Analysis and Control of LPV Systems using Hybrid Systems Methods

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1. (Short) introduction on LPV systems with some discussions
2. Stability analysis of a class of LPV systems with piecewise constant parameters
3. Control of a class of LPV systems with piecewise constant parameters
4. Hybrid systems formulation with some discussion
Introduction
LPV systems

\[ \dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t), \quad x(0) = x_0 \]  

where

- $x$ and $u$ are the state of the system and the control input
- $\rho(t) \in \mathcal{P}$, $\mathcal{P}$ compact, is the value of the parameter vector at time $t$
- the matrix-valued functions $A(\cdot)$ and $B(\cdot)$ are “nice enough”, i.e. continuous

Rationale

- Can be used to model a wide variety of real-world processes
- Convenient framework for the design gain-scheduled controllers/observers/filters
**Definition**

*The LPV system*

\[
\begin{align*}
\dot{x}(t) &= A(\rho(t))x(t) \\
x(0) &= x_0
\end{align*}
\]  

(2)

is said to be **quadratically stable** if \( V(x) = x^\top Px \) is a Lyapunov function for the system.

**Theorem**

The LPV system (2) is quadratically stable if and only if there exists a matrix \( P \in \mathbb{S}_{>0}^n \) such that the LMI

\[
A(\theta)^\top P + PA(\theta) < 0
\]  

(3)

holds for all \( \theta \in \mathcal{P} \).

**Remarks**

- All possible trajectories for the parameters are considered (with the restriction of existence of solutions)
- Semi-infinite dimensional LMI problem
Introduction
Stability analysis
Control design
Hybrid systems formulation
Conclusion

Robust stability

Definition
The LPV system

\[
\dot{x}(t) = A(\rho(t))x(t) \\
x(0) = x_0
\]  

with \( \rho \in C^1([0, \infty), \mathcal{P}) \) and \( \dot{\rho} \in \mathcal{D} \), for some given compact sets \( \mathcal{P}, \mathcal{D} \subset \mathbb{R}^N \), is said to be robustly stable if \( V(x, \rho) = x^\top P(\rho)x \) is a Lyapunov function for the system.

Theorem
The LPV system (4) is robustly stable if and only if there exists a differentiable matrix-valued function \( P : \mathcal{P} \to \mathbb{S}^n_{>0} \) such that the LMI

\[
\sum_{i=1}^{N} \theta_i' \partial_{\theta_i} P(\theta) + A(\theta)^\top P(\theta) + P(\theta)A(\theta) < 0
\]  

holds for all \( \theta \in \mathcal{P} \) and all \( \theta' \in \mathcal{D} \).

Remarks

- Trajectories of the parameters are continuously differentiable
- Infinite-dimensional LMI problem
Some remarks

- Two main classes of parameter trajectories associated with two main stability concepts
- But these classes are very far apart!
- Parameter trajectories are defined in a quite loose way
- Part of the success of periodic, switched and jump systems lies in the “tailoredness” of the tools
Some remarks

- Two main classes of parameter trajectories associated with two main stability concepts
- But these classes are very far apart!
- Parameter trajectories are defined in a quite loose way
- Part of the success of periodic, switched and jump systems lies in the “tailoredness” of the tools

Issues

- What if we consider piecewise constant or piecewise differentiable parameters?
- Robust stability not applicable and quadratic stability too conservative
- Need something else!
Stability analysis of a class of LPV systems with piecewise constant parameters
Two main classes of parameters

- Periodic jumps $\rightarrow$ constant dwell-time (time between two successive jumps)
- Aperiodic jumps $\rightarrow$ minimum dwell-time
LPV systems with PC parameters

Two main classes of parameters

- Periodic jumps → constant dwell-time (time between two successive jumps)
- Aperiodic jumps → minimum dwell-time

Stability results

- Discrete-time-like stability conditions
- Conditions obtained using lifting ideas
LPV systems with PC parameters

Two main classes of parameters

- Periodic jumps → constant dwell-time (time between two successive jumps)
- Aperiodic jumps → minimum dwell-time

Stability results

- Discrete-time-like stability conditions
- Conditions obtained using lifting ideas

Discussions

- Connections with quadratic and robust stability
- Example
Stability under constant dwell-time

Let us consider the LPV system

\[ \dot{x} = A(\rho)x, \quad x(0) = x_0 \]  

(6)

with piecewise-constant parameter vector \( \rho \in \mathcal{R}_T \) where

\[ \mathcal{R}_T = \left\{ \rho : \mathbb{R}_+ \rightarrow \mathcal{P} : \rho(t) = \alpha_k \in \mathcal{P}, \quad t \in [t_k, t_{k+1}), t_k = kT, \quad k \in \mathbb{N}_0 \right\} \]  

(7)

Defining then the discrete-time system

\[ x(t_{k+1}) = e^{A(\alpha_k)T}x(t_k) \]  

(8)

leads us to the following result:
Let us consider the LPV system

\[ \dot{x} = A(\rho)x, \quad x(0) = x_0 \quad (6) \]

with piecewise-constant parameter vector \( \rho \in \mathcal{R}_T \) where

\[ \mathcal{R}_T = \left\{ \rho : \mathbb{R}_\geq 0 \rightarrow \mathcal{P} : \rho(t) = \alpha_k \in \mathcal{P}, \quad t \in [t_k, t_{k+1}), t_k = kT, \quad k \in \mathbb{N}_0 \right\} \quad (7) \]

Defining then the discrete-time system

\[ x(t_{k+1}) = e^{A(\alpha_k)T}x(t_k) \quad (8) \]

leads us to the following result:

**Theorem**

*Assume that there exists matrix-valued function \( P : \mathcal{P} \rightarrow \mathbb{S}^n_{\succ 0} \) such that*

\[ e^{A(\theta)^	op T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0 \quad (9) \]

*holds for all \( \theta, \eta \in \mathcal{P}.*

*Then, the LPV system is asymptotically stable for all \( \rho \in \mathcal{R}_T. \)
Stability under minimum dwell-time

Let us consider the LPV system

\[ \dot{x} = A(\rho)x, \quad x(0) = x_0 \]  

(10)

with piecewise-constant parameter vector \( \rho \in P_{\geq T} \) where

\[ P_{\geq T} = \left\{ \rho : \mathbb{R}_{\geq 0} \to P : \rho(t) = \alpha_k \in P, \quad t \in [t_k, t_{k+1}), t_{k+1} \geq t_k + T, k \in \mathbb{N}_0 \right\}. \]  

(11)

The discrete-time system is given in this case by

\[ x(t_{k+1}) = e^{A(\alpha_k)T_k}x(t_k), \quad T_k \geq T \]  

(12)
Let us consider the LPV system

$$\dot{x} = A(\rho)x, \ x(0) = x_0$$  (10)

with piecewise-constant parameter vector $\rho \in \mathcal{P}_T$ where

$$\mathcal{P}_T = \left\{ \rho : \mathbb{R}_{\geq 0} \rightarrow \mathcal{P} : \rho(t) = \alpha_k \in \mathcal{P}, \ t \in [t_k, t_{k+1}), t_{k+1} \geq t_k + T, k \in \mathbb{N}_0 \right\}.$$  (11)

The discrete-time system is given in this case by

$$x(t_{k+1}) = e^{A(\alpha_k)T_k}x(t_k), T_k \geq T$$  (12)

**Theorem**

Assume that there exists a matrix-valued function $P : \mathcal{P} \rightarrow \mathbb{S}_{\succ 0}^n$ such that

$$e^{A(\theta)^\top T}P(\theta)e^{A(\theta)T} - P(\eta) \prec 0$$  (13)

and

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0$$  (14)

holds for all $\theta, \eta \in \mathcal{P}$. Then, the LPV system is asymptotically stable for all $\rho \in \mathcal{P}_T$. 

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Analysis and Control of LPV Systems using Hybrid Systems Methods
Conditions

\[ e^{A(\theta)^\top T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0, \quad A(\theta)^\top P(\theta) + P(\theta) A(\theta) \prec 0 \]
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Verification of the conditions

- Infinite-dimensional LMIs
- Exponential terms \( e^{A(\theta)T} \) difficult to consider even when \( A(\theta) \) is affine
- Gridding possible but inaccurate and computationally expensive
Conditions

\[ e^{A(\theta)^T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0, \quad A(\theta)^T P(\theta) + P(\theta) A(\theta) \prec 0 \]

Verification of the conditions

- Infinite-dimensional LMIs
- Exponential terms \( e^{A(\theta)T} \) difficult to consider even when \( A(\theta) \) is affine
- Gridding possible but inaccurate and computationally expensive

Control design

- Nonconvex at all since we would have something like \( e^{(A(\theta) + B(\theta) K(\theta))T} \)
Theorem

The following statements are equivalent:

(a) There exists a matrix-valued function $P : \mathcal{P} \to \mathbb{S}^n_>$ such that the condition

$$e^{A(\theta)^T} T P(\theta) e^{A(\theta)T} - P(\eta) < 0$$

holds for all $\theta, \eta \in \mathcal{P}$.
**Theorem**

The following statements are equivalent:

(a) There exists a matrix-valued function $P : \mathcal{P} \rightarrow \mathbb{S}^n_{\succ 0}$ such that the condition

$$e^{A(\theta)^T T} P(\theta) e^{A(\theta) T} - P(\eta) \prec 0$$

(15)

holds for all $\theta, \eta \in \mathcal{P}$.

(b) There exists a matrix-valued function $S : [0, T] \times \mathcal{P} \rightarrow \mathbb{S}^n$, $S(T, \theta) \succ 0$, such that the conditions

$$\partial_\tau S(\tau, \theta) + S(\tau, \theta) A(\theta) + A(\theta)^T S(\tau, \theta) \preceq 0$$

(16)

and

$$S(0, \theta) - S(T, \eta) \prec 0$$

(17)

hold for all $\theta, \eta \in \mathcal{P}$ and all $\tau \in T := [0, T]$. 

Corentin Briat Analysis and Control of LPV Systems using Hybrid Systems Methods
Theorem
The following statements are equivalent:

(a) There exists a matrix-valued function $P : \mathcal{P} \rightarrow \mathbb{S}^n_\succ 0$ such that the condition

$$e^{A(\theta)^\top T} P(\theta) e^{A(\theta)T} - P(\eta) \prec 0 \quad (15)$$

holds for all $\theta, \eta \in \mathcal{P}$.

(b) There exists a matrix-valued function $S : [0, T] \times \mathcal{P} \rightarrow \mathbb{S}^n$, $S(T, \theta) \succ 0$, such that the conditions

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and

$$S(0, \theta) - S(T, \eta) \prec 0 \quad (17)$$

hold for all $\theta, \eta \in \mathcal{P}$ and all $\tau \in T := [0, T]$.

Moreover, when one of the above statements holds, then the LPV system is asymptotically stable for all $\rho \in \mathcal{P}_T$. 
Sketch of the proof

(b) ⇒ (a)

- Integrating $\partial_\tau S(\tau, \theta) + \text{Sym}[S(\tau, \theta) A(\theta)] \preceq 0$ over $\tau \in [0, T]$ yields

$$e^{A(\theta)^T} T S(T, \theta) e^{A(\theta)T} \preceq S(0, \theta).$$

- Using now $S(0, \theta) - S(T, \eta) \prec 0$ yields the condition

$$e^{A(\theta)^T} T S(T, \theta) e^{A(\theta)T} - S(T, \eta) \prec 0.$$
Sketch of the proof

(b) ⇒ (a)

• Integrating $\partial_\tau S(\tau, \theta) + \text{Sym}[S(\tau, \theta)A(\theta)] \preceq 0$ over $\tau \in [0, T]$ yields

$$e^{A(\theta)^\top T} S(T, \theta)e^{A(\theta)T} \preceq S(0, \theta).$$

• Using now $S(0, \theta) - S(T, \eta) \prec 0$ yields the condition

$$e^{A(\theta)^\top T} S(T, \theta)e^{A(\theta)T} - S(T, \eta) \prec 0.$$

(a) ⇒ (b)

• Assume that there exists $P(\cdot) \succ 0$ such that $e^{A(\theta)^\top T} P(\theta)e^{A(\theta)T} - P(\eta) \prec 0$

• Pick $S^*(\tau, \theta) = e^{-A(\theta)^\top \tau} S^*(0, \theta)e^{-A(\theta)\tau}$, $S^*(T, \cdot) := P(\cdot) \succ 0$

• Then, we have that $\partial_\tau S^*(\tau, \theta) + \text{Sym}[S^*(\tau, \theta)A(\theta)] = 0$

• Moreover, we have that

$$S^*(0, \theta) - S^*(T, \eta) = e^{A(\theta)^\top T} S^*(T, \theta)e^{A(\theta)T} - S^*(T, \eta) \prec 0 \quad (18)$$
Theorem

The following statements are equivalent:

(a) There exists a matrix-valued function $P : \mathcal{P} \rightarrow \mathbb{S}_n^+ > 0$ such that the conditions

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0$$  \hspace{1cm} (19)

$$e^{A(\theta)^\top T}P(\theta)e^{A(\theta)T} - P(\eta) \prec 0$$  \hspace{1cm} (20)

hold for all $\theta, \eta \in \mathcal{P}$.

(b) There exists a matrix-valued function $S : [0, T] \times \mathcal{P} \rightarrow \mathbb{S}_n^+$, $S(T, \theta) > 0$, such that the conditions

$$A(\theta)^\top S(T, \theta) + S(T, \theta)A(\theta) \prec 0$$  \hspace{1cm} (21)

$$\partial_\tau S(\tau, \theta) + S(\tau, \theta)A(\theta) + A(\theta)^T S(\tau, \theta) \preceq 0$$  \hspace{1cm} (22)

$$S(0, \theta) - S(T, \eta) \prec 0$$  \hspace{1cm} (23)

hold for all $\theta, \eta \in \mathcal{P}$ and all $\tau \in T := [0, T]$.

Moreover, when one of the above statements holds, then the LPV system is asymptotically stable for all $\rho \in \mathcal{P} \supseteq T$. 
Connection with quadratic and robust stability

Theorem (Quadratic stability)

When $T \to 0$ in the minimum dwell-time theorem, then the quadratic stability condition

$$A(\theta)^\top P + PA(\theta) < 0$$

is recovered.
Connection with quadratic and robust stability

Theorem (Quadratic stability)

When \( T \rightarrow 0 \) in the minimum dwell-time theorem, then the quadratic stability condition

\[
A(\theta)\top P + PA(\theta) \prec 0
\]  \hspace{1cm} (24)

is recovered.

Theorem (Robust stability)

When \( T \rightarrow \infty \) in the minimum dwell-time theorem, then the robust stability condition

\[
A(\theta)\top P(\theta) + P(\theta)A(\theta) \prec 0
\]  \hspace{1cm} (25)

for constant parametric uncertainties is recovered.
Connection with quadratic and robust stability

Theorem (Quadratic stability)

When $T \to 0$ in the minimum dwell-time theorem, then the quadratic stability condition

$$A(\theta)^\top P + PA(\theta) \prec 0$$  \hspace{1cm} (24)

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Theorem (Robust stability)

When $T \to \infty$ in the minimum dwell-time theorem, then the robust stability condition

$$A(\theta)^\top P(\theta) + P(\theta)A(\theta) \prec 0$$  \hspace{1cm} (25)

for constant parametric uncertainties is recovered.
System

Let us consider the system (Wu, 1995, p. 55)

\[
\dot{x} = \begin{bmatrix}
\frac{3}{4} & 2 & \rho_1 & \rho_2 \\
0 & \frac{1}{2} & -\rho_2 & \rho_1 \\
-3\upsilon\rho_1/4 & \upsilon (\rho_2 - 2\rho_1) & -\upsilon & 0 \\
-3\upsilon\rho_2/4 & \upsilon (\rho_1 - 2\rho_2) & 0 & -\upsilon \\
\end{bmatrix} x
\]  

(26)

where \( \upsilon = \frac{15}{4} \) and \( \rho \in \mathcal{P} = \{z \in \mathbb{R}^2 : \|z\|_2 = 1\} \). This system is not quadratically stable.
**Example**

**System**

Let us consider the system (Wu, 1995, p. 55)

\[
\dot{x} = \begin{bmatrix}
  3/4 & 2 & \rho_1 & \rho_2 \\
  0 & 1/2 & -\rho_2 & \rho_1 \\
  -3v\rho_1/4 & v(\rho_2 - 2\rho_1) & -v & 0 \\
  -3v\rho_2/4 & v(\rho_1 - 2\rho_2) & 0 & -v
\end{bmatrix} x
\]  

where \( v = 15/4 \) and \( \rho \in \mathcal{P} = \{ z \in \mathbb{R}^2 : ||z||_2 = 1 \} \). This system is not quadratically stable.

**Results**

- Gridding (126 points) of the exponential conditions yields the lower bound on the minimum dwell-time given by \( T_{\ell} = 1.7594 \) (complexity 256032/1260)

- Sum of squares conditions yield the upper-bounds on the minimum dwell-time of 2.7282 for polynomials of order 2 (complexity 7228/1350) and 1.7605 for polynomials of order 4 (complexity 33200/3500).
Control of a class of LPV systems with piecewise constant parameters
System

• Let us consider the LPV system

\[
\begin{align*}
\dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\
x(0) &= x_0
\end{align*}
\]

where \(\{t_k\}_{k \in \mathbb{N}_0}\) is the sequence of time instants at which the parameter vector changes value.
System

- Let us consider the LPV system

\[
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\
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\]

where \( \{t_k\}_{k \in \mathbb{N}_0} \) is the sequence of time instants at which the parameter vector changes value.

Control laws

- Constant dwell-time case

\[
u(t) = K(t - t_k, \rho(t_k))x(t), \ t \in [t_k, t_{k+1}) \quad (27)
\]
System

- Let us consider the LPV system

\[
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\
x(0) = x_0
\]

where \( \{t_k\}_{k \in \mathbb{N}_0} \) is the sequence of time instants at which the parameter vector changes value.

Control laws

- Constant dwell-time case

\[
u(t) = K(t - t_k, \rho(t_k))x(t), \quad t \in [t_k, t_{k+1})
\]

- Minimum dwell-time case

\[
u(t) = \begin{cases} 
K(t - t_k, \rho(t_k))x(t), & t \in [t_k, t_k + T) \\
K(T, \rho(t_k))x(t), & t \in [t_k + T, t_{k+1})
\end{cases}
\]
State-feedback control - Minimum DT

**Theorem**

Assume that there exists matrix-valued functions $U : [0, T] \times \mathcal{P} \to \mathbb{R}^{m \times n}$ and $	ilde{S} : [0, T] \times \mathcal{P} \to \mathbb{S}^n$, $	ilde{S}(T, \theta) > 0$, such that the conditions

$$\text{Sym}[A(\theta)\tilde{S}(T, \theta) + B(\theta)U(T, \theta)] < 0, \quad (29)$$

$$-\partial_\tau \tilde{S}(\tau, \theta) + \text{Sym}[A(\theta)\tilde{S}(\tau, \theta) + B(\theta)U(\tau, \theta)] \preceq 0 \quad (30)$$

and

$$\tilde{S}(T, \eta) - \tilde{S}(0, \theta) < 0 \quad (31)$$

hold for all $\theta, \eta \in \mathcal{P}$ and all $\tau \in [0, T]$.

Then the closed-loop LPV system is asymptotically stable for all $\rho \in \mathcal{P}_{\geq T}$, and a suitable controller gain is moreover given by

$$K(\tau, \theta) = U(\tau, \theta)\tilde{S}(\tau, \theta)^{-1}. \quad (32)$$
Example

System

\[
\dot{x} = \begin{bmatrix}
3 - \theta & 1 \\
1 - \theta & 2 + \theta \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
1 + \theta \\
\end{bmatrix} u, \ \theta \in [0, 1].
\]  
(33)
Example

System

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\] (33)

Proposition

No control law of the form \( u = K(\theta)x \) can quadratically stabilize the system (33).
Example

System

$$\dot{x} = \begin{bmatrix} 3 - \theta & 1 \\ 1 - \theta & 2 + \theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 + \theta \end{bmatrix} u, \; \theta \in [0,1].$$ \quad (33)

Proposition

No control law of the form \( u = K(\theta)x \) can quadratically stabilize the system (33).

Proof

- Quadratically stabilizable if and only if the LMI (elimination lemma)

$$L(\theta) := B_\perp(\theta) [A(\theta)P + PA(\theta)^T] B_\perp(\theta)^T < 0$$

is feasible for all \( \theta \in [0,1] \) where \( B_\perp(\theta) = \begin{bmatrix} 1 + \theta & -1 \end{bmatrix} \).
Example

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\dot{x} = \begin{bmatrix} 3 - \theta & 1 \\ 1 - \theta & 2 + \theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 + \theta \end{bmatrix} u, \quad \theta \in [0, 1].
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- Quadratically stabilizable if and only if the LMI (elimination lemma)
  \[
  L(\theta) := B_\perp(\theta) [A(\theta) P + PA(\theta)^\top] B_\perp(\theta)^\top < 0
  \]
  is feasible for all \( \theta \in [0, 1] \) where \( B_\perp(\theta) = \begin{bmatrix} 1 + \theta & -1 \end{bmatrix} \).
- Assume it is stabilizable, then \( L(0) < 0 \) and \( L(1) < 0 \).
Example

System

\[ \dot{x} = \begin{bmatrix} 3 - \theta & 1 + \theta \\ 1 - \theta & 2 + \theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 + \theta \end{bmatrix} u, \; \theta \in [0, 1]. \] (33)

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- Quadratically stabilizable if and only if the LMI (elimination lemma)
  \[ L(\theta) := B_\perp(\theta)[A(\theta)P + PA(\theta)^\top]B_\perp(\theta)^\top < 0 \]
  is feasible for all \( \theta \in [0, 1] \) where \( B_\perp(\theta) = \begin{bmatrix} 1 + \theta & -1 \end{bmatrix} \).
- Assume it is stabilizable, then \( L(0) < 0 \) and \( L(1) < 0 \).
- This implies that there exists a \( p \in \mathbb{R} \) such that
  \[ f_1(p) = p^2 - 3p + 2 < 0 \quad \text{and} \quad f_2(p) = p^2 - 6p + 8 < 0. \] (34)
Introduction

Stability analysis

Control design

Hybrid systems formulation

Conclusion

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System

\[ \dot{x} = \begin{bmatrix} 3 - \theta & 1 \\ 1 - \theta & 2 + \theta \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 + \theta \end{bmatrix} u, \; \theta \in [0, 1]. \] (33)

Proposition

No control law of the form \( u = K(\theta)x \) can quadratically stabilize the system (33).

Proof

- Quadratically stabilizable if and only if the LMI (elimination lemma)

\[
L(\theta) := B_\perp(\theta) [A(\theta) P + P A(\theta)^\top] B_\perp(\theta)^\top < 0
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\[
f_1(p) = p^2 - 3p + 2 < 0 \quad \text{and} \quad f_2(p) = p^2 - 6p + 8 < 0. \] (34)

- But \( f_1(p) < 0 \iff p \in (1, 2) \) and \( f_2(p) < 0 \iff p \in (2, 4) \), a contradiction.
Example

- We pick $T = 0.05$, polynomials $S, U$ of order 1
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- Primal/dual variables: 551/120; computation time: less than 2sec
- We find

$$K(\tau, \theta) = \frac{1}{\text{den}(\tau, \theta)} \begin{bmatrix} K_1(\tau, \theta) & K_2(\tau, \theta) \end{bmatrix}$$

where

$$K_1(\tau, \theta) = 76.930 - 1109.596\tau + 14.343\theta + 1569.878\tau^2 + 170.469\tau\theta - 9.158\theta^2$$

$$K_2(\tau, \theta) = 24.445 - 739.302\tau - 17.004\theta + 1136.874\tau^2 + 159.427\tau\theta + 3.174\theta^2$$

$$\text{den}(\tau, \theta) = -23.189 + 483.241\tau - 0.934\theta - 947.359\tau^2 + 3.140\tau\theta + 1.066\theta^2$$
Example

- We pick \( T = 0.05 \), polynomials \( S, U \) of order 1
- Primal/dual variables: 551/120; computation time: less than 2sec
- We find

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K_1(\tau, \theta) = 76.930 - 1109.596 \tau + 14.343 \theta + 1569.878 \tau^2 + 170.469 \tau \theta - 9.158 \theta^2
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\[
K_2(\tau, \theta) = 24.445 - 739.302 \tau - 17.004 \theta + 1136.874 \tau^2 + 159.427 \tau \theta + 3.174 \theta^2
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\[
\text{den}(\tau, \theta) = -23.189 + 483.241 \tau - 0.934 \theta - 947.359 \tau^2 + 3.140 \tau \theta + 1.066 \theta^2
\]
Hybrid systems formulation
Main problems

Is it possible to unify the frameworks of quadratic/robust stability and those of piecewise constant/differentiable parameters?

Is it possible to generalize the approach to more complicated cases?
Stability analysis of hybrid systems

Let us consider the following hybrid inclusion

\[
\dot{x} \in F(x), \ x \in C \\
x^+ \in G(x), \ x \in D
\]  

where \( F \) and \( G \) are set-valued maps and a closed set \( \mathcal{A} \subset \mathbb{R}^n \).

Theorem \(^a\)

If the function \( V : C \cup D \cup G(D) \to \mathbb{R} \) is a Lyapunov function candidate for the system and there exist \( \alpha_1, \alpha_2 \in K_\infty \) and a positive definite function \( \alpha_3 \) such that

\[
\begin{align*}
(a) & \quad \alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A) \text{ for all } x \in C \cup D \cup G(D) \\
(b) & \quad \langle \nabla V(x), f \rangle \leq -\alpha_3(|x|_A) \text{ for all } x \in C \text{ and } f \in F \\
(c) & \quad V(g) - V(x) \leq -\alpha_3(|x|_A) \text{ for all } x \in D \text{ and } g \in G(D)
\end{align*}
\]

then \( \mathcal{A} \) is uniformly globally (pre-)asymptotically stable for the system.

Stability analysis of hybrid systems

Let us consider the following hybrid inclusion

\[
\begin{align*}
\dot{x} & \in F(x), \ x \in C \\
x^+ & \in G(x), \ x \in D
\end{align*}
\] (35)

where \( F \) and \( G \) are set-valued maps and a closed set \( \mathcal{A} \subset \mathbb{R}^n \).

**Theorem \(^{\text{(a)}}\)**

*If the function \( V : C \cup D \cup G(D) \to \mathbb{R} \) is a Lyapunov function candidate for the system and there exist \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) and a positive definite function \( \alpha_3 \) such that*

\( \alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \) for all \( x \in C \cup D \cup G(D) \)

\( \langle \nabla V(x), f \rangle \leq -\alpha_3(|x|_{\mathcal{A}}) \) for all \( x \in C \) and \( f \in F \)

\( V(g) - V(x) \leq -\alpha_3(|x|_{\mathcal{A}}) \) for all \( x \in D \) and \( g \in G(D) \)

then \( \mathcal{A} \) is uniformly globally (pre-)asymptotically stable for the system.

**Relaxed conditions**

- Persistent flowing \( \rightarrow V(g) - V(x) \leq 0 \) for all \( x \in D \) and \( g \in G(D) \)
- Persistent jumping \( \rightarrow \langle \nabla V(x), f \rangle \leq 0 \) for all \( x \in C \) and \( f \in F \)

Hybrid system formulation

Hybrid form

\[
\begin{align*}
\dot{x} &= A(\rho)x \\
\dot{\rho} &= 0 \\
\dot{\tau} &= 1 \\
x^+ &= x \\
\rho^+ &\in P \setminus \{\rho\} \\
\tau^+ &= 0
\end{align*}
\]

when \( \tau \in [0, T) \)

when \( \tau = T \)

- We augment the state with the parameter vector \( \rho \in P \) and a timer \( \tau \in [0, 1] \)
- State-space \( \mathbb{R}^n \times P \times [0, 1] \) where \( P := P_1 \times \ldots \times P_N \)
- Stability to the set \( \mathcal{A} = \{0\} \times P \times [0, 1] \)
- The jump part models parameters discontinuities
- The periodic case given above but the aperiodic case can be considered as well (e.g. \( \dot{\tau} \in [0, 1) \))
- Using the previous theorem on this system, we recover stability conditions that are very similar to those previously obtained.

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Deterministic parameters

Multiple clocks

\[
\begin{align*}
\dot{x} &= A(\rho)x \\
\dot{\rho} &= 0 \\
\dot{\tau} &= 1 \\
x^+ &= x \\
\rho_i^+ &\in P_i \setminus \{\rho_i\} \\
\rho_j^+ &= \rho_j, \ j \neq i \\
\tau_i^+ &= 0 \\
\end{align*}
\]

when \( \tau \in [0, T_1) \times [0, T_2) \times \ldots \times [0, T_N) \)  

when \( \tau_i = T_i, \ i = 1, \ldots, N \)
Deterministic parameters

Multiple clocks

\[
\begin{align*}
\dot{x} &= A(\rho)x \\
\dot{\rho} &= 0 \\
\dot{\tau} &= 1 \\
x^+ &= x \\
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\tau_i^+ &= 0
\end{align*}
\]

when \( \tau \in [0, T_1) \times [0, T_2) \times \ldots \times [0, T_N) \)

Piecewise differentiable parameters \( \rho, \dot{\rho} \in [-1, 1] \)

\[
\begin{align*}
\dot{x} &= A(\rho)x \\
\dot{\rho} &\in \begin{cases} 
[0, 1] & \text{if } \rho = -1 \\
[-1, 0] & \text{if } \rho = 1 \\
(-1, 1) & \text{otherwise}
\end{cases} \\
\dot{\tau} &= 1 \\
x^+ &= x \\
\rho^+ &\in [-1, 1] \\
\tau^+ &= 0
\end{align*}
\]

when \( \tau \in [0, T) \)

when \( \tau = T \)
Stochastic parameters

System (PDMP)

\[
\begin{align*}
\dot{x} &= A(\rho)x \\
\dot{\rho} &= 0 \\
x^+ &= x \\
\rho^+ &= \theta
\end{align*}
\]

with rate \( \lambda(\rho, \theta) \)  

\[ (39) \]

- \( \mathbb{P}[\rho(t + dt) = \eta | \rho(t) = \theta] = \lambda(\theta, \eta)dt + o(dt) \)
- Random jump times (exponentially distributed dwell-times)
- Generalization of Markov jump linear systems
Stochastic parameters

System (PDMP)

\[
\begin{align*}
\dot{x} &= A(\rho)x \\
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with rate $\lambda(\rho, \theta)$

- $\mathbb{P}[\rho(t + dt) = \eta | \rho(t) = \theta] = \lambda(\theta, \eta)dt + o(dt)$
- Random jump times (exponentially distributed dwell-times)
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Proposition

The above system is mean-square stable if (iff?) there exists a bounded matrix-valued function $P : \mathcal{P} \to \mathbb{S}_+^n$ such that the LMI

\[
A(\theta)^T P(\theta) + P(\theta) A(\theta) + \int_{\mathcal{P}} \lambda(\theta, \eta)[P(\eta) - P(\theta)]d\eta < 0
\]

holds for all $\rho \in \mathcal{P}$. 

Concluding remarks
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- We can capture discontinuities in the parameters trajectories in a tractable way
- Extend quadratic and robust stability
- Extends results on switched systems
- Unification of the results seem to be possible using the framework of hybrid systems
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What else can be done?

- Performance analysis, e.g. $L_2$-performance
- Nonlinear systems (implicit computation of the state-transition operator)
- Homogeneous Lyapunov functions (on the basis of the converse results in \(^1\))

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\(^1\) F. Wirth. A converse Lyapunov theorem for linear parameter-varying and linear switching systems, *SIAM Journal on Control and Optimization*, 2005
Concluding statements

- We can capture discontinuities in the parameters trajectories in a tractable way
- Extend quadratic and robust stability
- Extends results on switched systems
- Unification of the results seem to be possible using the framework of hybrid systems

What else can be done?

- Performance analysis, e.g. $L_2$-performance
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An open question

Is it possible to obtain tractable conditions for the design a dynamic output feedback?

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Thank you for your attention