$\mathcal{H}_\infty$ delay-scheduled control of linear systems with time-varying delays

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***Abstract*— This paper deals with $\mathcal{H}_\infty$ delay-scheduled control of linear systems with time-varying delays. First, a new model transformation is given which allows to provide a unified approach to stability analysis and state-feedback control synthesis for time-delay systems represented in 'LFT' form. A new type of controller is then synthesized, where the state-feedback is scheduled by the value of the delay (delay-scheduled controllers). The results are provided in terms of Linear Matrix Inequalities (LMIs) which are known to be efficiently solvable.**

***Index Terms*— Time-delay systems, LPV control, robust control, LFT**

I. Introduction

Since several years, many papers have been devoted to the study of time-delay systems (TDS) with constant delays (see for instance [1]–[3] and references therein). More recently, systems with time-varying delays arising for instance in network controlled systems, have attracted more and more attention (See [4]–[6] and references therein). Indeed, lags in communication channels may destabilize such systems, or at least, deteriorate performance. Different approaches devoted to the study of time-delay systems stability have been developed in the literature. Let us mention, among others, the use of Lyapunov-Krasovskii functionals (or Lyapunov-Razumikhin functions) [1], [6], [7], robust analysis [6], [8], well-posedness of feedback systems [9], spectral approaches [6], [8], etc.

The approaches developed in this paper are mainly based on the notions of robust stability, robust and LPV control of linear dynamical systems. Some new results for the stability and control of such systems are provided and suggest that the robust control approach can be used to derive Linear Parameter Varying (LPV) controllers for LTI time-delay systems.

In the last two decades, LPV systems have been of growing interest since they allow to approximate nonlinear and LTV systems [10]–[16]. Three main approaches are usually considered in the study and control of LPV systems involving LMIs: the polytopic approach [17], the use of parameter-dependent LMIs [18] or the Linear Fractional Transformation (LFT) approach [10]–[12].

While both first approaches have been recently extended to time-delay systems [19]–[22], the latter has been very few used in the context of time-delay systems and especially for bounded time-varying delays. This is due to the fact that the LFT formulation for LPV/uncertain time-delay systems is difficult to apply and may result in untractable conditions. Indeed, by applying classical robust control theorems (such as projection lemma) on bounded-real lemmas obtained from Lyapunov-Krasovskii functionals yields, in many cases, nonlinear matrix inequalities (due to the supplementary decision matrices) whose solving is known to be an NP-hard problem.

This paper proposes a new approach to study delay-dependent stability and control of LTI time-delay systems, within a unified framework involving robust control theory for LPV systems. The contributions of the paper are the following:

- First, a new model transformation for TDS with time-varying delays is introduced which turns a time-delay system into an uncertain LPV system in 'LFT' form. Using this formulation, a stability test based on the scaled bounded real lemma allows to conclude on asymptotic stability of the time-delay system. The interest of such a reformulation resides in the fact that many classical robust control tools are available and can be used in order to derive sufficient conditions for the controller existence.
- A new kind of controller, which has been referred to as delay-scheduled state-feedback controller, is developed when an approximate value of the delay is known or estimated. The error on the delay knowledge value is taken into account to ensure the robustness of the closed-loop system with respect to this uncertainty.

The interest and advantage of the provided methodology rely on the fact that, for the first time, a unique LFT formulation to design different types of controllers for different classes of time-delay systems is proposed. Even if the present paper is devoted to (delay-scheduled) state-feedback controllers only, the approach can be easily extended to the case of (delay-scheduled) dynamic output feedback [11]. Moreover, even if only the single-delay problem is addressed, the methodology is also valid in the multiple-delay case and for LPV TDS. Finally, this method describes a new original way to control time-delay systems and the authors stress that this approach may be of great interest for systems with large variation of the delay since a controller gain adaptation will be provided accordingly.

The paper is structured as follows, section II gives the paper objectives and preliminary results. Section III introduces the new model transformation. In section IV the new LPV based control method for time-delay systems is exposed. Finally Section V concludes on the paper and gives future works.

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Along the paper, the notation is standard and $Ker[A]$ stands for a basis of the null-space of A.

II. Paper Objectives and Preliminary Results

Let us consider in this section LPV/uncertain linear systems represented in 'LFT' form:

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}(t) \\ z_0(t) \\ z_1(t) \end{bmatrix} & = & \begin{bmatrix} A & B_0 & B_1 \\ C_0 & D_{00} & D_{01} \\ C_1 & D_{10} & D_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ w_0(t) \\ w_1(t) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{0u} \\ D_{1u} \end{bmatrix} u(t) \\ w_0(t) & = & \Theta(z_0(t)) \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w_1 \in \mathbb{R}^r$, $z_1 \in \mathbb{R}^s$ are respectively the state, the control input, the exogenous inputs and the controlled outputs. The signals w_0/z_0 are 'virtual' signals describing the interconnection of the LTI system and Θ.

In this work, Θ will contain both linear operators and multiplicative parameters. In consequence, it is chosen to have the following diagonal structure:

$$\Theta := \begin{bmatrix} \text{diag}_i[\theta_i(t) I_{p_i}] & 0 \\ 0 & \text{diag}_i[\mathcal{L}_i(\cdot) I_{o_i}] \end{bmatrix} \tag{2}$$

where p_i and o_i are respectively the number of occurrences of the i^{th} parameter θ_i and operator $\mathcal{L}_i(\cdot)$. The parameters $\theta_i(t)$ are assumed to belong to the interval $[-1, 1]$ and the linear operators to have an $\mathcal{H}_\infty$ norm (or $\mathcal{L}_2$-induced norm) less than 1. With such a formulation it is possible to consider both polynomial and rational dependence on parameters and operators.

The upcoming results in this section are developed for LPV systems only (without time-delay) and will be used in Section III. Let us consider now the general uncertain LPV system

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} & = & \begin{bmatrix} \bar{A} & \bar{B} & \bar{B}_u \\ \bar{C} & \bar{D} & \bar{D}_u \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \\ w_0(t) & = & \Theta(z_0(t), t) \\ w_1(t) & = & \Delta(z_1(t), t) \end{array} \tag{3}$$

with $z = col(z_0, z_1, z_3)$, $w = col(w_0, w_1, w_3)$ and

$$\bar{D} = \begin{bmatrix} D_{00} & D_{01} & D_{02} \\ D_{10} & D_{11} & D_{12} \\ D_{20} & D_{21} & D_{22} \end{bmatrix} \quad \begin{array}{rcl} \bar{B} & = & \begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix} \\ \bar{C} & = & \begin{bmatrix} C_0^T & C_1^T & C_2^T \end{bmatrix}^T \\ \bar{D}_u & = & \begin{bmatrix} D_{0u}^T & D_{1u}^T & D_{2u}^T \end{bmatrix}^T \end{array} \tag{4}$$

where w_0/z_0, w_1/z_1 and w_3/z_3 are respectively the LPV channel, the uncertain channel and the performance channel which has to be optimized. Θ are scheduling parameters/operators defined in (2) and Δ the uncertainties obeying to

$$\Delta := \left\{ \operatorname*{diag}_{i=1}^{n_u} \Delta_i, \ ||\Delta_i||_\infty < 1, \ i = 1, \dots, n_u \right\} \tag{5}$$

Definition 2.1: The aim of the paper is to design a gain-scheduled state-feedback (GSSF) control law of the form:

$$\begin{bmatrix} u(t) \\ z_c(t) \end{bmatrix} = K \begin{bmatrix} x(t) \\ w_c(t) \end{bmatrix} \qquad w_c(t) = f(h(t)) z_c(t) \tag{6}$$

where $f(\cdot)$ is a scheduling function to be defined/determined, w_c/z_c the controller scheduling channel, such that the closed-loop LPV time-delay system (3) is asymptotically stable and $||z_3||_{\mathcal{L}_2} \leq \gamma ||w_3||_{\mathcal{L}_2}$.

The following theorem gives a new sufficient condition for the existence of a gain-scheduling state-feedback controller (similar to the sufficient condition for the existence of a dynamic output feedback provided in [11]).

Theorem 2.1: Consider the LPV system (3)-(4) of order n. If there exist symmetric positive definite matrices X, L_3, J_3 and a scalar $\gamma > 0$ such that

$$\mathcal{K}_i^T (N_i + T_i^T M_i T_i) \mathcal{K}_i < 0, \ i = 1, 2 \tag{7}$$

$$\begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0 \quad L_3, J_3 \in L_\Theta$$

with $N_1 = -diag(L_3, I_{n_1}, \gamma I_{w_3})$, $N_2 = \begin{bmatrix} \bar{A}X + X\bar{A}^T & X\bar{C}^T \\ \star & -V_J \end{bmatrix}$, $V_J = diag(J_3, I_{n_1}, \gamma I_{z_3})$, $T_1 = \bar{D}$ $\mathcal{K}_1 = I$, $T_2 = \begin{bmatrix} \bar{B}^T & \bar{D}^T \end{bmatrix}$, $M_2 = diag(J_3, I_{n_1}, \gamma I_{z_3})$, $\mathcal{K}_2 = Ker \begin{bmatrix} \bar{B}_u^T & \bar{D}_u^T L \end{bmatrix}$ and L_Θ is the set of positive definite matrices commuting with Θ. Then there exists a gain-scheduled state-feedback of the form (6) which stabilizes (3)-(4) and ensures an $\mathcal{H}_\infty$ performance index lower than γ, according to Definition 2.1.

Proof: A sketch of a proof is presented in appendix A. ■

Since the conditions of theorem 2.1 are stabilizability conditions which do not depend explicitly on the controller matrix, it is then computed separately. The computing technique is described in [11] and recalled in Appendix B for simplicity.

III. A new model transformation for delay-dependent stability analysis

This section introduces a new model transformation which turns a time-delay system with time-varying delays into an uncertain LPV system represented in an 'LFT' form. This transformation allows to use classical robust stability analysis and control synthesis on the transformed system, in order to derive a delay-dependent stability test obtained from the scaled-bounded real lemma. Similar approaches can be found for instance in [23] where the maximal value of the delay appears explicitly in the comparison model. In this paper, the comparison model is an uncertain parameter varying system which is then studied in the robust/LPV framework. This is a real novelty in the analysis and control of time-delay systems with time-varying delays.

A. Model transformation

Let us consider the LTI time delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t - h(t)) + B w_3(t) + B_u u(t) \\ z_3(t) & = & Cx(t) + C_h x(t - h(t)) + D w_3(t) + D_{3u} u(t) \\ x(\eta) & = & \phi(\eta), \ \eta \in [-h_M, 0] \end{array} \tag{8}$$

where x, w_3, z_3, u and ϕ are respectively the state, the exogenous inputs, the controlled outputs, the control input and the functional initial condition. The delay $h(t)$ is assumed to belong to the set

$$\mathscr{H} := \left\{ h(t) \in \mathscr{C}^1(\mathbb{R}^+, [h_m, h_M]), 0 \leq h_m < h_M < +\infty \right\} \tag{9}$$

where $\mathscr{C}^1(J_1, J_2)$ is the set of differentiable functions mapping J_1 to J_2. Let us consider next the operator

$$\mathcal{D}_h \quad : \quad w(t) \quad \rightarrow \quad \int_{t-h(t)}^{t} (h(\eta) + h_M - h_m)^{-1} w(\eta) d\eta \tag{10}$$

where $h \in \mathscr{H}$.

Lemma 3.1: The operator $\mathcal{D}_h$ enjoys the following properties

1) $\mathcal{D}_h(\cdot)$ is linear
2) $\mathcal{D}_h(\cdot)$ has an $\mathcal{L}_2$-induced norm less than 1.

Proof: The proof is similar to the one given in [6]. ■

By the mean of this model transformation and the use of the scaled bounded real lemma, it is possible to provide the first main result of the paper.

Proposition 3.1: Assume $u \equiv 0$, the LPV delay free system (11) is a comparison system for the LTI time-delay system (8).

$$\begin{array}{rcl} \dot{x}(t) & = & (A + A_h)x(t) - A_h w_1(t) + B w_3(t) \\ z_1(t) & = & \alpha(t)(A + A_h)x(t) - \alpha(t) A_h w_1(t) \\ & & +\alpha(t) B w_3(t) \\ z_3(t) & = & (C + C_h)x(t) - C_h w_1(t) + D w_3(t) \\ w_1(t) & = & \mathcal{D}_h(z_1(t)) \\ x(\eta) & = & \psi(\eta), \ \eta \in [-2h_M, 0] \\ \alpha(t) & = & h(t) + h_M - h_m \end{array} \tag{11}$$

where $\psi(\theta)$ is the new functional initial condition which coincides with $\phi(\theta)$ over $[-h_M, 0]$.

Proof: First note that the dynamical equation of system (8) can be rewritten as

$$\dot{x}(t) = (A + A_h)x(t) - A_h w_1(t) + B w_3(t) \tag{12}$$

where $w_1 = x - x_h$. Then defining

$$\begin{array}{rcl} z_1(t) & = & \alpha(t)[(A + A_h)x(t) - A_h w_1(t) + B w_3(t)] \\ & = & \alpha(t)\dot{x}(t) \end{array}$$

with $\alpha(t) = h(t) + h_M - h_m$ we have

$$\begin{array}{rcl} \int_{t-h(t)}^{t} \alpha^{-1}(\tau) z_1(\tau) d\tau & = & \int_{t-h(t)}^{t} \dot{x}(\tau) d\tau \\ & = & x(t) - x(t - h(t)) \\ & = & w_1(t) \end{array} \tag{13}$$

Thus we obtain the first and second line of system (11). The third line is obtained by the same way. ■

Remark 3.1: Since (11) requires an initial functional condition on $[-2h_M, 0]$ then, under some particular conditions, unstable additional dynamics may be created, making the comparison system unstable even if the original one is stable. This will result in conservatism and indicates that the stability of system (11) is only a sufficient condition for the stability of (8); this fact is usual when model transformations are used. However as emphasized in [6] the study of additional dynamics is not easy in the time-varying delay case and remains an open problem.

The interest of the new model transformation is to turn the LTI time-delay system (8) into an uncertain linear parameter varying system (11) where the operator $\mathcal{D}_h(\cdot)$ plays the role of a norm-bounded uncertainty, and the delay $h(t)$ the role of a time-varying parameter through the term $\alpha(t)$. Then LPV/robust control tools can be used in order to study such system.

B. Delay-Dependent Stability test

The comparison model (11) is used in this section to develop a delay-dependent stability test with guaranteed $\mathcal{L}_2$ performances. This test is based on the application of the scaled-bounded real lemma [11] and is useful to obtain stabilization results.

Theorem 3.1 (Delay dependent scaled-bounded real lemma):

The system (11) with $h \in \mathscr{H}$ is asymptotically stable with an $\mathcal{L}_2$ performance index on channel $w_3 \rightarrow z_3$ lower than γ if there exist symmetric positive definite matrices X_1, X_2 and a positive scalar γ such that the LMIs

$$\begin{bmatrix} \hat{A}^T X_1 + X_1 \hat{A} & -X_1 A_h & X_1 B & \alpha_i \hat{A}^T X_2 & \bar{C}^T \\ \star & -X_2 & 0 & -\alpha_i A_h^T X_2 & -C_h^T \\ \star & \star & -\gamma I_w & \alpha_i B^T X_2 & D^T \\ \star & \star & \star & -X_2 & 0 \\ \star & \star & \star & \star & -\gamma I_z \end{bmatrix} < 0 \tag{14}$$

hold for $i = 1, 2$ with $\alpha_1 = h_M$, $\alpha_2 = 2h_M - h_m$, $\hat{A} = A + A_h$ and $\bar{C} = C + C_h$.

Proof: The proof is given in appendix C ■

The latter theorem concludes on the stability of system (11) for a delay taking values in the separated interval $0 \cup [h_m, h_M]$. Indeed, if the LMIs (14) are satisfied then the left-upper term is negative definite and finally the system with zero delay is asymptotically stable.

IV. Delay-scheduled controller synthesis

In this section, a new way to control linear time-delay systems of the form (8) is provided under the assumption that an approximate delay value is known. This approach differs radically from the common state-feedback with delayed state which consists in adding to the instantaneous state-feedback, a supplementary term involving the delayed state (e.g. $u(t) = Kx(t) + K_h x(t - h(t))$). The implementation of such a control law is difficult since it is generally difficult to know the exact value of the delay in real time, and the robustness analysis with respect to delay uncertainties is also not an easy task. Moreover, since past states need to be stored in memory, then the controller with memory is more expensive from a memory space point of view.

This motivates the introduction of the delay-scheduled controller which is midway between the instantaneous state-feedback and the state-feedback with delayed state. Indeed, while the delay-scheduled state-feedback depends only on the instantaneous state, it also depends on the delay $h(t)$ in a scheduling fashion:

$$u(t) = K(h(t))x(t) \tag{15}$$

This method does not need any dynamical model of the delay nor its exact value but only an approximative measurement. Indeed, since in the comparison model (11) the delay is viewed as a parameter, then the robust analysis with respect to delay uncertainties on the implemented delay is brought back to a perturbation on the parameters, which is simple to deal with. Hence, using this approach it is possible to ensure the robust stability of the closed-loop system in presence of delay uncertainties.

To design these controllers, the role of the model transformation defined in section III is crucial since it turns a linear time-delay system into an uncertain LPV system (11). Due to the structure of this system (finite-dimensional), it is possible to use available tools in analysis and control of LPV systems (and hence theorem 2.1).

A. Controller Existence

The time-delay is assumed to belong to the set $\mathscr{H}$ defined in (9). In order to apply the scaled-small gain theorem, the first step is to normalize the set of values of the parameters such that it coincides with the closed unit ball. Note first that the system delay $h(t)$ can be expressed as $h(t) = h_c(t) + h_e(t)$ where $h_c(t)$ is the measured delay that will be used in the controller and $h_e(t)$ the bounded measurement error satisfying $|h_e(t)| < \delta_e$. Then denoting respectively $\bar{h}$ and δ_h the mean value and the radius of the delay interval $[h_m, h_M]$ defined as $\bar{h} = (h_m + h_M)/2$ and $\delta_h = h_M - \bar{h}$, the normalized delay parameter $h_n(t)$, which takes value in $[-1, 1]$, is defined by $h_c(t) = \delta_h h_n(t) + \bar{h}$.

The following proposition provides a comparison model for (8) where the above defined notations are used instead of 'absolute' notations used in (11). Indeed, a new comparison model is necessary to derive controller existence conditions by taking into account the delay uncertainty.

Proposition 4.1: The system (16) is a comparison system for (8) (a control input u has been added)

$$\begin{array}{rcl} \begin{bmatrix} \dot{x} \\ \hline z \end{bmatrix} &=& \left[\begin{array}{c|c|c} \bar{A} & \bar{B}_w & \bar{B}_u \\ \hline \bar{C}_z & D_w & \bar{D}_u \end{array}\right] \begin{bmatrix} x \\ \hline w \\ \hline u \end{bmatrix} \\ w_0(t) &=& h_n(t) z_0(t) \\ w_1(t) &=& \mathcal{D}_h(z_1(t)) \\ w_2(t) &=& \dfrac{h_e(t)}{\delta_e} z_2(t) \end{array} \tag{16}$$

where $z = col(z_0, z_1, z_2, z_3)$, $w = col(w_0, w_1, w_2, w_3)$

$$\begin{array}{rcl} \bar{A} &=& A + A_h \qquad \bar{B}_w = \begin{bmatrix} -A_h & 0 & B_3 \end{bmatrix} \\ \bar{C}_z^T &=& \begin{bmatrix} A + A_h & \alpha(A + A_h) & A + A_h & C_3 + C_{3h} \end{bmatrix}^T \\ \bar{D}_u^T &=& \begin{bmatrix} B_u & \alpha B_u & B_u & D_{3u} \end{bmatrix}^T \qquad \bar{B}_u = B_u \\ D_w &=& \begin{bmatrix} 0 & -A_h & 0 & B_3 \\ \delta_h I & -\alpha A_h & \delta_e I & \alpha B_3 \\ 0 & -A_h & 0 & B_3 \\ 0 & -C_{3h} & 0 & D_{33} \end{bmatrix} \end{array}$$

with $\alpha = h_M - h_m + \bar{h}$ and where $w_3(t)$ and $z_3(t)$ are respectively the exogenous inputs and controlled outputs.

Proof: The proof consists in computing the generalized system by substituting input/output signals w_0/z_0, w_1/z_1, w_2/z_2 by their explicit expression, in order to show that the computed generalized system finally coincides with system (8). First, note that $h(t) = h_c(t) + h_e(t)$, $h_c(t) = \delta_h h_n(t) + \bar{h}$ and let $\rho(t) := \dot{x}(t) = z_0(t) = z_2(t)$. We have $w_0(t) = h_n(t) z_0(t) = h_n(t)\rho(t)$ and $w_2(t) = \dfrac{h_e(t)}{\delta_e} z_2(t) = \dfrac{h_e(t)}{\delta_e}\rho(t)$. Then according to (16)

$$\begin{array}{rcl} z_1(t) &=& \alpha\rho(t) + \delta_h w_0(t) + \delta_e w_2(t) \\ &=& (\alpha + \delta_h h_n(t) + h_e(t))\rho(t) \\ &=& (h_M - h_m + h(t))\rho(t) \end{array}$$

Finally as $\rho(t) = \dot{x}(t)$ we get $w_1(t) = \mathcal{D}_h(z_1(t)) = x(t) - x(t - h(t))$ which implies, from the state equation of Proposition 4.1

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + A_h x(t - h(t)) + B_3 w_3(t) + B_u u(t) \\ z_3(t) &=& C_3 x(t) + C_{3h} x(t - h(t)) + D_{33} w_3(t) \\ && + D_{3u} u(t) \end{array} \tag{17}$$

Since (17) has the same form as (8), this shows that if system (16) is asymptotically stable then (17) and therefore (8) are hence asymptotically stable. ■

Using Proposition 4.1 and Theorem 2.1 the second main result of the paper, which provides the solution to the delay-scheduled state feedback design problem, can be given:

Lemma 4.1: Consider system (8). If there exist symmetric positive definite matrices $Z, L_3, J_3 > 0$ and a positive scalar γ such that the LMIs

$$\mathcal{K}_i^T (N_i + T_i^T M_i T_i) \mathcal{K}_i < 0, \text{ for } i = 1, 2 \qquad \begin{bmatrix} L_3 & I \\ I & J_3 \end{bmatrix} \geq 0 \tag{18}$$

hold with $N_1 = -\operatorname{diag}(L_3, I_1, I_2, \gamma I_{w_3})$, $M_1 = \operatorname{diag}(L_3, I_1, I_2, \gamma^{-1} I_{z_3})$, $U_J = \operatorname{diag}(J_3, I_1, I_2, \gamma I_{z_3})$, $M_2 = \operatorname{diag}(J_3, I_1, I_2, \gamma^{-1} I_{w_3})$, $N_2 = \begin{bmatrix} Z\bar{A}^T + \bar{A}Z & Z\bar{C}^T \\ \star & -U_J \end{bmatrix}$, $T_1 = \bar{D}$, $\mathcal{K}_1 = I$, $T_2 = \begin{bmatrix} \bar{B}^T & \bar{D}^T \end{bmatrix}$, $\mathcal{K}_2 = \begin{bmatrix} \bar{B}_u^T & \bar{D}_u^T \end{bmatrix}$ then there exists a stabilizing delay scheduled state-feedback according to definition 2.1 with an $\mathcal{H}_\infty$ performance index on channel $w_3 \to z_3$ lower than γ.

Proof: The proof is a straightforward application of theorem 2.1 with system (16). ■

B. Controller Computation

The controller can be constructed using the methodology of Appendix B where $X = Z$, $\bar{A} = A + A_h$, $\bar{B} = \begin{bmatrix} 0 & -A_h & 0 & B_3 \end{bmatrix}$, $\bar{B}_u = B_u$, $D_{3u}^T = \begin{bmatrix} B_u & \alpha B_u & B_u & D_{3u} \end{bmatrix}^T$

$$\bar{C} = \begin{bmatrix} A + A_h \\ \alpha(A + A_h) \\ A + A_h \\ C_3 + C_{3h} \end{bmatrix} \quad \bar{D} = \begin{bmatrix} 0 & -A_h & 0 & B_3 \\ \delta_h I & -\alpha A_h & \delta_e I & \alpha B_3 \\ 0 & -A_h & 0 & B_3 \\ 0 & -C_{3h} & 0 & D_{33} \end{bmatrix} \tag{19}$$

where $\alpha = h_M - h_m + \bar{h}$, $\bar{h} = (h_m + h_M)/2$ and $\delta_h = h_M - \bar{h}$. Using the following lemma, the controller can be finally constructed:

Lemma 4.2 (Controller construction): The controller construction is obtained by applying Algorithm 1.1 where Y corresponds to the matrix (24) in which the matrices defined above are substituted and $Z = X = P^{-1}$.

C. Example

Consider system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} x(t-h(t)) \\ &\quad + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \end{aligned} \tag{20}$$

with $(h_m, h_M) = (0.1, 0.3)$. When the delay value is exactly known, we find an $\mathcal{H}_\infty$ closed-loop performance lower than $\gamma^* = 4.9062$. Since a finite γ has been found, this means that the closed loop system can be stabilized by a delay-scheduled state-feedback controller. A suitable controller is given by the expression

$$\begin{aligned} \begin{bmatrix} u(t) \\ z_c(t) \end{bmatrix} &= K \begin{bmatrix} x(t) \\ w_c(t) \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \\ w_c(t) &= h_n(t) z_c(t) \end{aligned}$$

$$\begin{aligned} K_{11} &= \begin{bmatrix} -30.1953 & -14.9936 \end{bmatrix} \\ K_{12} &= \begin{bmatrix} -0.3196 & 1.1417 \end{bmatrix} \\ K_{21} &= \begin{bmatrix} -5.2534 & -2.7139 \\ 0.3303 & 0.1755 \end{bmatrix} \\ K_{22} &= \begin{bmatrix} -0.0330 & 0.1899 \\ 0.0617 & -0.0289 \end{bmatrix} \end{aligned}$$

V. Conclusion

A new model transformation allowing to turn an LTI time-delay system with time-varying delays into an uncertain LPV system in 'LFT' form has been developed. From this reformulation, a new delay-dependent stability test, based on the application of the scaled-bounded real lemma, has been derived. The interest of such formulation resides in the similarities with the bounded-real lemma for finite dimensional systems which can be used with many robust control tools.

The stability test is extended to address a new control synthesis problem for time-delay systems. In this original approach, the controller gains are scheduled by the current value of the delay and this structure has motivated the name of 'delay-scheduled controller' in reference to gain-scheduled controllers arisings in the control of LPV systems. A certain interest of the approach is the simple robustness analysis with respect to uncertainty on the delay knowledge which is actually a difficult problem when the delay is considered as an operator. In the provided approach, the uncertainty is uniquely characterized by a parameter variation and can be very easily handled due to the similarities with robustness analysis with respect to uncertain system matrices. Even if the results are presented for LTI systems with single time-delay, it can be easily generalized to the LPV case with multiple delays.

Further works will be devoted to the reduction of conservatism of the approach by finding better model transformations and tighter bound on the norm of the operators.

Appendix

A. Proof of Lemma 2.1

Consider the LPV system (3). The augmented system (see [11]) is given by

$$\begin{bmatrix} \dot{x} \\ \hline z_c \\ z_0 \\ z_1 \\ z_3 \\ \hline x \\ w_c \end{bmatrix} = \left[\begin{array}{c|cccc|cc} A & 0 & B_0 & B_1 & B_2 & B_u & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I \\ C_0 & 0 & D_{00} & D_{01} & D_{02} & D_{u0} & 0 \\ C_1 & 0 & D_{10} & D_{11} & D_{12} & D_{u1} & 0 \\ C_2 & 0 & D_{20} & D_{21} & D_{22} & D_{u2} & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w_c \\ w_0 \\ w_1 \\ w_3 \\ \hline u \\ z_c \end{bmatrix} \tag{23}$$

The signals w_c and z_c represent the controller parameter-scheduled channel. Let K be the state-feedback matrix, as defined in definition 2.1. Then computing the closed-loop system and substituting its expression into the scaled-bounded real lemma

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & C_{cl}^T \\ \star & -L_w & D_{cl}^T \\ \star & \star & -L_z^{-1} \end{bmatrix} < 0 \tag{24}$$

with $L_w = \operatorname{diag}(L, I_1, \gamma I_{w_3})$ and $L_z = \operatorname{diag}(L, I_1, \gamma^{-1} I_{z_3})$ where L scales the controller and system parameter-scheduled channels. Isolating the expression containing the state-feedback matrix from (24), this inequality may be rewritten into

$$\Psi(P, L_w, L_z^{-1}) + \mathcal{P}^T U^T K V + V^T K^T U \mathcal{P} < 0 \tag{25}$$

where $\mathcal{P} = \operatorname{diag}(P, I, I)$, $U = \begin{bmatrix} I & | & 0 & | & 0 \end{bmatrix}$, $V = \left[\begin{array}{c|cccc|ccc} B_u^T & 0 & D_{u0}^T & D_{u1}^T & D_{u2}^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^T$ and the matrix $\Psi(P, L_w, L_z^{-1})$ is given by the substitution of the matrices of system (23) into (24). Note that this inequality is nonlinear due to the product between the matrix variables P, K and the presence of both matrices L and L^{-1}. However, as shown in [11], it turns out that it is possible to find an equivalent LMI formulation by the mean of the projection lemma and then the following equivalent underlying problem is obtained

$$K_U^T \Phi K_U < 0 \qquad K_V^T \Psi K_V < 0 \tag{26}$$

where $K_U = Ker(U)$, $K_V = Ker(V)$, $\Phi = \operatorname{diag}(P^{-1}, I, I) \Psi \operatorname{diag}(P^{-1}, I, I)$ and Ψ is given by the substitution of the matrices of system (23) into (24). Permuting lines and columns of matrix inequalities (26) yields

$$K_U'^T \Phi' K_U' < 0 \qquad K_V'^T \Psi' K_V' < 0$$

where $K_U' = \operatorname{diag}(\mathcal{N}_U, I)$ and $K_V' = \operatorname{diag}(\mathcal{N}_V, I)$. Due to the identity block in the matrices K_U' and K_V', it is possible to simplify the expression using Schur's complement. To this aim, let us decompose the previous matrices as $\Psi' = [\Psi_{ij}]_{i,j=1,2}$ $\quad \Phi' = [\Phi_{ij}]_{i,j=1,2}$ and therefore we get

$$\begin{aligned} \begin{bmatrix} \mathcal{N}_U^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \star & \Phi_{22} \end{bmatrix} \begin{bmatrix} \mathcal{N}_U & 0 \\ 0 & I \end{bmatrix} < 0 \\ \begin{bmatrix} \mathcal{N}_V^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \star & \Psi_{22} \end{bmatrix} \begin{bmatrix} \mathcal{N}_V & 0 \\ 0 & I \end{bmatrix} < 0 \end{aligned} \tag{27}$$

which are in turn equivalent to

$$\begin{aligned} \mathcal{N}_U^T (\Phi_{11}' - \Phi_{12}' \Phi_{22}^{-1} \Phi_{12}'^T) \mathcal{N}_U < 0 \\ \mathcal{N}_U^T (\Psi_{11}' - \Psi_{12}' \Psi_{22}^{-1} \Psi_{12}'^T) \mathcal{N}_V < 0 \end{aligned} \tag{28}$$

$$Y = \left[\begin{array}{c|c|c} \bar{A}^T X^{-1} + X^{-1}\bar{A} & X^{-1}\begin{bmatrix}0 & \bar{B}\end{bmatrix} & \begin{bmatrix}0 & \bar{C}^T\end{bmatrix} \\ \hline \star & -diag(L, I_{n_1}, \gamma I_{w_3}) & \begin{bmatrix}0 & 0 \\ 0 & \bar{D}^T\end{bmatrix} \\ \hline \star & \star & -diag(L^{-1}, I_{n_1}, \gamma I_{z_3}) \end{array}\right] \tag{21}$$

$$\Pi_1 = \begin{bmatrix} (A+A_h)^T X_1 + X_1(A+A_h) & -X_1 A_h & X_1 B & (h_M - h_m)(A+A_h) & (C+C_h)^T \\ \star & -X_2 & 0 & -(h_M - h_m)A_h^T & -C_h^T \\ \star & \star & -\gamma I_w & (h_M - h_m)B^T & D^T \\ \star & \star & \star & -X_2^{-1} & 0 \\ \star & \star & \star & \star & -\gamma I_z \end{bmatrix} \tag{22}$$

which are exactly the LMIs of Theorem 2.1 and the proof of the controller existence is completed. The form of K is given in Appendix B.

B. State-Feedback computation

The state-feedback matrix K may be computed using the following algorithm:

Algorithm 1.1: Controller computation

1) Compute whole matrices $L = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$ and $J = L^{-1} = \begin{bmatrix} J_1 & J_2 \\ J_2^T & J_3 \end{bmatrix}$ from the values of L_3 and J_3 using a singular value decomposition on the relations $L_2^T J_2 = I - L_3 J_3$ and then L and J are solutions of the equations:

$$\begin{bmatrix} 0 & I \\ \Xi_2^T & \Xi_3 \end{bmatrix} \Upsilon = \begin{bmatrix} \Upsilon_2^T & \Upsilon_3 \\ 0 & I \end{bmatrix} \tag{29}$$

where $(\Xi, \Upsilon) \in \{(L, J), (J, L)\}$.

2) The computation of the controller matrix K consists in solving the LMI (as in [11]).

$$Y + \tilde{U}^T K V + V^T K^T \tilde{U} < 0$$

where Y is given in (21) and

$$\tilde{U} = \left[\begin{array}{cc} 0 & X^{-1}\bar{B}_u \\ \hline 0 & 0 \\ \hline 0 & I \\ \bar{D}_u & 0 \end{array}\right]^T \qquad V = \left[\begin{array}{cc} I & 0 \\ \hline 0 & 0 \\ 0 & I \\ \hline 0 & 0 \end{array}\right]$$

C. Proof of theorem 3.1

First substitute matrices governing system (11) into the scaled-bounded real lemma [11] and performing a congruence transformation with respect to $\mathrm{diag}(I, I, I, X_2, I)$ leads to the semi-infinite LMI $\Pi_1 + h(t)\Pi_2 < 0$ with Π_1 defined in (22) and

$$\Pi_2 = \begin{bmatrix} 0 & 0 & 0 & (A+A_h) & 0 \\ \star & 0 & 0 & -A_h^T & 0 \\ \star & \star & 0 & B^T & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix} \tag{30}$$

Since this LMI is affine in $h(t)$ then it is necessary and sufficient to check the feasibility of the LMI at the vertices only (i.e. at $h(t) \in \{h_{min}, h_{max}\}$) and we get LMIs (14).

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