Dynamic equations on time-scale: application to stability analysis and stabilization of aperiodic sampled-data systems

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Outline

- Introduction
- Problem statement and Preliminaries
- Stability analysis
- Stabilization
- Conclusion and Future Works
Introduction

Problem Statement

Stability analysis

Stabilization

Aperiodic sampled-data systems

- Discrete-time systems with varying sampling period
- Several frameworks
  - Time-delay systems [Yu et al., Fridman et al.]
  - Impulsive systems [Naghshtabrizi et al., Seuret]
  - Sampled-data systems [Mirkin]
  - Robust techniques [Fusioka], [Hetel et al., Oishi et al., Ariba et al.]
  - Functional-based approaches [Seuret]
Problem statement
System and Problem definition

- Continuous-time LTI system
  \[
  \begin{align*}
  \dot{x}(t) &= Ax(t) + Bu(t) \\
  x(0) &= x_0
  \end{align*}
  \]  
  with state \( x \) and control input \( u \).

- Sampled-data control law
  \[
  u(t) = Kx(t_k), \ t \in [t_k, t_{k+1})
  \]  
  where \( T_k := t_{k+1} - t_k \leq T, \ k \in \mathbb{N} \).

- Stability analysis problem: given \( K \), find the set \( T \) of admissible \( T > 0 \) for which stability still holds.

- Find controller gain \( K \) maximizing the maximal sampling period.
Systems over time-scales [Bohner]

- Unification/generalization of continuous-time and discrete-time systems
- Examples: $T = \mathbb{R}_+, T = \mathbb{Z}_+, T = \{0\} \cup \{1/k\}_{k \in \mathbb{N}}, T = \bigcup_{k \in \mathbb{N}} [t_{2k}, t_{2k+1}]$
- Forward jump operator: $\sigma(t) = \{\inf s \in T : t < s\}$
- Graininess: $\mu(t) = \sigma(t) - t$ (distance)
- Dynamical system on time-scale:
  \[
  x(\Delta)(t) = Ax(t) + Bu(t) \\
  x(0) = x_0
  \]

- $\Delta$ operator [Goodwin]:
  \[
  f(\Delta)(t) := \begin{cases} 
  \lim_{s \to t, s \in T} \frac{f(t) - f(s)}{t - s} & \text{if } \mu(t) = 0 \\
  \frac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if } \mu(t) > 0.
  \end{cases}
  \]
Stability analysis via Lyapunov functions

- Linear systems \( V(x) = x^T P x \)
- Stability condition
  \[ A^T P + PA + \mu A^T PA < 0, \quad P = P^T \succ 0 \]  
  \( \mu \) fixed: equivalent to a DT criterion
  \( \mu \to 0 \): equivalent to a CT criterion
- Spectrum condition: \( \lambda(A) \subset \mathbb{D}(-1/\mu, 1/\mu) \)
  \( \mu \to 0 \): \( \mathbb{D}(-1/\mu, 1/\mu) \to \mathbb{C}_- \)
  \( \mu = \mu_0 \neq 0 \): \( \mathbb{D}(1/\mu_0, 1/\mu_0) \) analogous to the unit disc.
Representation of sampled-data systems

- DT system:
  \[
  x(t) = \left[ e^{A(t-t_k)} + \int_{t_k}^{t} e^{A(t-s)} ds BK(t_k) \right] x(t_k)
  \]  

- System on TS:
  \[
  z^{\Delta}(t_k) = A^{\Delta}(\mu(t_k)) z(t_k)
  \]

where the new state \( z \) coincide with \( x \) at sampling instants and

\[
A^{\Delta}(\mu(t_k)) = \mu(t_k)^{-1} \left( e^{A\mu(t_k)} + \Phi(\mu(t_k)) BK(t_k) - I \right)
\]

\[
\Phi(\mu(t_k)) = \int_{0}^{t_k} e^{A(\mu(t_k)-s)} ds
\]
Stability analysis
A general stability result

**Theorem**
The dynamical system $z^\Delta(t_k) = A^\Delta(t_k)z(t_k)$, $z(t_0) = z_0$, $(t_0, t_k) \in \mathbb{T}^2$, $t_k \geq t_0$ is robustly exponentially stable for $\mu(t_k) \in \mu$ if the following statements hold:

1. $A(t_k)$ is rd-continuous and regressive, i.e. $\det(I + \mu(t_k)A(t_k)) \neq 0$ for all $t_k \in \mathbb{T}$ and $\mu(t_k) \in \mu$.

2. There exist $P : \mathbb{T} \to \mathbb{S}^{n+}$ verifying $\theta_1 I \preceq P(t_k) \preceq \theta_2 I$ for some $0 < \theta_1 < \theta_2 < +\infty$ and $\beta \in (0, 1/\sup\{\mu\})$ such that

$$
\mathcal{M}_{\mu(t_k)}(P(\sigma(t_k)), A^\Delta(t_k), P^\Delta(t_k) + \beta P(t_k)) \preceq 0
$$

holds for all $\mu(t_k) \in \mu$ and all $t_k \in \mathbb{T}$. 
Graininess dependent Lyapunov function

Theorem
The dynamical system $z^\Delta(t_k) = A_\Delta(\mu(t_k))z(t_k)$, $z(t_0) = z_0$, $(t_0, t_k) \in \mathbb{T}^2$, $t_k \geq t_0$ is robustly exponentially stable for $\mu(t_k) \in \mu$, $\inf\{\mu\} > 0$, if the following statements hold:

1. $A(\mu(t_k))$ is rd-continuous and regressive, i.e. $\det[I + \mu A(\mu)] \neq 0$ for all $\mu \in \mu$.
2. There exist a bounded matrix function $P : \mu \rightarrow \mathbb{S}^n_+$ such that

\[ M_\mu(P(\mu_n), A_\Delta(\mu_c), S(\mu_c, \mu_n)) < 0 \]  

holds for all $(\mu_n, \mu_c) \in \mu^2$ and where $S(\mu_c, \mu_n) = \mu_c^{-1}(P(\mu_n) - P(\mu_c))$. 

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Example 1

- Scalar system ([Mirkin], [Fridman], [Fujioka])

\[
\dot{x}(t) = -2x(t) + x(t_k).
\]  

(10)

- TS formalism

\[
A_{\Delta}(\mu) = \frac{3}{2\mu} \left( e^{-2\mu} - 1 \right).
\]

(11)

- Lyapunov condition

\[
\frac{3}{\mu} \left( e^{-2\mu} - 1 \right) \left( 1 + \frac{3}{4} \left( e^{-2\mu} - 1 \right) \right) < 0
\]

- True for \( \mu > 0 \)

- When \( \mu \to 0 \), the LHS tends to -6.
Example 2

Let us consider the system

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} \] (12)

Ref. | Maximal varying sampling period
--- | ---
[Fridman,04] | 0.8696
[Yue,05] | 0.8871
[Ariba,07] | 1.009
[Naghshtabrizi,08] | 1.1137
[Mirkin,07] | 1.3659
[Seuret,09b] | 1.6894
[Oishi,09] | 1.7294
Proposed result | 1.72941
Theoretical | 1.7294194 (constant case)
Example 3

\[ A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]

- Pathological sampling periods: \{2.2228, 4.4457, 6.6685, 8.8913, 11.1142, \ldots\},
- Constant sampling period
  - \( P(\mu) = P_0 \rightarrow \mu = [0.5004, 1.9203] \)
  - \( P(\mu) = P_0 + P_1 \mu \rightarrow \mu = [0.2013, 2.0204] \)
  - Other disjoint intervals

\[ \mu \in \{ [2.4706, 3.6963], [5.4307, 6.3447], [7.0277, 7.7249], [10.3916, 10.7559], [11.4973, 11.7179] \} . \]

- Aperiodic case:

\[ \mu \in \{ [0.2187, 1.0031], [0.500, 1.9256], [2.47, 3.6], [2.77, 3.6963], [5.4584, 5.8004], [5.8172, 6.3447], [7.0339, 7.5009], [7.5000, 7.7070] \} . \]
Stabilization
Robust state-feedback design

Theorem
There exists a quadratically stabilizing switching sampling-period-dependent state-feedback control law if there exist $X \in \mathbb{S}^n_{++}$ and a bounded continuous matrix function $U : \mu \rightarrow \mathbb{R}^{n \times m}$ such that the LMI

$$
\begin{bmatrix}
\Xi_{11}(\mu) & \Xi_{12}(\mu) \\
* & -\mu^{-1}X
\end{bmatrix} \prec 0
$$

(14)

holds for all $\mu \in \mu$ with

$$
\begin{align*}
\Xi_{12}(\mu) &= \mu^{-1}[A_e(\mu)X + \Phi(\mu)BU(\mu)]^T \\
\Xi_{11}(\mu) &= \Xi_{12}(\mu) + \Xi_{12}(\mu)^T \\
A_e(\mu) &= \exp(A\mu) - I
\end{align*}
$$

(15)

In such a case, the controller matrix is given by $K(\mu) = U(\mu)X^{-1}$. 
Sampling-period dependent controller

Theorem
There exists a robustly stabilizing sampling-period-dependent state-feedback control law if there exist a matrix $Z \in \mathbb{R}^{n \times n}$, bounded continuous matrix functions $P : \mu \to \mathbb{S}^{++}$, $U : \mu \to \mathbb{R}^{n \times m}$ and a sufficiently large positive scalar function $\epsilon : \mu^2 \to \mathbb{R}^{++}$ such that the matrix inequality

$$
\begin{bmatrix}
\Xi_{11}(\mu_c, \mu_n) & \Xi_{12}(\mu_c, \mu_n) & Z \\
* & \Xi_{22}(\mu_c, \mu_n) & 0 \\
* & * & \Xi_{33}(\mu_c, \mu_n)
\end{bmatrix} \prec 0
$$

(16)

holds for all $\mu \in \mu$, $\inf\{\mu\} > 0$, with $S(\mu_c, \mu_n) = \mu_c^{-1}(P(\mu_n) - P(\mu_c))$ and

$$
\begin{align*}
\Xi_{11}(\mu_c, \mu_n) &= -Z - Z^T + \mu_c P(\mu_n) \\
\Xi_{12}(\mu_c, \mu_n) &= \mu_c^{-1} [A_e(\mu_c)X + \Phi(\mu_c)BU(\mu_c)] + P(\mu_n) \\
\Xi_{22}(\mu_c, \mu_n) &= -\epsilon(\mu_c, \mu_n)P(\mu_n) + S(\mu_c, \mu_n) \\
\Xi_{33}(\mu_c, \mu_n) &= -P(\mu_n)/\epsilon(\mu_c, \mu_n) \\
A_e(\mu_c) &= \exp(A\mu_c) - I
\end{align*}
$$

(17)

In such a case, the controller matrix is given by $K(\mu_c) = U(\mu_c)Z^{-1}$. 
Examples

- System 1:
  \[ A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  \hfill (18)

- System 2:
  \[ A = \begin{bmatrix} 7 & 4 \\ 5 & 11 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]  \hfill (19)
### Examples

<table>
<thead>
<tr>
<th>degree of $K(\mu)$</th>
<th>$\mu^+$ for System 1</th>
<th>$\mu^+$ for System 2</th>
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Conclusion
Conclusion

- Time-scale approach for stability analysis of sampled-data systems
- Stability analysis via sampling-period dependent Lyapunov Functions
- Stabilization via sampling-period dependent controllers
- Extension to dynamic output feedback possible
- Future works will be devoted to the theory of systems on time-scales
Thank you for your attention!