Nonlinear state-dependent delay modeling and stability analysis of Internet congestion control

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December 16th 2010
CDC 2010, Atlanta, USA
The congestion problem in networks

Network Elements
- Buffers/Servers/Routers
- Media
- Users

Congestion problem - QoS deterioration
- Data loss
- Too large delay

Need for protocols for congestion control
Problem definition

Modeling problem

- Precisely represent each element (user, medium, buffer)
- Provide precise adapters/interfaces

Modular construction of networks models

Congestion control problem

- Define efficient protocols achieving performance specifications
  - Fairness
  - Efficiency
  - Cross-traffic adaptation
- Analyze network stability and dynamic performance
  - Local vs. global stability
  - Static and dynamic performance
  - Delays effects on stability
How to model networks?

**Packet level**
- Asynchronous discrete-time large-scale (hybrid) system
- Perfect for simulation (NS-2)
- Few tools available for analysis

**Flow level**
- Valid when packets size small w.r.t. transfer speed (e.g. kbit vs Mbit/s)
- Fluid-flow models, continuous-time
- Difficult to transpose packets level effects to a flow level
- Good models for representation and analysis
Buffer model (infinite capacity)

\[ \dot{\tau}(t) = \frac{1}{c} [\phi(t) - r(\phi(t), \tau(t))] \]

\[ r(\phi(t), \tau(t)) = \begin{cases} 
  c & \text{if } C(t) \\
  \phi(t) & \text{otherwise}
\end{cases} \]

\[ C(t) = [\phi(t) > c] \text{ or } [\tau(t) > 0] \]

\[ \dot{\phi}(t) \in [-1, +\infty) \]

- Hybrid linear model with linear constraints
- Flow integrator
- Aggregated flows
- Queue saturation \(\rightarrow\) 3rd mode
QUEUING DELAY MAP

The time is not universal
- Temporal order of reaction
- Different models for different reference times

Sending time $t_s$ as a reference
- Forward operator $f(t) = t + \tau(t)$.
- Reception time $t_r = f(t_s)$
- Intuitive, easy for modeling

Reception time $t_r$ as a reference
- Backward operator $g = f^{-1}$ exists iff $\phi > 0$
- Sending time $t_s = g(t_r)$
- Less intuitive but better for analysis
**Problem of flow separation**

- Crucial for interconnections description
- Flows are aggregated in the previous model
- How to split up \( r \) into a sum of atomic \( r_i \) (if possible)?
Extended buffer model (2)

Output flows - Closed form expression

\[
\dot{\tau}(t) = \frac{1}{c} \left[ \sum_i \phi_i(t) - \sum_i r_i(\phi_t, \tau_t) \right]
\]

\[
r_i(\phi_t, \tau_t) = \begin{cases} 
\phi_i(g(t)) \frac{c}{\sum_j \phi_j(g(t))} & \text{if } C(g(t)) \\
\phi_i(t) & \text{otherwise}
\end{cases}
\]

- \( g(t) = t - \tau(g(t)) \)
- Delayed flow proportion
- Can be extended to more complex buffers, e.g. multiple output capacities
Forward vs. Backward protocol model

**Forward protocol model -** $t_s$ is the reference

\[
\begin{align*}
\dot{z}(t_s + RTT_{t_s}) &= P(z(t_s + RTT_{t_s}), \tau(t_s + T_f), T) \\
w(t_s + RTT_{t_s}) &= h(z(t_s + RTT_{t_s})) \\
RTT_{t_s} &= T_f + \tau(t_s + T_f) + T_b
\end{align*}
\]

- Not very easy to work with

**Backward protocol model -** $t_r$ is the reference

\[
\begin{align*}
\dot{z}(t_r) &= P(z(t_r), \tau(g(t_r - T_b)), T) \\
w(t_r) &= h(z(t_r)) \\
\tau(g(t_r - T_b)) &= \tau(t_r - T_b - \tau(g(t_r - T_b)))
\end{align*}
\]

- Standard form for dynamical systems
- Implicit state-dependent delay!
Complete network model

**Single-buffer/Single-user**

\[
\begin{align*}
\dot{r}(t) &= \frac{1}{c} \left[ \phi(t) - r(\phi(t), \tau(t)) \right] \\
\dot{z}(t) &= \mathcal{P}(z(t), \tau(g(t - T_b)), T_f, T_b) \\
w(t) &= h(z(t)) \\
\phi(t) &= \Phi(w(t), \tau(t), T_b, T_f)
\end{align*}
\]

- Functional $\Phi$ converts windows sizes into flows [Jacobsson]

\[
\phi(t) = \frac{w(t - T_f)}{T + \tau(t)}, \quad \dot{\phi}(t) = \frac{w(t - T_f)}{T + \tau(t)} + \dot{w}(t - T_f)
\]
Delays properties

**Propagation delays** $T_f, T_b$
- Constant delays
- Bounded

**Queuing delay** $\tau(g(t - T_b))$ - Single buffer case
- Bounded in practice
- Derivative belongs to $(-\infty, 1)$:

$$D^+[\tau(g)](t) = \begin{cases} 1 - \frac{c}{\phi(g(t))} & \text{if } C(g(t)) > 0 \\ 0 & \text{otherwise} \end{cases}$$
- Well-posedness problems do not occur ($\dot{\tau}(t) < 1$)
- Many results on time-delay systems can be applied (Lyapunov-Krasovskii Theory)
Queuing delay $\tau(g(t - T_b))$ - Multiple buffer case

- Two delays: $\tau_1(g_1(g_2(t)))$ and $\tau_2(g_2(t))$
- $D^+[\tau_1(g_1(g_2))](t)$ given by

$$
\begin{cases}
\frac{(\phi_1(g_1(g_2)) + \phi_2(g_1(g_2)) - c_1)c_2}{c_1}
\quad \text{if } C_1(g_1(g_2)) \text{ and } C_2(g_2)
\
1 - \frac{\phi_1(g_1(g_2)) + \phi_2(g_1(g_2))}{\phi_1(g_1(g_2)) + \phi_2(g_1(g_2))}
\quad \text{if } C_1(g_1(g_2)) \text{ and not } C_2(g_2)
\
0
\quad \text{otherwise}
\end{cases}
$$

- May exceed one when $c_2 > c_1$. 
FAST-TCP protocol

\[ \dot{w}(t) = \gamma \left( -\frac{\tau(g(t - T_b))}{T + \tau(g(t - T_b))} w(t) + \alpha \right) \]

- \( \gamma \) tunes the bandwidth of the protocol (control sense)
- \( \alpha \) rules out the bandwidth-delay product (communication network sense)
- Network equilibrium point (buffer+user):
  \[
  \tau^* = \frac{\alpha}{c(1 - \delta^*)}, \quad w^* = \alpha(1 + T/\tau^*), \quad \phi^* = \alpha/\tau^*
  \]
- Equilibrium flow independent of the propagation delay (also in the multiple users case)
The single-user/single-buffer network is

- globally exponentially stable without delays (Lyapunov theory)
- locally exponentially stable independently of the delay when
  \[ \tau^* > T \text{ where } \tau^* = \frac{\alpha}{c} \text{ (small gain)} \]
- locally delay-dependent exponentially stable when
  \[ \tau^* < T, \quad \tau^*(T - 1) + T^2 \leq 0 \text{ (quasipolynomials)} \]
- locally delay-dependent exponentially stable when
  \[ \tau^* < T, \quad \tau^*(T - 1) + T^2 > 0, \quad \gamma < \frac{1}{\tau^*(T - 1) + T^2} \text{ (quasipolynomials)} \]
Conclusion and Future Works

**Conclusion**

- Accurate models for FIFO buffers
- Theoretical delay modeling and analysis
- Local stability analysis based on time-delay systems theory

**Future works**

- More general buffer models (priorities, multiple output links/capacities)
- Nonlinear stability analysis
- Study of limit cycles in more complex topologies
Thank you for your attention