Positive systems analysis via integral linear constraints

Sei Zhen Khong\textsuperscript{1}, Corentin Briat\textsuperscript{2}, and Anders Rantzer\textsuperscript{3}

\textsuperscript{1}Institute for Mathematics and its Applications  
University of Minnesota

\textsuperscript{2}Department of Biosystems Science and Engineering  
Swiss Federal Institute of Technology Zürich (ETH Zürich), Switzerland

\textsuperscript{3}Department of Automatic Control  
Lund University, Sweden

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Positive systems analysis

- Quadratic forms are widely used for systems analysis: Lyapunov inequality, Kalman-Yakubovich-Popov Lemma, integral quadratic constraints etc.

- Analysis can be simplified if systems are known to be positive

  **Lyapunov inequality:**
  - $\exists P \succ 0$ such that $A^TP + PA \prec 0$
  - $\exists z > 0$ (element-wise) such that $Az < 0$

  **Kalman-Yakubovich-Popov Lemma:**
  - $\exists x, u, p \geq 0$ such that
    $$Ax + Bu \leq 0 \quad \text{and} \quad Q \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} A^T \\ B^T \end{bmatrix} p \leq 0$$

  - The theory of **integral linear constraints** (ILCs)?

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Outline

1. Positive closed-loop systems
2. Robust stability
3. Geometric intuition
4. Example
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1. Positive closed-loop systems
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4. Example
A system $G$ is said to be positive if
\[
u(t) \geq 0 \ orall t \geq 0 \implies y(t) = (Gu)(t) \geq 0 \ orall t \geq 0\]

Given a positive feedback interconnection of two positive systems $G_1$ and $G_2$, is the closed-loop map $(d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)$ always positive?

No!
A system $G$ is said to be positive if

$$u(t) \geq 0 \quad \forall t \geq 0 \implies y(t) = (Gu)(t) \geq 0 \quad \forall t \geq 0$$

Given a positive feedback interconnection of two positive systems $G_1$ and $G_2$, is the closed-loop map $(d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)$ always positive?

No!
Positive systems

A simple counterexample:

\[ d_2 = 0 \]

\[ d_1 \rightarrow u_1 = \frac{1}{1 - 2} = -1 \]
Feedback interconnections

\[ \hat{G}_1(s) = C_1 (sI - A_1)^{-1} B_1 + D_1 \]
\[ \hat{G}_2(s) = C_2 (sI - A_2)^{-1} B_2 + D_2 \]

- \( A_1 \) and \( A_2 \) are Metzler and \( B_1 \geq 0, B_2 \geq 0, C_1 \geq 0, C_2 \geq 0, D_1 \geq 0, \) and \( D_2 \geq 0 \) (element-wise) implies \( G_1 \) and \( G_2 \) are positive.

**Positivity of closed-loop map [Ebihara et. al. 2011]**

If \( \rho(D_1D_2) < 1 \), then \((d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)\) is **positive**.
Feedback interconnections

Suppose (nonlinear) $G_i : \mathbb{L}_{1e} \to \mathbb{L}_{1e}$ are causal and positive, define

$$
\alpha(G_i) := \sup_{T > 0} \inf_{\Delta T > 0} \sup_{x,y \in \mathbb{L}_{1e} : P_T x = P_T y : P_T + \Delta T (x-y) \neq 0} \frac{\|P_{T+\Delta T} (G_i x - G_i y)\|_1}{\|P_{T+\Delta T} (x - y)\|_1}
$$

Positivity of closed-loop map

If $\alpha(G_1) \alpha(G_2) < 1$, then $(d_1, d_2) \mapsto (u_1, y_1, u_2, y_2)$ is positive
Outline

1 Positive closed-loop systems

2 Robust stability

3 Geometric intuition

4 Example
Robust stability of feedback systems

Integral quadratic constraints (IQC) [Megretski & Rantzer 97]

Given bounded, causal $G_1 : \mathbf{L}_2 \rightarrow \mathbf{L}_2$ and $G_2 : \mathbf{L}_2 \rightarrow \mathbf{L}_2$, suppose there exists linear $\Pi : \mathbf{L}_2 \rightarrow \mathbf{L}_2$ such that

- $[\tau G_1, G_2]$ is well-posed for all $\tau \in [0, 1]$;
- $\int_0^\infty v(t)^T (\Pi v)(t) \, dt \geq 0 \quad \forall v \in \mathcal{G}(\tau G_1) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{L}_2 : y = \tau G_1 u \right\}, \tau \in [0, 1]$;
- $\int_0^\infty w(t)^T (\Pi w)(t) \, dt \leq -\epsilon \int_0^\infty |w(t)|^2 \, dt \quad \forall w \in \mathcal{G}^'(G_2)$,

then $[G_1, G_2]$ is stable.
## Integral quadratic constraint (IQC) examples

<table>
<thead>
<tr>
<th>Structure of $G_1$</th>
<th>$\Pi$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$ is passive</td>
<td>$\begin{bmatrix} 0 &amp; I \ I &amp; 0 \end{bmatrix}$</td>
<td>$x(j\omega) \geq 0$</td>
</tr>
<tr>
<td>$|G_1| \leq 1$</td>
<td>$\begin{bmatrix} x(j\omega)I &amp; 0 \ 0 &amp; -x(j\omega)I \end{bmatrix}$</td>
<td>$X = X^* \geq 0, Y = -Y^*$</td>
</tr>
<tr>
<td>$G_1 \in [-1, 1]$</td>
<td>$\begin{bmatrix} X(j\omega) &amp; Y(j\omega) \ Y(j\omega)^* &amp; -X(j\omega) \end{bmatrix}$</td>
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<tr>
<td>$G_1(t) \in [-1, 1]$</td>
<td>$\begin{bmatrix} X &amp; Y \ Y^T &amp; -X \end{bmatrix}$</td>
<td>$X = X^* \geq 0, Y = -Y^*$</td>
</tr>
<tr>
<td>$G_1(s) = e^{-\theta s} - 1$, for $\theta \in [0, \theta_0]$</td>
<td>$\begin{bmatrix} x(j\omega)\rho(\omega)^2 &amp; 0 \ 0 &amp; -x(j\omega) \end{bmatrix}$</td>
<td>$\rho(\omega) = 2 \max_{</td>
</tr>
</tbody>
</table>

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Integral linear constraints
Robust stability of positive feedback systems

Integral linear constraints

Given bounded, causal, linear $G_1 : \mathcal{L}_{1e}^m \to \mathcal{L}_{1e}^P$ and $G_2 : \mathcal{L}_{1e}^P \to \mathcal{L}_{1e}^m$, suppose there exists $\Pi \in \mathbb{R}^{1 \times m+p}$ such that

- $[\tau G_1, G_2]$ is well-posed and positive for all $\tau \in [0, 1]$;
- $\int_0^\infty \Pi v(t) \, dt \geq 0 \quad \forall v \in \mathcal{G}_+(\tau G_1) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{L}_{1+} : y = \tau G_1 u \right\}, \tau \in [0, 1]$;
- $\int_0^\infty \Pi w(t) \, dt \leq -\epsilon \int_0^\infty |w(t)| \, dt \quad \forall w \in \mathcal{G}'_+(G_2)$,

then $[G_1, G_2]$ is stable.

When $G_1$ and $G_2$ are LTI, conditions can be stated as

- $\Pi \begin{bmatrix} I \\ \hat{G}_1(0) \end{bmatrix} \geq 0$ and $\Pi \begin{bmatrix} \hat{G}_2(0) \\ I \end{bmatrix} < 0$
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Geometric interpretation of integral quadratic constrains

\[ \mathcal{G}(G_1) + \mathcal{G}'(G_2) = L_2; \]

\[ \mathcal{G}(G_1) \cap \mathcal{G}'(G_2) = \{0\} \]
Geometric interpretation of integral quadratic constraints

Integral quadratic constraints (IQCs)

\[ \int_0^\infty v(t)^T (\Pi v)(t) \, dt \geq 0 \quad \forall v \in \mathcal{G}(G_1); \]

\[ \int_0^\infty w(t)^T (\Pi w')(t) \, dt \leq -\epsilon \int_0^\infty |w(t)|^2 \, dt \quad \forall w \in \mathcal{G}'(G_2) \]
Geometric interpretation of integral linear constraints

$$\mathcal{G}_+(G_1) + \mathcal{G}_+(G_2) = \mathbf{L}_{1+};$$

$$\mathcal{G}_+(G_1) \cap \mathcal{G}_+(G_2) = \{0\}$$
Geometric interpretation of integral linear constraints

\[ \mathcal{G}_+(G_1) \]

\[ \mathcal{G}_+',(G_2) \]

Integral linear constraints

- \[ \int_0^\infty \Pi v(t) \, dt \geq 0 \quad \forall v \in \mathcal{G}_+(G_1) ; \]
- \[ \int_0^\infty \Pi w(t) \, dt \leq -\epsilon \int_0^\infty |w(t)| \, dt \quad \forall w \in \mathcal{G}_+',(G_2) \]
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LTI systems

\[
\hat{G}_1(s) = C_1(sI - A_1)^{-1}B_1 + D_1 \\
\hat{G}_2(s) = C_2(sI - A_2)^{-1}B_2 + D_2
\]

- \( A_1 \) and \( A_2 \) are Metzler, Hurwitz and \( B_1 \geq 0, B_2 \geq 0, C_1 \geq 0, C_2 \geq 0, D_1 \geq 0, \) and \( D_2 \geq 0 \)

Robust stability [Ebihara et. al. 2011] [Tanaka et. al. 2013]

If \( \rho(\hat{G}_1(0)\hat{G}_2(0)) < 1 \), then \([G_1, G_2]\) is stable

Can be recovered with integral linear constraint theorem with

\[ \Pi := z^T [\hat{G}_1(0) \ \ -I] , \]

where \( z^T (\hat{G}_1(0)\hat{G}_2(0) - I) < 0 \)
Conclusions:

- Sufficient condition for positivity to be preserved under feedback
- Developed integral linear constraints theory for analysis of feedback interconnections with positive closed-loop mappings
- Many extensions possible:
  - Positive coprime factorisations
  - Integral linear constraints with time-varying multipliers
  - LMI conditions for verifying integral linear constraints
  - Stabilisation of open-loop unstable dynamics?