplus I$. That leads to a semi-infinite LMI written $F_1 + h(t)F_2 < 0$ where $F_1, F_2$ are respectively the LMI constant part and the parameter (the delay) dependent part. As the relation in $h(t)$ is affine then via a convex argument we can state that if we check the feasibility of the LMI at the vertices of the convex set where $h(t)$ evolves, then the parameter dependent LMI will be negative definite for every $h(t)$ in the convex set. Hence, it remains two LMIs only for $h_m$ and $h_M$ which corresponds to the LMI (15).

Remark 3.1: The above theorem permits to conclude on the system stability only onto the separated delay interval $[0) \cup [h_m, h_M]$. If the LMI (15) is satisfied then that means that the upper-left term is negative definite and hence that the system without delay is asymptotically stable.

In the remaining parts of this section, we consider the following LPV time-delay system (see figure 3-a)

$$
\begin{align*}
\dot{x} &= Ax + A_h x_h + B_0 w_0 + B_2 w_2 + B_4 u \\
\dot{z}_0 &= C_0 x + D_{00} w_0 + D_{20} w_2 + D_{0u} u \\
\dot{z}_2 &= C_2 x + D_{20} w_0 + D_{22} w_2 + D_{2u} u
\end{align*}
$$

(16)

where $x$ is the system state, $x_h$ the delayed state, $u$ the control input, $z_0/w_0$ the LPV channel, $z_2/w_2$ the performance channel with the exogenous inputs $w_2$ and the controlled outputs $z_2$.

Using the model transformation presented in III-A, the above model is transformed into a LPV uncertain (memory-less) system (fig. 3-b) where $\Delta$ is the operator $D_h(\cdot)$

$$
\begin{align*}
\dot{x} &= \bar{A} x + \bar{B} w + \bar{B}_u u \\
\dot{z} &= \bar{C} x + \bar{D} w + \bar{D}_u u
\end{align*}
$$

(17)

with $w = [w^T_0 \ w^T_1 \ w^T_2]^T$ and $z = [z^T_0 \ z^T_1 \ z^T_2]^T$ where we add the uncertain channel $z_1/w_1$.

![Fig. 3. (a) LPV time delay system (b) LPV uncertain system](image)

We consider now systems of the form (17) with matrices

$$
\bar{A} = A + A_h, \quad \bar{D}_u = \begin{bmatrix} D_{0u}^T & \alpha_1 \bar{B}_u^T \ D_{2u}^T \end{bmatrix}^T
$$

$$
\bar{B} = \begin{bmatrix} B_0 & -A_h & B_2 \end{bmatrix}, \quad \bar{C}_y = I
$$

$$
\bar{C}_t = \left[ (C_0 + C_{0h})^T \ \alpha_1 \bar{A}^T \ (C_2 + C_{2h})^T \right]^T
$$

$$
\bar{D}_t = \begin{bmatrix} -C_{0h} & D_{01} & D_{02} \\
-\alpha_1 C_{0h} & \alpha_1 B_1 & \alpha_1 B_2 \\
-C_{2h} & D_{21} & D_{22} \end{bmatrix}, \quad \bar{D}_{yw} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}
$$

The above system is represented in a similar representation than system (7).

C. Delay dependent LPV state feedback result

We propose in this part a control law of the form $u(t) = K(\theta(t))x(t)$ in a delay dependent framework. The delay is assumed to belong to a compact set $[h_m, h_M]$. We consider here a single delayed state system but this method can be easily extended to multiple delayed state system.

Lemma 3.1: Delay-dependent LPV state feedback existence lemma

Consider the state-delayed system (16). If there exist some symmetric positive definite matrices $Y$, $L_3$, $J_3$ and a positive scalar $\gamma$ such that these LMIs hold

$$
K_j^T \left( L_j + T_j^T M_j T_j \right) K_j < 0
$$

(18)

for $j = 1, 2$ and $i = 1, 2$

$$
L_3, J_3 \in L_\Theta \begin{bmatrix} L_3 & I \\
I & J_3 \end{bmatrix} \geq 0
$$

(19)
with
\[
\begin{align*}
\Lambda_1 &= -L_3 \oplus -I_1 \oplus -\gamma I_{w_2} \\
M_1 &= L_3 \oplus I_1 \oplus -\gamma I_{z_2} \\
\Lambda_2 &= \begin{bmatrix} YA^T + \lambda YC^T \end{bmatrix} + V_j, \quad V_j = J_3 \oplus I_1 \oplus \gamma I_{z_2} \\
T_2 &= \begin{bmatrix} \bar{B}_i^T \bar{D}_t \end{bmatrix}^T \\
K_i &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\
T_1 &= \bar{D}_i
\end{align*}
\]

then there exist a GSSF of the form (9) which stabilizes (16) with an achieved performance index lower than \( \gamma \). The controller matrix can be found by solving the two following LMIs
\[
\Psi_{sf_i} + P_{sf_i}^T K Q_{sf_i} + P_{sf_i} K^T Q_{sf_i}^T < 0
\]

where \( K = [K_{10} \ K_{11}] \) and \( \Psi_{sf_i} \) is the matrix (6) with matrices
\[
\tilde{A} = A \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\
\lambda \Lambda I & \gamma I_{w_2} \\
\gamma I_{z_2} \end{bmatrix} \\
\tilde{C} = \begin{bmatrix} 0 & 0 \end{bmatrix} \\
\tilde{D} = \begin{bmatrix} 0 & 0 \end{bmatrix} \\
P_{sf_i} = \begin{bmatrix} 0 & Y^{-1} B_u \\
0 & 0 \\
0 & I \\
D_u & 0
\end{bmatrix} \\
Q_{sf_i} = \begin{bmatrix} I & 0 \end{bmatrix}
\]

IV. DELAY-SCHEDULED CONTROLLERS

In the literature, two kinds of controllers are studied. Controllers with memory which contain delayed terms (for example \( u(t) = K_0 x(t) + K_1 x(t - h(t)), \ldots \)) and memoryless controllers which do not contain the types of terms. Here we propose a method for the design of delay-scheduled memoryless controllers for time delay system when approximate value of the delay can be measured or estimated in real-time. This means that the delay value will appear in the controller parameter set but no delayed state is used. Moreover, we take into account in the synthesis a possible error on the delay measurement.

In this section, we provide sufficient conditions to existence of stabilizing delay-scheduled state feedback for both LTI and LPV time-delay systems.

A. Problem Formulation

We consider here that the system delay \( h(t) \) can be expressed as \( h(t) = h_a(t) + h_c(t) \) where \( h_a(t) \) is the measured delay used in the controller and \( h_c(t) \) is the measurement error such that \( 0 \leq |h_c(t)| < \delta_e \).

Now consider another outer representation of the system (13) where \( \Theta(t) = h_a(t) \) and \( \Delta = D_h(h(t) \oplus h_c(t)/\delta_e I) \) (see figure 4)

\[
\dot{x} = \tilde{A} x + \tilde{B} u + \tilde{C} \Theta + \tilde{D} \Delta
\]

\[
\dot{\Theta} = \tilde{A} \Theta + \tilde{B} u + \tilde{C} \Theta + \tilde{D} \Delta
\]

\[
\dot{\Delta} = \tilde{A} \Delta + \tilde{B} u + \tilde{C} \Theta + \tilde{D} \Delta
\]

\[
\dot{z}_0 = \tilde{A} z_0 + \tilde{B} u + \tilde{C} \Theta + \tilde{D} \Delta
\]

\[
\dot{z}_1 = \tilde{A} z_1 + \tilde{B} u + \tilde{C} \Theta + \tilde{D} \Delta
\]

\[
\dot{z}_2 = \tilde{A} z_2 + \tilde{B} u + \tilde{C} \Theta + \tilde{D} \Delta
\]

B. Delay-scheduled state feedback result

For that kind of controller, we must define \( \tilde{C}_y = I \) and \( \tilde{D}_{yu} = 0 \). Then applying theorem 2.1 we obtain the following lemma

Lemma 4.1: Delay-scheduled state feedback existence lemma

Consider the system (21) then if there exist symmetric
positive definite matrices $L_3$, $J_3$, $Y$ and a scalar $\gamma > 0$ such that the LMIs hold
\[ K_i^T (A_i + T_i^T M_i T_i) K_i < 0, \quad \text{for } i = 1, 2 \] (23)

and
\[ \begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0, \quad L_3, J_3 \in \mathcal{L}_2 \] (24)

\[ K_i = \begin{bmatrix} Y A_i^T \Phi \big(Y C_i^T \Phi & -U_J \big) \\ Y C_i \end{bmatrix}, \quad T_i = \begin{bmatrix} D_i \end{bmatrix} \]

\[ K_i = \begin{bmatrix} \Phi & 0 \end{bmatrix}, \quad T_i = \begin{bmatrix} D_i \end{bmatrix} \]

and
\[ v \in \mathcal{L}_2 \]

then there exist a delay scheduled state-feedback of the form (22) which stabilizes the time-delayed system (13) with a $\mathcal{H}_\infty$ performance achievement lower than $\gamma$ on the channel $w_3 \rightarrow z_3$. The controller construction method is the same as stated in the theorem 2.1.

V. CONCLUSION

We have presented a LPV state-feedback design for LPV time-delay systems. The present method has been extended to a LPV dynamic output feedback design in [7]. They have been applied for uncertain time-delay systems, after model transformation, in the delay-dependent framework. We have also developed an outer approximation of LTI state-delayed systems represented as uncertain LPV systems which can be used for delay-dependent stability criteria using a classical bounded real lemma. The latter controller uses the approximate knowledge of the delay value to schedule the controller. The error between the delay real value and the known one is taken into account into the synthesis via a robustness analysis.

APPENDIX

A. Proof of state feedback existence theorem - matrices $L$ and $K$ construction

First of all the augmented system is given by
\[
\begin{bmatrix} x \\ z_c \\ z_0 \\ z_1 \\ z_2 \\ x \\ w_c \end{bmatrix} = \begin{bmatrix} A & 0 & B_0 & B_1 & B_2 & -I \\ 0 & C_0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & D_0 & 0 & 0 \\ 0 & C_2 & 0 & D_1 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w_c \end{bmatrix}
\]

The signals $w_c$ and $z_c$ represent the controller parameter-scheduled channel. Let $\Omega$ be the controller state-space "static" matrices representation. Then compute the closed-loop system and inject into the matrix inequality (6). We consider here matrices $L_0 = L \oplus I_1 \oplus \gamma I_{w_2}$ and $L_z = L \oplus I_1 \oplus \gamma I_{z_2}$ where $L$ scales the controller and system parameter-scheduled channels. It is straightforward to obtain that the inequality (6) is equivalent to the following one
\[ \Psi + P^T \Omega Q + \Omega P^T Q^T > 0 \] (26)

Note that this inequality is not linear in the matrix variables $X$, $\Omega$, $L_w$ and $L_z^{-1}$. However, it can be exactly relaxed without inducing any conservatism. Using the projection lemma we obtain the equivalent underlying problem
\[ K_p^T \Psi K_p < 0 \quad K_q^T \Psi K_q < 0 \]

where $K_p = Ker(P)$ and $K_q = Ker(Q)$. Then performing a Schur complement onto both inequalities leads to the LMIs used in the theorem 2.1 and the proof of the controller existence is completed.

The second part concerns the controller reconstruction. First, compute the matrices $L (J = L^{-1})$. We have computed in theorem 2.1 two matrices $L_3$ and $J_3$ which are blocks of bigger matrices defined as $L = \begin{bmatrix} L_1 & L_2 \\ * & L_3 \end{bmatrix}$ and $J = \begin{bmatrix} J_1 & J_2 \\ * & J_3 \end{bmatrix}$ where $LJ = I$. The algorithm is then

- Perform a SVD on $I - L_3 J_3$ to obtain the product $L_2^T J_2$ and isolate the two matrices.
- Then solve the matrix equation

\[
\begin{bmatrix} 0 & I \\ J_2^T & J_3 \end{bmatrix} L = \begin{bmatrix} L_2^T & L_3 \\ 0 & I \end{bmatrix}
\]

- then it is straightforward to obtain $J$.

Now inject the matrices describing the system (25) into the inequality (26) which is convex in $\Omega$. Hence it is easy to solve it with convex additional constraints if necessary.

B. Proof of the $\mathcal{H}_\infty$ norm of the operator $D_h(\cdot)$

Consider an input signal $w \in \mathcal{L}_2$ and an output signal $z \in \mathcal{L}_2$ such that $z(t) = O(w(t))$ is a stable operator. If that operator has an $\mathcal{L}_2$ induced norm of 1 then the input and output signal must satisfy the inequality
\[
\int_0^t z^T(\tau) Z z(\tau) d\tau \geq \int_0^t w^T(\tau) Z w(\tau) d\tau
\]

for some $Z = Z^T > 0$.

The aim of this proof is first to show stability of $D_h$ to prove $z \in \mathcal{L}_2 \Rightarrow w \in \mathcal{L}_2$. Secondly, one shows that the operator $D_h(\cdot)$ satisfies unitary $\mathcal{H}_\infty$-norm property.

Define $h(t) \in [h_m, h_M], 0 < h_m < h_M < +\infty$ and
\[ w(t) = \frac{h_m}{h_M} \int_{t-h(t)}^t \hspace{-1cm} h^{-1}(\beta) z(\beta) d\beta \in \mathcal{L}_2 \]

Inject the expression of $z(t)$ into $z$ and applying Jensen’s inequality we obtain
\[ \mathcal{I} \leq \frac{h_m^2}{h_M^2} \int_{t-h(t)}^t \hspace{-1cm} h(\tau) \int_{\tau-h(\tau)}^{\tau} \hspace{-1cm} h^{-2}(\beta) Z^T(\tau) Z z(\beta) d\beta d\tau \]

As $z \in \mathcal{L}_2$ and $h^{-1}(\tau)$ is always finite then $\int_{\tau-h(\tau)}^{\tau} \hspace{-1cm} h^{-2}(\beta) Z^T(\tau) Z z(\beta) d\beta$ is finite. In fact, this integral term represents a band of energy from $\tau - h(\tau)$ to $\tau$. Finally, we sum all these bands weighted by $h(\tau)$ over the interval $[0, t]$. This sum is obviously finite since $h(\cdot)$ is finite. This proves that $D_h$ is a stable operator from $\mathcal{L}_2$ to $\mathcal{L}_2$. 
Bounding the term $h^{-1}(\cdot)$ on the interval $[h_m, h_M]$ leads to

$$I \leq \frac{1}{h_M} \int_0^t h(\tau) \int_{\tau-h(\tau)}^{\tau} z^T(\beta) Z z(\beta) d\beta d\tau$$

Exchange the order of integration, considering zero initial condition, to obtain

$$I \leq h^{-2}_M \int_0^{t-h(t)} \left[ \int h(\tau)d\tau \right] z^T(\beta) Z z(\beta) d\beta$$

but

$$\int \tau d\tau \leq h^2_M$$

and hence we have

$$I \leq \int_0^{t-h(t)} z^T(\tau) Z z(\tau) d\tau \leq \int_0^t z^T(\tau) Z z(\tau) d\tau$$

which completes the proof.

A similar proof can be found in [8].

REFERENCES