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Titre:

Commande et Observation Robustes des Systèmes LPV Retardés

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Summary

This thesis is concerned with the stability analysis, observation and control of LPV time-delay systems. The main objectives of the thesis are

- the development of adapted and possibly low conservative stability sufficient conditions for LPV time-delay systems.
- the development of new advanced control/observation strategies for such systems using new tools developed in the thesis, such as specific relaxation techniques of Linear and Nonlinear Matrix Inequalities.

For that purpose, this thesis is subdivided is three parts:

- The first part, composed of Chapters 1 and 2, aims at providing a sufficiently detailed state of the art of the representation and stability analysis of both LPV and time-delay systems. In both cases, the importance of LMI in stability analysis is strongly emphasized. Several fundamental results are bridged in order to show the relations between different theories and this constitutes the first part of the contributions of this work.
- The second part, composed of Chapter 3, consists in a presentation of several (new) preliminary results that will be used along the thesis. This part contains most of the contributions of this work.
- Finally, the third part, composed of Chapters 4 and 5, uses results of the second part in order to derive efficient observation, filtering and control strategies for LPV time-delay systems.

Introduction and Structure of the Thesis

Context of the Thesis

This thesis is the fruit of a three years work (2005-2008) spent in the GIPSA-Lab[1] (former LAG[2]) in the SLR[3] Team. The topic of the thesis is on **Robust/LPV Control and Observation of LPV Time-Delay Systems** under supervision of Olivier Sename (Professor at Grenoble-INP[4]) and Jean-François Lafay (professor at Centrale Nantes, IRCCyN[5]), Nantes, France).

This thesis is in the continuity of works of Annas Fattouh [Fattouh, 2000, Fattouh et al., 1998], Olivier Sename [Sename, 2001, 1994, Sename and Fattouh, 2005] and more deeply Jean-François Lafay who was the Olivier Sename's thesis supervisor (the thesis was on the controllability of time-delay systems).

During the thesis the Rhônes-Alpes Region granted me of a scolarship in order to travel and collaborate in a foreign laboratory. I went to School of Electrical and Computer Engineering (ECE) in GeorgiaTech (Georgia Institute of technology) to work with Erik I. Verriest on the topic of time-delay systems with applications in the control of disease epidemics. The collaboration gave rise to a conference paper '**A New Delay-SIR Model for Pulse Vaccination**' [Briat and Verriest, 2008] and potentially to a journal version according to the invitation of the editor of the new Elsevier journal: 'Biomedical Signal Processing and Control'. A copy of the conference paper is given in Appendix K.

Finally, thanks to Emmanuel Witrant (GIPSA Lab), I incorporated the project on the control of unstable modes in plasmas in Tokamaksn cores of the promising future (?) energy production technology exploiting nuclear fusion. The collaboration is done with Erik Olofsson and Per Brunsell(KTH[6]) and S-I. Niculescu (LSS[7]). The work has led to the conference paper [Olofsson et al., 2008] given in Appendix L.

[1]Grenoble Image Parole Signal Automatique Laboratory - Grenoble Image Speech Signal Control Systems

[2]Laboratoire d'Automatique de Grenoble - Grenoble Control Systems Laboratory

[3]Systèmes Linéaires et Robustesse - Linear Systems and Robustness

[4]Grenoble INstitut Polytechnique - Grenoble Institute of Technology

[5]Institut de Recherche en Communications et Cybernétique de Nantes - Nantes Research Institute in Communications and Cybernetics

[6]Kungliga Tekniska högskolan - Royal Institute of Technology, Sweden

[7]Laboratoire des Signaux et Systèmes - Signals and Systems Laboratory

Introduction and Motivations

At the beginning of the century, Emile Picard (french mathematician) wrote an interesting remark in the proceedings of the 4th International Mathematician Congress in Rome (the complete text will be provided at the end of this section). He noticed that while classical mechanics equations, the future is immediately predicted using current information (speed and position), while considering living beings, the future cannot be predicted in the same way. Indeed, the future evolution would depend on the current information but also on past events. Mathematically speaking, the evolution would consider integral term taken from past to current time and would describe the heredity. In the 1970s such equations began to be studied and these studies give rise to several books in the 1980s. Since then, time-delay systems (which is the common denomination) have gained more and more attention in a wide variety of problems such as the stability analysis, control and observation design... Even if it was at the beginning only coming from mathematical interests and ideas, it turned out that many physical, biological, economic systems can be modeled as time-delay systems (examples will be given in Section 2.1) and strengthened their importance in modern theories of dynamical systems, both in mathematical and control systems frameworks.

Due to the particular structure of these systems, lots of specific approaches have been developed and generalized in order to study their stability, controllability and many other properties. As a fundamental example, Lyapunov theory has been extended to this type of systems through two celebrated theorems, namely the Lyapunov-Krasovskii and Lyapunov-Razumikhin theorems. From these results a lots of advanced has been made but many problems remain open.

On the other hand, in the stability analysis of linear systems, a great problem is the robustness of the stability. In few words, it consists in determining the stability of a linear system whose constant coefficients belong to a certain interval. Several tools have been deployed to study these systems such as μ-analysis and has led to good results and many applications, notably in aerospace. Furthermore, robust stabilization is also an important research framework and is still an open problem.

One main problem is the case of systems which are not robustly stabilizable and, in this case, another strategy should be developed. Here comes LPV control, the idea behind LPV control is to measure some parameters and use them in the control law. It turns out that, using such a control strategy, the class of systems which are stabilizable is then enlarged.

Moreover, LPV systems can be used to approximate nonlinear systems and hence systematic and generic 'LPV tools' can be then applied to derive nonlinear control laws for nonlinear systems. An other interest of LPV control is the design of tunable controllers: external parameters can be added in the design in order to characterize different modes of working.

The idea of merging time-delay systems and LPV systems is not new but is rather marginal. Indeed, only a few work are based on the stability analysis and control synthesis. No work exists on the observation and few results are provided for the filtering problems. At first sight, it seems straightforward to find solutions to problems involving LPV time-delay systems since it would be enough to merge the theories. Actually, many results deployed in robust stability and robust control for finite dimensional systems do not work with time-delay system and this makes the main difficulty of the study of LPV time-delay systems.

Emile Picard's original text [Kolmanovskii and Myshkis, 1999]:

"Les équations différentielles de la mécanique classique sont telles qu'il y en résulte que

le mouvement est déterminé par la simple connaissance des positions et des vitesses, c'est à dire par l'état à un instant donné et à l'instant infiniment voisin".

"Les états antérieurs n'y intervenant pas, l'hérédié y est un vain mot. L'application de ces équations où le passé ne se distingue pas de l'avenir, où les mouvements sont de nature réversibles, sont donc inapplicables aux êtres vivants".

"Nous pouvons rêver d'équations fonctionelles plus compliquées que les équations classiques parce qu'elles renfermeront en outre des intégrales prises entre le temps passé très éloigné et le temps actuel, qui apporteront la part de l'hérédité".

Emile Picard, "La mathématique dans ses rapports avec la physique, Actes du IV^e congrès international des Mathématiciens, Rome, 1908

English translation:

Differential equations of classical mechanics are such that the movement is determined by the only knowledge of positions and speeds, that is to say by the state at a given instant and at the instant infinitely nearby.

Since the anterior states are not involved, heredity is a vain word. The application of these equations where the past and future are not distinguishable, where the movements are by nature reversible, are hence unapplicable to living beings.

We may dream about more complex functional equations than classical equations since they shall contain in addition integral terms taken from a distant past time instant and the current time instant, which shall bring the share of heredity.

Structure of the Thesis

Chapter 1 provides an overview of different types of representation for a LPV system. For each model, several adapted stability tests are presented and are compared between each others.

Chapter 2 gives an insight of different representations of time-delay systems and several physical examples show the interest of such systems. Then a large part is concerned with the stability analysis of these systems in the time domain in which several methods of the literature are presented and compared. A last section address the problem of the stability in presence of uncertain delay.

Chapter 3 is devoted to preliminary notions and results used along the thesis. First of all, spaces of delays and parameters are clearly defined. Second, new methods of relaxation of parameter dependent LMI and matrix inequalities with concave nonlinearity are developed and analyzed. Then a method to compute explicit expression of parameter derivatives in LPV polytopic systems is given using linear algebra. Finally, several Lyapunov-Krasovskii based techniques are given in order to show asymptotic stability of LPV systems.

Chapter 4 presents results in observation and filtering of LPV systems using results provided in Chapter 3. Several types of observers and filters are studied in both certain and uncertain frameworks.

Chapter 5 concludes on the stabilization of LPV time-delay systems. Several structures of controllers are explored according to the presence of a delayed term in the control law; both state-feedback and dynamic output feedback controllers are synthesized.

This chapter also presents a new type of controllers which is called 'delay-scheduled' controllers whose gains are smoothly scheduled by the delay value.

Contributions

The contribution of the thesis is plural:

- Methodological contributions
- Theoretical contributions

Methodological contribution

The methodological contribution is based on a common remark by reading journal and conference papers. Why most of the paper addresses stability of time-delay systems ? Why there only few papers on the control and observation or filtering ? The main reason comes the fact that, while considering time-delay systems, it is not sufficient to substitute the closed-loop system expression in the stability condition to derive efficient and easy to compute constructive stabilization conditions (taking generally the form of a set of LMIs). This is mainly due to the presence of a high number of decision matrices in the stability conditions.

A global method is to perform a relaxation after substitution of the closed-loop system (which is the direct and efficient method used for finite dimensional systems). We emphasize in this thesis that this may be not the right choice since this alters the efficiency of the initial result. So we preconize to perform a relaxation technique on the initial problem in order to turn the original stability condition into a form which is more suitable for synthesis purposes. Hence a step as added in the methodology of design and results in better results and a great interest of the relaxation is its adaptability on a wide variety of different LMI stability conditions. Indeed, it will be shown that this relaxation applies on every stability tests developed in this thesis.

Theoretical contributions

The theoretical contributions are multiple and address several different topics:

- A new method for relaxation of polynomially parameter dependent LMI is provided. This approach allows to turn the polynomial dependence on the parameters into a linear one by introducing a supplementary decision variable, generally called 'slack' variable. An example is given to show the effectiveness of the approach.
- Concave nonlinearities (involving inverse of matrices) in matrix inequality are quite difficult to handle and their simplification (or removal) generally results in conservative conditions. Bounds involving completion by the squares and using the cone complementary algorithm can be used but while the former is too conservative, the latter cannot be used with parameter dependent matrices. To solve this problem we have deployed a new exact relaxation which turns the rational dependence into a bilinear one and allows for the application of simple iterative algorithm.
- Several LMI tests have been generalized to the LPV case and the relaxation method have been applied in order to provide new LMI tests more suitable for design strategies.
- A new Lyapunov-Krasovskii functional has been developed in order to consider systems with two delays in which the delays satisfy a algebraic constraint. This functional

addresses well the problem of stabilizing a time-delay system with a controller with memory embedding a delay which is different from the system one.

- A new strategy to control time-delay systems has been introduced and has been called 'delay-scheduled' controllers. This type of controllers are designed using a LPV formulation of time-delay systems. Then using LPV design tools, it is possible to derive controllers whose gain is smoothly scheduled by the delay value, provided that it is measured or estimated. Since the delay is viewed as a parameter, then it is possible to consider uncertainties on the delay and perform robust stabilization in presence of measurement/estimation errors.

- Finally, the last contributions are based on the application of new and adapted stability tests to observation, filtering and control. Such methods will be shown to lead to interesting results.

How to read the thesis

This thesis has been written in order to be sufficient unto oneself. Hence, leveled readers should consider only the core of the thesis including Chapters 3, 4 and 5. Readers who are not really familiar with LPV or time-delay systems should read Chapters 1 or 2 respectively.

The profane reader should consider the appendices first in order to learn the fundamentals of automatic control. Fundamentals on linear algebra are given in Appendix A. Their presentation is motivated by the fact that linear algebra is extensively used in linear dynamical system theory, introduced in Appendix B, in which the concept of dynamical systems, stability of dynamical systems with Lyapunov theory is presented. Since in modern control theory, new more representative spaces of signals are used and allow to elaborate powerful result, they are described in Appendix C. In Appendix D, a brief history of Linear Matrix Inequalities (LMIs) is provided with examples and a simple algorithm to solve them. Then lots of important and widely used technical results on robust/LPV control and LMIs are provided in Appendix E while interesting technical results on time-delay systems are given in Appendix F. Simple frequency domain methods for stability analysis of time-delay systems are introduced in Appendix G. Appendices H and I provides results on the controllability and observability of LPV and time-delay systems respectively. Finally Appendix J completes, with less relevant results, Chapter 4 on the observation and filtering of LPV time-delay systems.

Contents

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List of Publications

This thesis gave rise to several papers

Journal Papers

- C. Briat, O. Sename and J.F. Lafay, '$\mathcal{H}_\infty$ delay-scheduled control of linear systems with time-varying delays', submitted to IEEE Transactions on Automatic Control (2nd round review)
- C. Briat, O. Sename and J.F. Lafay, 'Delay-Scheduled State-Feedback Design for Time-Delay Systems with Time-Varying Delays - A LPV Approach', Submitted to Systems & Control Letters (2nd round review)

International Conference Papers with Proceedings

- E. Olofsson, E. Witrant, C. Briat, S-I. Niculescu, P. Brunsell, 'Stability Analysis and Model-Based Control in EXTRAP-T2R with Time-Delay Compensation', Accepted in 47th IEEE Conference on Decision and Control, Cancun, Mexico, 2008
- C. Briat, E. I. Verriest, 'A new delay-SIR Model for Pulse Vaccination', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, 'Delay-Scheduled State-Feedback Design for Time-Delay Systems with Time-Varying delays', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, 'Parameter dependent state-feedback control of LPV time-delay systems with time-varying delays using a projection approach', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, 'A Full-Block S-procedure application to delay-dependent $\mathcal{H}_\infty$ state-feedback control of uncertain time-delay systems', IFAC World Congress, Seoul, South Korea, 2008
- C. Briat, O. Sename and J.F. Lafay, A LFT/$\mathcal{H}_\infty$ state feedback design for linear parameter varying time-delay systems', European Control Conference, Kos, Greece, 2007
- O. Sename, C. Briat, '$\mathcal{H}_\infty$ observer design for uncertain time-delay systems', European Control Conference, Kos, Greece, 2007

- C. Briat, O. Sename and J.F. Lafay, 'Full order LPV/$\mathcal{H}_\infty$ Observers for LPV Time-Delay Systems', IFAC Conference on System, Structure and Control, Foz do Iguacu, Brazil, 2007
- O. Sename, C. Briat, 'Observer-based $\mathcal{H}_\infty$ control for time-delay systems: a new LMI solution', IFAC Conference on Time-Delay Systems, L'Aquila, Italy, 2006

National Conference and Workshop Papers with Proceedings

- C. Briat, O. Sename, J.F. Lafay, 'Filtrage $\mathcal{H}_\infty$/LPV de systèmes LPV incertains à retards' ($\mathcal{H}_\infty$/LPV filtering of uncertain LPV Time-Delay Systems), CIFA Conference, Bucharest, Romania, 2008

National Conference and Workshop Papers without Proceedings

- C. Briat, O. Sename, J.F. Lafay, 'Delay-scheduled controllers for time-delay systems with time-varying delays', GDR MACS MOSAR, Grenoble, France, 2008
- C. Briat, O. Sename, J.F. Lafay, 'Delay-scheduled controllers for time-delay systems with time-varying delays', GDR MACS SAR, Lille, France, 2007
- C. Briat, O. Sename, J.F. Lafay, 'LPV/LFT Control for Time-delay systems with time-varying delays - A delay independent result', GDR MACS SAR, Paris, France, 2006

Notations and Acronyms

$\mathbb{N}$	Set of integers
$\mathbb{Z}$	Set of rational integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^{+*}$	Set of real numbers, nonnegative real numbers, strictly positive real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{K}^{p\times m}$	Matrix algebra (of dimension $p \times m$) with coefficients in $\mathbb{K}$
$\mathbb{S}^n$	Set of symmetric matrices of dimension n
$\mathbb{S}^n_+$	Cone of positive symmetric semidefinite matrices of dimension n
$\mathbb{S}^n_{++}$	Cone of positive symmetric definite matrices of dimension n
$\mathbb{H}^n$	Set of skew-symmetric matrices of dimension n
$\imath$	Imaginary unit (i.e. $\imath^2 = -1$)
I_n	Identity matrix of dimension n
A^T	Transpose matrix of A
A^*	Conjugate transpose of A
s A^H	Hermitian operator such that $A^H := A + A^T$
	($A^H := A + A^*$ for complex matrices)
A^{-1}	Inverse of matrix A
A^{-T}	Transpose of the inverse of A
A^+	Moore-Penrose pseudoinverse of A
trace(A)	Trace of A
det(A)	Determinant of A
Adj(A)	Adjugate matrix of A
Ker(A)	Basis of the null-space of A
$Im(A)$	Image set of A
Null(A)	Null space of the operator A (i.e. $Im(\mathrm{Ker}[A])$)
$\lambda(A)$	Set of the eigenvalues of A
$\sigma(A)$	Set of singular values of A
$\lambda_{max}(A)$, $\lambda_{min}(A)$	Maximal and minimal eigenvalues of A
$\bar{\sigma}(A)$	Maximal singular value of A (i.e. $\sqrt{\lambda_{max}(A^TA)}$)
$\|\|w\|\|_q$	q Euclidian norm of vector w, i.e. $\|\|w\|\|_q = (\|w_1\|^q + \ldots + \|w_n\|^q)^{1/q}$
$\|\|w\|\|_{\mathcal{L}_q}$	$\mathcal{L}_q$ norm of signal defined as $\left(\int_0^{+\infty} \|\|w(t)\|\|_q^q dt\right)^{1/q}$
$\|\|T\|\|_{\mathcal{H}_q}$	$\mathcal{H}_q$ norm of system T

$\mathcal{C}^1(J,K)$	Set of continuously differentiable functions from set J to K
$\mathcal{F}(J,K)$	Set of functions from J to K
$\mathbb{D}$	Unit open disc
$\bar{\mathbb{D}}$	Closure of $\mathbb{D}$
$\partial\mathbb{D}$	Boundary of $\mathcal{D}$
I	Interior of I
$h(t)$	Time-Delay
μ^-, μ^+	Bounds on the derivative of the delay such that $\dot{h} \in [\mu^-, \mu^+]$
ρ	vector of parameters
U_ρ	Space of parameter values $U_\rho := \times_{i=1}^p [\rho_i^-, \rho_i^+] \subset \mathbb{R}^p$
U_ν	Set of vertices of the polytope containing $\dot{\rho}$ defined as $U_\nu := \times_{i=1}^p \{\nu_i^-, \nu_i^+\}$ compact of $\mathbb{R}^{2p}$
hull$[U]$	Convex hull of the set U
$\mathcal{P}_\nu^p$	set of parameters with bounded derivatives defined as $\{\rho : \mathbb{T} \to U_\rho : \dot{p} \in \text{hull}\, U_\nu\}$
$\mathcal{P}_\infty^p$	set of parameters with unbounded derivatives $\{\rho : \mathbb{T} \to U_\rho\}$
$\blacksquare$	End of Theorem, Lemma or Corollary
$\square$	End of proof

LTI	Linear Time-Invariant
LTV	Linear Time-Varying
LMI	Linear Matrix Inequality
pLMI	parametrized Linear Matrix Inequality
NMI	Nonlinear Matrix Inequality
BMI	Bilinear MAtrix Inequality
LFT	Linear Fractional Transformation
LFR	Linear Fractional Representation
TDS	Time-Delay System
PSD	Positive Symmetric Definite
CCA	Cone Complementary Algorithm
SF	State-Feedback
SOF	Static Output Feedback
DOF	Dynamic Output Feedback
BIBO	Bounded-Input Bounded-Output

Chapter 1

Overview of LPV Systems

Linear Parameter Varying (LPV) systems belong to the general class of Linear Time-Varying Systems. The main difference stems from the particularity that the time-dependence is, in some words, 'hidden' into parameters. Indeed, while the evolution of the time-varying coefficients are a priori known (e.g. $\sin(t)$), the evolution over time of the parameters may be unknown. Actually, the boundary of the inclusion between these types is absolutely unclear but is relative to the fields of application and techniques applied to analyze (and finally control) these systems. A strict analysis does not fall into the context of this introduction and only LPV systems will be considered in the remaining of this chapter. But, before introducing the interests and motivations for studying LPV systems, let us provide the expression of a generalized LPV system, taking the form of a non-autonomous non-stationary system of linear differential equations with vectorial equalities:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho(t))x(t) + B(\rho(t))u(t) + E(\rho(t))w(t) \\ z(t) & = & C(\rho(t))x(t) + D(\rho(t))u(t) + F(\rho(t))w(t) \\ y(t) & = & C_y(\rho(t))x(t) + F_y(\rho(t))w(t)\end{array} \tag{1.1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n\times n}$, $u \in \mathcal{U} \subset \mathbb{R}^m$, $w \in \mathcal{W} \subset \mathbb{R}^p$, $z \in \mathcal{Z} \subset \mathbb{R}^q$ and $y \in \mathcal{Y} \subset \mathbb{R}^t$ are respectively the state of the system, the control input, the exogenous input, the controlled output and the measured output. For more details on dynamical systems and related fundamental results, the reader should refer to Appendix B.

It is clear, from the expressions, that the behavior of output signals depends on input signals and on parameters acting themselves in an internal fashion on the system.

The parameters ρ are always assumed to be bounded:

$$\rho \in U_\rho \subset \mathbb{R}^k \text{ and } U_\rho \text{ compact} \tag{1.2}$$

From these considerations the questions of stability, controllability and observability are not as 'easy' as in the LTI case and remain important problems beginning to be solved efficiently by recent techniques, mainly using LMIs. Questions on controllability and observability are treated in Appendix H.

The great interest of LPV systems is their ability to model a wide variety of systems, from nonlinear to LTV systems passing through switched systems; this will be illustrated in Section 1.1. For instance, we may think to an automotive process where the dampers have to be controlled. In this case, possible parameters may be the speed of the car and position/orientation of the car since their are consequences of the driver and road behaviors. It is clear that the behavior of the vehicle is dissimilar for different speeds and road configuration. Hence it would be more efficient if the control to be applied to the dampers would depend on these parameters.

The second interest, illustrated in the latter small scenario, resides in the control of LPV systems: the flexibility and adaptability that LPV control suggests. Indeed, the fact that some parameters could be used internally in the control law gives rise to an interesting opportunity of improving system stability and performances. Coming back to our little scenario, if one wishes to synthesize a control law without any information on the speed and determine a single LTI controller, this falls into the robust control framework and the stabilization of the process may be difficult to obtain or may lead to bad performances. On the other hand, if the speed is measured and 'internally' used in the control law, the stabilization would be a more simple task and the closed-loop system would certainly have better performances; this is the advantage of the LPV control over the robust control, provided that real-time measurements of potential parameters are possible. We provide below some applications of the LPV modeling and LPV control for a wide variety of systems. It is important to note that LPV control techniques can be easily combined with recent results on $\mathcal{H}_\infty$, $\mathcal{H}_2$, μ norms, to give enhanced control laws with performances and robustness specifications.

We will end this succinct introduction by examples provided in the literature. Since, in many cases, heavy computations are performed to turn the nonlinear system formulation into a LPV dynamical system, only a simple case is detailed hereunder while others are briefly enumerated with corresponding references.

Inverted Pendulum - robust control and performances

This application has been provided in Kajiwara et al. [1999] where a model is given in the LPV form using a change of variable. The inverted pendulum is constituted of two arms moving in the vertical plane. The LPV model is given by:

$$\frac{d}{dt}\begin{bmatrix} z \\ \dot{z} \\ r_x \\ \dot{\phi}_1 \end{bmatrix} = A(\rho)\begin{bmatrix} z \\ \dot{z} \\ r_x \\ \dot{\phi}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{K_a}{T_a} \end{bmatrix} u \tag{1.3}$$

with

$$A(\rho) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{T_a} \end{bmatrix} + \frac{3}{4\ell_2}g\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} + \rho\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

where ϕ_1 is the angle of the first arm, $\phi_2 + \phi_1$ is the angle of the second arm (with respect to the ground), $r_y = 2\ell_1 \sin(\phi_1)$, $r_x = 2\ell_1 \cos(\phi_1)$, ℓ_1 is the half of the length of the arm 1, ℓ_2 is the half of the length of the arm 2, g is the gravitational acceleration, the parameter $\rho = r_y$, K_a, T_a are constant parameters of the actuator (a motor here) and $z := r_x \frac{4}{3}\ell_2\phi_2$ is the change of variable used to formulate the model as a LPV system.

According to Kajiwara et al. [1999], the obtained control law leads to encouraging results (the paper is from 1999, the beginning of the LPV trend) for the LPV formulation. The LPV approach has led in this application to an enhancement of the stability and performances.

Automotive Suspension System[1]

Another application of LPV control is the performance adaptation that can be performed. Indeed, parameters can be introduced in weighting functions in Loop Shaping strategies in order to modify in real time the bandwidth, the weight on the control law.

For instance, in [Poussot-Vassal, 2008], control of semi-active suspensions is addressed in view of performing a global chassis control. Since semi-active suspensions, in which the damper coefficient is controlled, can only absorb energy but not supply it, the control input is constrained to belong to a specific set depending on the deflection speed which is the derivative of the difference the sprung mass (z_s) and the unsprung mass z_{us}, i.e. $\dot{z}_s - \dot{z}_{us}$. Figures 1.1 and 1.2 represent different kind of suspension systems with associated characteristics. Ideally, the force produced by the suspension must be positive (negative) if the deflection speed is positive (negative).

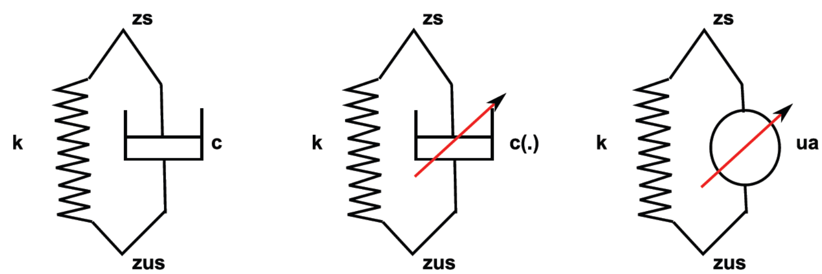

Figure 1.1: Different types of suspensions, from left to right: passive, semi-active and active suspensions

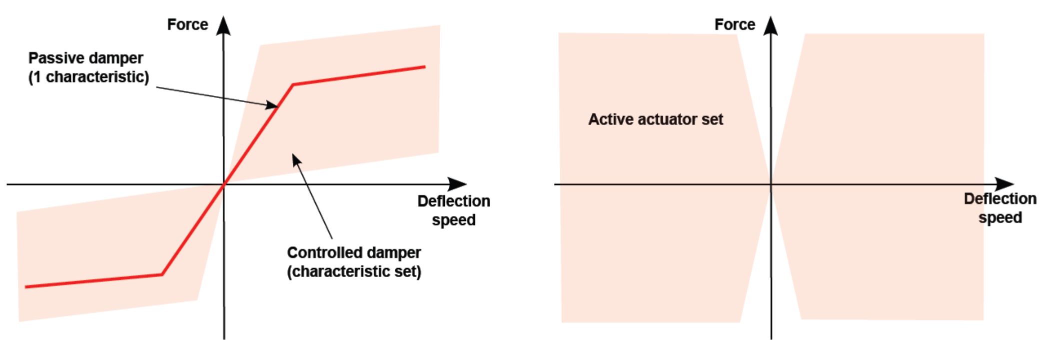

Figure 1.2: Characteristics of passive, semi-active (left) and active (right) suspensions

Since in the $\mathcal{H}_\infty$ control framework such constraint cannot be explicitly specified then the

[1]Thanks to Charles Poussot-Vassal for providing material on this topic

idea is to use a parameter dependent weighting functions on the control input of the form

$$W_u(s,\rho) = \rho(u-v)\frac{1}{s/1000+1}$$

where u is the computed force and v is the achievable force which satisfies the quadrant constraint. The parameter ρ is chosen to satisfy the following relation

$$\rho(\varepsilon) = 10\frac{\mu\varepsilon^4}{\mu\varepsilon^4+1/\mu})$$

for sufficiently large $\mu > 0$, e.g. 10^8. In this case, the parameter belongs to $[0, 10]$ and has the form depicted on Figure 1.3 and the bode diagram of the inverse of the weighting function is plotted on Figure 1.4 where it is shown that if ρ is high (i.e. the computed force is far from the achievable force) the gain applied by the inverse of the filter on the control input is very low. This has the effect of having a control input which is close to 0, value which is always achievable.

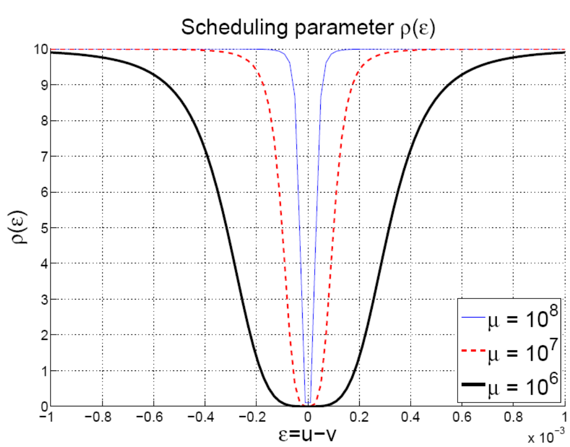

Figure 1.3: Graph of the parameter ρ with respect to $u-v$

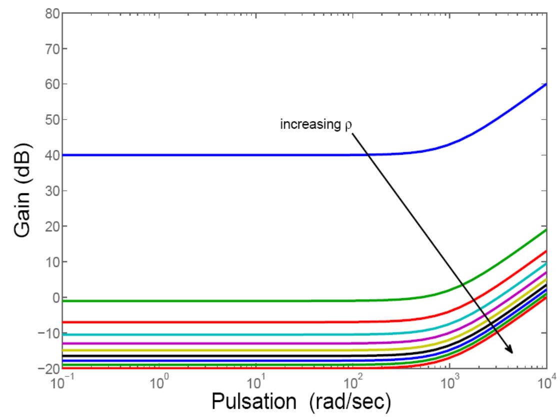

Figure 1.4: Bode diagram of $1/W_u(s,\rho)$ for different values of ρ

This example shows the interest of parameter varying systems and parameter varying control; many other applications of such technique may be developed, for instance let us mention parameter varying bandwidth of the closed-loop system, parameter dependent disturbance rejection where the parameter would correspond to the pulsation of the disturbance, and so on...

A wide range of applications

We give here a non-exhaustive list of application of LPV modeling and control in the literature. In Wei and del Re [2007], the modeling and control of the air path system of diesel engines in view of reducing polluting gas is addressed. This paper shows that a LPV formulation leads to interesting results in terms of simplicity of implementation and system performances. The control of elements in diesel engines is considered in Gauthier et al. [2005, 2007a,b], He and Yang [2006], Jung and Glover [2006] where the air flow, the fuel injection and/or the power unit are controlled. In Gilbert et al. [2007], Reberga et al. [2005], LPV modeling and synthesis are applied to turbofan engines. Electromagnetic actuators are piloted in Forrai et al. [2007] while a robotic application is presented in Kwiatkowski and Werner [2005]. In Liu et al. [2006a,b], LPV controller is applied to power system regulator. In Lim and How [1999], Tan and Grigoriadis [2000], White et al. [2007] applied in the synthesis of missile autopilot. In Lu et al. [2006], the control of the performances of the attitude of an F-16 Aircraft in response of the pilot orders for different angles of attack is addressed. LPV vehicle suspensions modeling and control is presented in Gaspard et al. [2004], Poussot-Vassal et al. [2006, 2008a,b] while global chassis control (attitude control) is handled in Gáspár et al. [2007], Poussot-Vassal et al. [2008c]. Finally, the control of nonuniform sampled-data systems is treated in a LPV fashion in Robert et al. [2006]. This list shows the efficiency and wide applicability of LPV control on theoretical and practical applications and motivates further studies on this topic. It will be shown later in this thesis that LPV methods can be used to control, in a novel fashion, time-delay systems with time-varying delays (see also [Briat et al., 2007a, 2008b]).

1.1 Classification of parameters

The behavior of LPV systems highly depends on the behavior of the parameters. Indeed, the global system is defined over a continuum of systems induced by a continuum of parameters. If the parameters take discrete values (the set of values is finite) or are piecewise constant continuous, the system would have a specific behavior and, in general, a specific denomination is given for these particular kinds of systems over these peculiar parameter trajectories; this will be deeper detailed further. This motivates the needs for classifying parameters in order to differentiate every behavior and therefore, any system that may arise. Two proper viewpoints can be adopted: either a mathematical one, centering the analysis on mathematical properties of the parameters trajectories such as continuity and differentiability; or a physical point of view, focusing on the physical meaning of parameters such as measurability and computability. Such a classification aims at discussing on the validity and the meaning of LPV modeling in order to apply control strategies. It is important to note that while the first classification is important for a theoretical point of view the second is crucial to apply LPV analysis and design methods on real physical systems.

1.1.1 Physical Classification

In general, the parameters can be sorted in three types, depending on their content type and their origin.

1.1.1.1 Parameters as functions of states

The parameters may be defined as functions of states, and such cases arise when LPV models are used to approximate nonlinear models; for instance the nonlinear system

$$\dot{x}(t) = -x(t)^3 \tag{1.4}$$

can be approximated by the LPV system

$$\dot{x}(t) = -\rho(t)^2 x(t) \tag{1.5}$$

where $\rho(t) := x(t)$.

LPV system is which states appear in the parameters expressions are called *Quasi-LPV systems*; see [He and Yang, 2006, Jung and Glover, 2006, Liberzon et al., 1999, Shin, 2002, Tan and Grigoriadis, 2000, Wei and del Re, 2007, White et al., 2007] for some applications of quasi-LPV systems.

The main difficulty of quasi-LPV comes from the fact that theoretically, the states are unbounded, while by definition, the parameters are. If, by chance, the functions mapping the states to the parameter values are bounded for every state values, the problem would not occur (but this assumption is too strong to be of interest). On the contrary, if the functions are unbounded, then a supplementary condition should be added in order to satisfy the boundedness property of the parameters values. Fortunately, in practice, the states are generally bounded and such problem only occurs in theoretical considerations.

It is worth noting that generally, several LPV systems correspond to a nonlinear system and finding the 'best' LPV model remains a challenging open problem [Bruzelius et al., 2004, Mehendale and Grigoriadis, 2004, Shin, 2002]. Indeed, in the latter example, $\rho(t) = x(t)^2$ would have be chosen. But the latter example is a simple one since the origin (i.e. $x = 0$) is globally asymptotically stable attractive point and hence any parametrization would give an asymptotically stable LPV system. On the contrary, let us consider the Van-der-Pol equation (with reverse vector field) considered in [Bruzelius et al., 2004]:

$$\begin{array}{rcl} \dot{x}_1(t) & = & -x_2(t) \\ \dot{x}_2(t) & = & x_1(t) - a(1 - x_1(t)^2)x_2(t) \end{array} \tag{1.6}$$

with $a > 0$. It is well-known that a limit cycle exists for such systems and for systems with reverse vector field, each trajectory starting inside the limit-cycle converges to 0 while each trajectory starting outside diverges. In [Bruzelius et al., 2004], it is shown that a 'good' LPV approximation, giving the exact stability region (i.e. interior of the limit cycle), is difficult to obtain.

1.1.1.2 Internal Parameters

The parameters may be used to model time-varying parts involved in the system expression (assuming that time-varying terms are bounded), in view of simplifying the stability analysis and/or using them in advanced control laws. For instance, the LTV system:

$$\dot{x}(t) = (a(t) + b\sin(t))x(t) \quad , \ a(t) \text{ bounded over time} \tag{1.7}$$

can be represented by

$$\dot{x}(t) = (\rho_1(t) + b\rho_2(t))x(t) \tag{1.8}$$

where $\rho_1(t) := a(t)$ and $\rho_2(t) := \sin(t)$.

The term 'internal parameters' means that the information used to compute the parameter values is part of the system dynamical model and elapsed time. This is put in contrast with the last class of parameters exposed hereunder.

1.1.1.3 External parameters

External parameters are involved in control and observation design problems only. Such 'virtual' parameters can be added in the design (for instance in frequency weighting functions in $\mathcal{H}_\infty$ control/observation) in order to modify, in real-time, the behavior of the closed-loop system. These external signals may stem from a monitoring system and model state of emergency or anything else, in view of modifying the behavior of the system, such as the system bandwidth, gains...

Let us consider the SISO LTI system

$$\dot{x}(t) = x(t) + u(t) \tag{1.9}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are respectively the system state and the control input. It is proposed to determine a control law such the system has a variable and controlled bandwidth, hence the following control law is suggested:

$$u(t) = -(1 - \rho(t))x(t) + \rho(t)r(t), \ \rho(t) > 0$$

where r is the reference to be tracked.

The interconnection yields:

$$\dot{x}(t) = \rho(t)(r(t) - x(t)), \ \rho(t) > 0$$

From this latter expression, the external parameter $\rho(t)$ controls the bandwidth of the closed-loop system and tries to maintain the tracking error to 0. In this scenario, a monitoring system including heuristics would be able to manage the parameter value with respect to higher-level data.

1.1.2 Mathematical Classification

On the other hand, the mathematical ordering aims at sorting the parameters behavior while considering mathematical properties of the trajectories. Consequently, these properties will be taken into account in stability tests in order to provide less conservative results than by ignoring these characteristics.

1.1.2.1 Discrete vs. Continuous Valued Parameters

The first idea is to isolate the parameters with respect to the type of values (or more precisely the type of the image set of the mapping) that they take. Indeed, parameters must be viewed as functions of time $t \in \mathbb{R}_+$:

$$\rho : \mathbb{R}_+ \to \rho(\mathbb{R}_+) \tag{1.10}$$

where $\rho(\mathbb{R}_+)$ is the image set of $\mathbb{R}_+$ by the vector valued function $\rho(\cdot)$.

Recall that the image set of the parameters is always bounded, then one can easily imagine that the image set is continuous or discrete, for instance

$$\rho : t \to \sin(t) \tag{1.11}$$

maps $t \in \mathbb{T}$ into $[-1, 1]$ continuously while

$$\rho : t \to [\sin(t)]_r \tag{1.12}$$

where $[\alpha]_r$ is the rounding of α, maps $\mathbb{T}$ into $\{-1, 0, 1\}$.

The main difference between these image sets is that, while the first one contains an infinite number of values, the second contains only three. Discrete valued image sets are more simple to consider since one has to verify the stability at a finite number of points only. Systems for which parameters take discrete values are called *Switched Systems* or *Systems with jump parameters* (See Blanchini et al. [2007], Cheng et al. [2006], Colaneri et al. [2008], Daafouz et al. [2002], Ghaoui and Rami [1997], Hespanha and Morse [1999], Liberzon et al. [1999], Verriest [2005], Xie et al. [2002], Xu and Antsaklis [2002] and references therein for more details on switched systems). It is clear, from the definition of discrete valued image sets, that the parameters trajectories are discontinuous (more precisely they are piecewise constant continuous) while for parameters with continuous image sets, continuity properties are not imposed. This brings us to the idea of considering continuity as a second criterium of classification of the parameters.

1.1.2.2 Continuous vs. Discontinuous Parameters

Values of the parameters with continuous image set may evolve within the image set, in two different ways: either in a continuous or a discontinuous fashion.

Definition 1.1.1 *A continuous function f, defined over $\mathbb{R}_+$ such that*

$$f : \mathbb{R}_+ \to U$$

satisfies the following well-known statement:

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ |t - t_0| \leq \eta \Rightarrow |f(t) - f(t_0)| \leq \varepsilon,\ \forall\ t_0 \in \mathbb{R}_+ \tag{1.13}$$

It is worth noting that there exists a large difference between switched systems (systems with discrete valued parameters) and systems with continuously valued discontinuous parameters, e.g.

$$\alpha(t) = \sum_{i=0}^{+\infty} a_n(\Gamma(t - t_n) - \Gamma(t - t_{n+1}))$$

where $a_n \in [a^-, a^+] \subset \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_n < \ldots < t_{n+1}$ and Γ is the step function.

Indeed, the latter ones cannot be reduced to a finite number of systems and results in a greater complexity than switched systems; this will be detailed further. The advantage of continuous parameters is their potential differentiability and will be the last criterium to classify parameters from a mathematical viewpoint.

1.1.2.3 Differentiable vs. Non-Differentiable Parameters

The final criterium is the first order differentiability of the parameters of some continuously valued continuous parameters. By considering bounds on the parameter derivatives, it is then possible to characterize the time-varying nature of the parameters in terms of speed of variation.

Definition 1.1.2 *A continuous differentiable function f, defined over $\mathbb{R}_+$ such that*

$$f : \mathbb{R}_+ \to U$$

satisfies the well-known statement

$$\exists f' : \forall t_0 \in \mathbb{R}_+ : \lim_{\delta t \to \{0^-, 0^+\}} \frac{f(t_0 + \delta t) - f(t_0)}{t - t_0} = f'(t_0) \tag{1.14}$$

Note that in the classical definition of the derivative, the limit from each side of 0 must coincide. This is clear that discontinuous functions do not satisfy such a condition and hence have unbounded derivative at discontinuity. Therefore, no global bounds can be given for discontinuous parameters. Moreover, from the differentiability property above, the parameter $\rho(t)$ defined by

$$\rho : t \to |\sin(t)|,\ t \geq 0 \tag{1.15}$$

does not admit a derivative at points $t_i = k\pi,\ k \in \mathbb{N}$. Indeed, the derivative value take the value -1 and 1 respectively by computing the limit from the left and the right, and hence no function f' exists. This is a consequence to the fact that the absolute value function is not differentiable at 0.

The non-existence of the function f' is apparently annoying since the global differentiability property is lost because of a finite number of isolated points only. Fortunately, since bounds on the parameters derivatives are necessary only, it is possible to show that this obtrusive troublesome particularity does not introduce any supplementary difficulty.

In these cases (continuous functions with non-differentiable points), it is possible to affect two bounded values of the derivative at each point for which the function is non-differentiable, assuming of course, that the function is continuous at these points. For continuous parameters, these two values of the extended derivative are always bounded and it is possible, by extension, to consider that the 'derivative' takes simultaneously all values in an bounded interval (in the preceding example, the interval is $[-1, 1]$). For discontinuous functions, the fact that their derivative values are unbounded is retrieved since the 'derivative' takes all values of $\mathbb{R}$. This 'exotic' version is not without reminding us of the definition of the subgradient in nonsmooth analysis [Clarke, 1983], defined presently in less formal fashion. Since we are only interesting in bounds of the derivative, this definition is sufficient to provide them. This gives rise to the following propositions.

Proposition 1.1.3 *For a smooth function $f : \mathbb{R}_+ \to U$, U compact of $\mathbb{R}$, the bounds on the derivative is defined by an interval $[a, b]$ where $a = \min_{t\in\mathbb{R}_+} f'(t)$ and $b = \max_{t\in\mathbb{R}_+} f'(t)$.*

Proposition 1.1.4 *For a continuous nonsmooth (Lipschitz) function we have*

$$a = \min\{a_1, a_2\} \text{ and } b = \max\{b_1, b_2\}$$

where

$$\begin{array}{lllllll} a_1 & = & \min_{t\in\mathbb{T}-\{t_i\}_i} f'(t) & \quad & b_1 & = & \max_{t\in\mathbb{T}-\{t_i\}_i} f'(t) \\ a_2 & = & \min\{\min\{U_1\}, \ldots, \min\{U_N\}\} & \quad & b_2 & = & \max\{\max\{U_1\}, \ldots, \max\{U_N\}\} \end{array} \tag{1.16}$$

where $\{t_i\}$ is the set of points where f is nonsmooth and U_i the set interval corresponding to all values of the 'derivative' at t_i.

As a simple example, the derivative of the parameter defined by $\rho(t) = |\sin(t)|$ is bounded by -1 and 1.

It is important to give an extra discussion on quasi-LPV systems. It is clear that, generally, the functions involving the states of the system are continuously differentiable with respect to them. Then, since the states are also differentiable, it is possible to tackle bounds on the parameter derivatives. Note that these bounds would certainly depend on the bounds on the derivatives of the states, but bounding derivative of the states in a problematic task. Let us consider, for instance, that in the synthesis we fix $\dot{x} \in [a, b]$, where x is the state of the system. Then the controller is computed and the closed-loop system exhibits state derivatives going out of the bounds a and b; for instance the state derivative belongs to $[a - 1, b + 1]$. This incoherent situation invalidates the synthesis and it cannot be proved exactly that the system is stable for state derivative in $[a-1, b+1]$. Hence the synthesis should be made again with an enlargement of the bounds of the state derivative bounds, e.g. $[a - 2, b + 2]$. On the other hand, by expanding to much the derivative bounds (or even considering infinite values), this may result in a too high conservatism in the approach culminating in bad performances of the closed-loop system. This is one of the main difficulty while dealing with quasi-LPV systems which does not occur in any other types of parameters (i.e. internal and external). Another problem arising in LPV system is the desynchronization between parameters and system state in the stability analysis; this will be detailed later in the stability analysis of LPV systems.

1.2 Representation of LPV Systems

The aim of this section is to present different frameworks used to represent LPV systems with their respective tools for stability analysis.

1.2.1 Several Types of systems...

Amongst the large variety of LPV systems, it is possible to isolate three main types of LPV systems based on the dependence on the parameters:

1. Affine and multi-affine systems
2. Polynomial Systems
3. Rational systems

It is worth noting that every LPV systems can be brought back to one of these latter types by mean of a suitable change of variable (e.g. $\rho_1' \leftarrow e^{\rho_1}$).

1.2.1.1 Affine and Multi-Affine Systems

Affine and multi-affine systems are the most simple LPV systems that can be encountered. Their general expression is given by

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + F(\rho)w(t) \end{array} \tag{1.17}$$

where

$$\left[\begin{array}{c|c} A(\rho) & E(\rho) \\ \hline C(\rho) & F(\rho) \end{array}\right] = \left[\begin{array}{c|c} A_0 & E_0 \\ \hline C_0 & F_0 \end{array}\right] + \sum_{i=1}^{N} \left[\begin{array}{c|c} A_i & E_i \\ \hline C_i & F_i \end{array}\right] \rho_i \tag{1.18}$$

Due to the affine dependence, stability of such systems can be determined with a low degree of conservatism (in some cases there is no conservatism). This will be detailed further in Section 1.3.2.

1.2.1.2 Polynomial Systems

Polynomial systems are the generalization of affine systems to a polynomial dependence with respect to the parameters. Their general expression is given below:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + F(\rho)w(t) \end{array} \tag{1.19}$$

where

$$\left[\begin{array}{c|c} A(\rho) & E(\rho) \\ \hline C(\rho) & F(\rho) \end{array}\right] = \left[\begin{array}{c|c} A_0 & E_0 \\ \hline C_0 & F_0 \end{array}\right] + \sum_{i=1}^{N} \left[\begin{array}{c|c} A_i & E_i \\ \hline C_i & F_i \end{array}\right] \rho^{\alpha_i} \tag{1.20}$$

where $\alpha_i = \left[\begin{array}{ccc} \alpha_i^1 & \dots & \alpha_i^N \end{array}\right]$ and $\rho^{\alpha_i} = \rho_1^{\alpha_i^1} \rho_2^{\alpha_i^2} \dots \rho_N^{\alpha_i^N}$.

Such systems are slightly more complicated to analyze, but recently, several approaches brought interesting solutions to stability analysis and control synthesis for this kind of systems. This will be detailed in Section 1.3.3.

1.2.1.3 Rational Systems

Rational systems are the most complicated LPV systems that can be imagined and their general expression is given hereunder

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + F(\rho)w(t) \end{array} \quad (1.21)$$

where $A(\rho)$, $E(\rho)$, $C(\rho)$ and $F(\rho)$ are rationally polynomially parameter dependent matrices.

Such systems have the advantage to be able to model the largest set of systems and multi-affine/polynomial systems are special case of this kind. Due to the rational dependence, they are also the most complicated type of LPV systems to deal with. This will be detailed further.

1.2.2 But essentially three global frameworks

Even if a LPV system can be classified in several families depending on how the parameters enter the system, only three global techniques are commonly used (at this time) to deal with LPV systems.

1.2.2.1 Polytopic Formulation

Polytopic systems are really spread in robust analysis and robust control. They have been studied in many papers (see for instance Apkarian and Tuan [1998], Borges and Peres [2006], Geromel and Colaneri [2006], Jungers et al. [2007], Oliveira et al. [2007], Peaucelle et al. [2000] for recent results on polytopic systems and references therein).

A time-varying polytopic system is a system governed by the following expressions

$$\begin{array}{rcl} \dot{x}(t) & = & A(\lambda(t))x(t) + E(\lambda(t))w(t) \\ z(t) & = & C(\lambda(t))x(t) + F(\lambda(t))w(t) \end{array} \quad (1.22)$$

where

$$\begin{bmatrix} A(\lambda) & E(\lambda) \\ C(\lambda) & F(\lambda) \end{bmatrix} = \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} A_i & E_i \\ C_i & F_i \end{bmatrix} \quad (1.23)$$

and $\sum_{i=1}^{N} \lambda_i(t) = 1$, $\lambda_i(t) \geq 0$.

The term polytopic comes from the fact that the vector $\lambda(t)$ evolves over the unit simplex (which is a polytope) defined by

$$\Gamma := \left\{ \underset{i}{\mathrm{col}}(\lambda_i(t)) : \sum_{i=1}^{N} \lambda_i(t) = 1,\ \lambda_i(t) \geq 0 \right\} \quad (1.24)$$

This set is depicted on Figure 1.5 for values $N = 2$ and $N = 3$. For $N = 2$, the set takes the form of a segment on a line; for $N = 3$, the set is a closed surface on a plane; and so on.

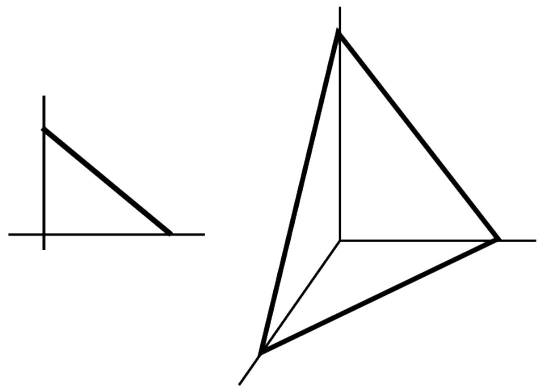

Figure 1.5: Set Γ for $N = 2$ and $N = 3$

The polytope Γ can be defined uniquely from the set of vertices:

$$\mathcal{V} = \bigcup_{i=1}^{N} \mathcal{V}_i \tag{1.25}$$

where

$$\mathcal{V}_i = \text{col}(\underbrace{0, \ldots, 0}_{i-1 \text{ terms}}, \underbrace{1}_{i^{th} \text{ term}}, \underbrace{0, \ldots, 0}_{N-i \text{ terms}}) \tag{1.26}$$

Indeed, in this case, the convex hull of $\mathcal{V}$, denoted hull[$\mathcal{V}$] coincides with Γ. Recall that the convex hull is the convex envelope of $\mathcal{V}$ and is the smallest convex set containing $\mathcal{V}$. The notion of convex hull is illustrated on Figure 1.6.

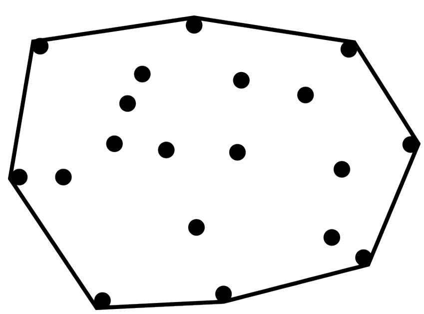

Figure 1.6: Convex hull of a set of points on the plane

Polytopic systems enjoy a nice property coming from the fact that a polytope is a convex polyhedral and, as we will see later, the stability of a polytopic system can be characterized in terms of stability of 'vertex' systems.

It is important to note that each parameter dependent system of any type can be coarsely turned into a polytopic system formulation. Affine and multi-affine systems can be equivalently represented as polytopic systems, this is illustrated in the following example:

Example 1.2.1 *Let us consider the LPV system with two parameters ρ_1, ρ_2:*

$$\dot{x}(t) = (A_1\rho_1(t) + A_2\rho_2(t))x(t) \tag{1.27}$$

with $\rho_i(t) \in [\rho_i^-, \rho_i^+]$ for $i = 1, 2$. The corresponding equivalent polytopic system is then given by

$$\dot{x} = [A_1[(\lambda_1 + \lambda_3)\rho_1^- + (\lambda_2 + \lambda_4)\rho_1^+] + A_2[(\lambda_1 + \lambda_2)\rho_2^- + (\lambda_3 + \lambda_4)\rho_2^+]]x \tag{1.28}$$

with $\lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) = 1$, $\lambda_i(t) \geq 0$.

It is clear that the polytopic model is not interesting in this case since it involves 4 time-varying parameters instead of 2 for the original system. This is an obvious fact in multi-affine systems. However, the transformation of the above multi-affine system into a polytopic formulation allows to provide somewhat nonconservative stability conditions (depending on the notion of stability which is considered); this will be detailed in Section 1.3.

On the other hand, the transformation of other LPV systems which are not multi-affine may be interesting but remains conservative as demonstrated in the following example.

Example 1.2.2 *Let us consider the polynomially parameter dependent system*

$$\dot{x}(t) = (A_0 + A_1\rho + A_2\rho^2)x(t) \tag{1.29}$$

where $\rho \in [\rho^-, \rho^+]$. It can be converted into the polytopic system

$$\dot{x}(t) = [A_0 + A_1 f_1(\lambda(t)) + A_2 f_2(\lambda(t))]x(t) \tag{1.30}$$

with

$$\begin{aligned} f_1(\lambda(t)) &= (\lambda_1(t) + \lambda_3(t))\rho^- + (\lambda_2(t) + \lambda_4(t))\rho^+ \\ f_2(\lambda(t)) &= \lambda_3(t)(\rho^-)^2 + \lambda_4(t)(\rho^+)^2 \end{aligned}$$

Indeed, we have considered

$$\begin{pmatrix} \rho \\ \rho^2 \end{pmatrix} = \lambda_1 \begin{pmatrix} \rho^- \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} \rho^+ \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} \rho^- \\ (\rho^-)^2 \end{pmatrix} + \lambda_4 \begin{pmatrix} \rho^+ \\ (\rho^+)^2 \end{pmatrix} \tag{1.31}$$

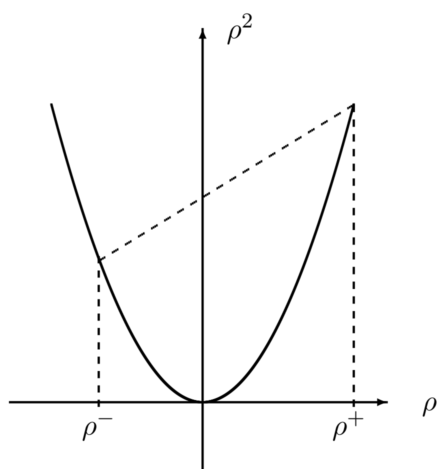

Figure 1.7: Comparison between exact set of values (the parabola) and the approximate set (the interior of the trapezoid)

To see that systems (1.29) and (1.30) are not equivalent it suffices to show that the polytopic parametrization can generate aberrant parameter values. This is easily visualized on Figure 1.7, aberrant values lie inside the trapezoid but not on the parabola. Then dealing with the polytopic systems would results in conservative stability conditions. The drawback of polytopic system as approximants comes from the fact that they decorrelate parameters and functions of them. Indeed, in the previous example, the dependence between ρ and ρ^2 is lost in the parametrization (1.31); only extremal points are correlated.

In order to reduce this conservatism, it is interesting to reduce the size of the polytope. This can be done by adding new vertices in order to shape the non-convex dependence between parameters. For the curve $f(x) = x^2$ it is possible to add new points below the curve to approximate the curve by tangent straight lines as seen on Figure 1.9. Nevertheless, it is not possible to approximate (asymptotically) exactly the parameter set (ρ, ρ^2). Indeed, since the domain has to remain convex, the surface above the curve $f(x) = x^2$ (the epigraph) must be convex too, and thus cannot be reduced more.

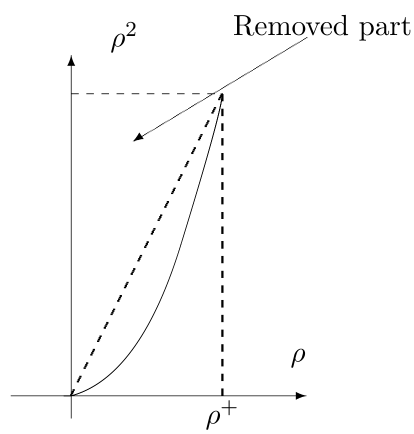

Figure 1.8: Illustration of Polytope Reduction using epigraph reduction

With the assumptions that $\rho^- = 0$ and $\rho^+ > 0$, the polytope can be reduced by removing a part of the epigraph. The surface above the line joining the points $(0, 0)$ and $(\rho^+, (\rho^+)^2)$ can be removed. In this case the new domain is a triangle instead of a rectangle, as depicted in Figure 1.8.

1.2.2.2 Parameter Dependent Formulation

This formulation is the most direct one, the system is considered in his primal form. The stability analysis or control synthesis are performed directly with specific tools. This formulation is better suited for polynomially parameter dependent systems but can be used with any type of LPV systems:

$$\dot{x}(t) \quad = \quad A(\rho)x(t) \tag{1.32}$$

where

$$A(\rho) = A_0 + \sum_i A_i \rho^{\alpha_i}$$

with $\alpha_i = \begin{bmatrix} \alpha_i^1 & \ldots & \alpha_i^N \end{bmatrix}$ and $\rho^{\alpha_i} = \rho_1^{\alpha_i^1} \rho_2^{\alpha_i^2} \ldots \rho_N^{\alpha_i^N}$.

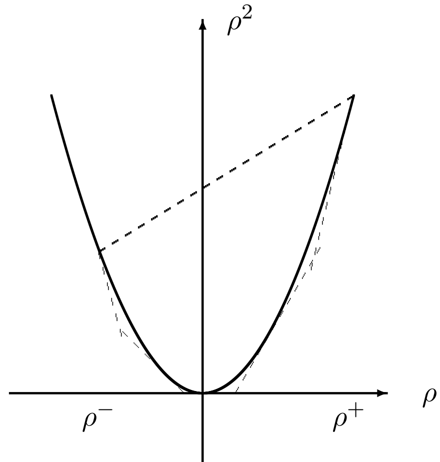

Figure 1.9: Illustration of Polytope Reduction by straight lines

It is obvious that the multi-affine case is a special case of that more general formulation. Moreover, even if the definition is given for systems with polynomial dependence on parameters only, it also applies to systems with rational dependence on parameters. However, a more more suitable formulation for such systems is given in the next section.

1.2.2.3 'LFT' Formulation

The last formulation for LPV systems is called, with a slight abuse of language, 'LFT' systems. Indeed, the term 'LFT' means 'Linear Fractional Transformation' and is the transformation used to convert a LPV/uncertain system into a Linear Fractional Form (LFR). The interest of this formulation for LPV systems have been emphasized in [Packard, 1994] and has given rise to many papers, let us mention for instance [Apkarian and Adams, 1998, Apkarian and Gahinet, 1995, Scherer, 2001]. The major interest of such a formulation is to embed a large variety of systems in a single class, englobing in a unified way systems with polynomial and rational dependence on parameters.

The key idea of this representation is to split the system in two parts, the parameter-varying and the constant part to analyze them separately. It is worth noting that the idea of separating the system in two connected independent parts is not new. It actually brings us back to the 50's when the nonlinearities on the actuators were dealt with such a representation and lead to Lu're systems. In robust stability analysis, such a transformation is extensively used as shown in [Scherer and Wieland, 2005, Zhou et al., 1996].

As an introductive example, let us consider the LPV system

$$\dot{x}(t) = A(\rho)x(t) \tag{1.33}$$

which is rewritten into an interconnection of two systems

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \\ w(t) & = & \Theta(\rho)z(t) \end{array} \tag{1.34}$$

as depicted in figure 1.10. Note that the matrices of the lower system $(\tilde{A}, B, C, D)$ are constant while the parameter varying part is located in the upper system.

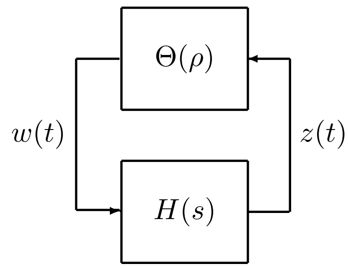

Figure 1.10: System (1.33) written in a 'LFT' form corresponding to description (1.34) where $H(s) = C(sI - \tilde{A})^{-1}B + D$

Example 1.2.3 *Let us consider the LPV system*

$$\dot{x} = \left(\frac{\rho_2}{\rho_1^2 + 1} - 3\right) x \tag{1.35}$$

It is possible to rewrite it in a 'LFT' form as shown below

$$\begin{bmatrix} \dot{x} \\ \hline z_0 \\ z_1 \\ z_2 \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A} & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x \\ \hline w_0 \\ w_1 \\ w_2 \end{bmatrix} \tag{1.36}$$

with

$$\left[\begin{array}{c|c} \tilde{A} & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|ccc} -3 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] \tag{1.37}$$

and

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \rho_2 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \tag{1.38}$$

Once the LPV system is split in two parts, the stability of the system or any other property can be determined using theorems applying on system interconnections; this will be detailed in Section 1.3. Such an idea has been extensively developed in the literature to deal with systems with nonlinearity leading to analysis of Lu're Systems with sector/norm-bounded nonlinearities.

The matrix $\Theta(\rho)$ is assumed, generally, to gather diagonally all the parameters involved in the LPV model (as illustrated in Example 1.2.3):

$$\Theta(\rho) = \mathrm{diag}(I_{n_1} \otimes \rho_1, \ldots, I_{n_p} \otimes \rho_p) \tag{1.39}$$

where n_i is the number of occurrences of parameter ρ_i in $\Theta(\rho)$. Each parameter is repeated enough times as needed to turn system (1.33) into system (1.34). A complete discussion on the construction of the interconnection is given in [Scherer and Wieland, 2005, Zhou et al., 1996].

It is generally assumed, for simplicity, that $\Theta(\rho)^T\Theta(t) \leq I$ (or equivalently $||\Theta(\rho)||_{\mathcal{L}_\infty} \leq 1$) meaning that the parameters ρ belong to the hypercube $[-1,1]^p$ where p is the number of parameters. It is worth noting that, by a simple change of variable, every real parameter can be modified to belong to the interval $[-1,1]$.

To emphasize the correspondence between both systems, we will turn the LFT formulation into a 'one-block' formulation.

From (1.34), we have

$$\begin{array}{rcl} w(t) & = & \Theta(\rho)z(t) \\ & = & \Theta(\rho)(Cx(t)+Dw(t)) \end{array} \tag{1.40}$$

and then

$$(I-\Theta(\rho)D)w(t) = \Theta(\rho)Cx(t) \tag{1.41}$$

If the problem is well-posed (i.e. the matrix $I-\Theta(\rho)D$ is nonsingular for all $\rho \in [-1,1]^p$) then we get

$$w(t) = (I-\Theta(\rho)D)^{-1}\Theta(\rho)Cx(t) \tag{1.42}$$

and finally

$$\dot{x}(t) = (\tilde{A}+B(I-\Theta(\rho)D)^{-1}\Theta(\rho)C)x(t) \tag{1.43}$$

showing that we have

$$\begin{array}{rcl} A(\rho) & = & \tilde{A}+B(I-\Theta(\rho)D)^{-1}\Theta(\rho)C \\ & = & \tilde{A}+B\Theta(\rho)(I-D\Theta(\rho))^{-1}C \end{array} \tag{1.44}$$

Example 1.2.4 *We will show here the equivalence between system (1.35) and (1.36)-(1.38). Applying formula $A(\rho) = \tilde{A}+B(I-\Theta(\rho)D)^{-1}\Theta(\rho)C$ yields*

$$\begin{array}{rcl} A(\rho) & = & -3+\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\rho_1\\ 0 & \rho_1 & 1\end{bmatrix}^{-1}\begin{bmatrix}\rho_2\\0\\0\end{bmatrix} \\ & = & -3+\dfrac{1}{1+\rho_1^2}\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \rho_1\\ 0 & -\rho_1 & 1\end{bmatrix}\begin{bmatrix}\rho_2\\0\\0\end{bmatrix} \\ & = & -3+\dfrac{\rho_2}{\rho_1^2+1} \end{array}$$

1.3 Stability of LPV Systems

Three frameworks have been introduced in the latter section which cover the wide variety of LPV systems: affine, polynomial and rational systems. These past years, specific tools have been developed to deal with stability analysis of systems belonging to each class and have lead to interesting results. The aim of the current section is to present these tools and their most important associated results, but first of all, some preliminary results on stability of LPV systems are necessary.

1.3.1 Notions of stability for LPV systems

It is convenient, for the reader ease, to introduce several notions of stability of LPV systems. Since LPV system are defined over a (smooth) continuum of systems, hence the stability may take several forms at the difference of LTI systems. For more details on stability of dynamical systems, the reader should read Appendix B.4. This section is devoted to show the complexity of the stability analysis of LPV systems and introduces ad-hoc notions of stability for this type of systems.

Before giving specific definitions of stability for LPV systems, it is convenient to introduce two fundamental definitions of stability for uncertain systems. These definitions are also of interest in the framework of LPV systems. In modern system and control theory, the stability of a dynamical system is determined by mean of a Lyapunov function and which has given rise to Lyapunov's theory [Lyapunov, 1992]. The key idea behind this theory is that if it is possible to find a nonnegative function, measuring the energy contained into the system, which is decreasing over time, then the system is said to be stable. This is explained more in details in Appendices B.4 and B.5.

The notions of stability and Lyapunov function are illustrated in the following

Example 1.3.1 *Let us consider an asymptotically stable LTI system (i.e. A has all its eigenvalues with negative real part)*

$$\dot{x}(t) = Ax(t),\ x(0) = x_0$$

From the assumption on A, it is clear that the system is stable (even exponentially stable) since the explicit solution of the latter system is

$$x(t) = e^{At}x_0$$

and converges to 0 as t grows to $+\infty$.

A Lyapunov function for such system is given by

$$V(x(t)) = x(t)^T Px(t),\ P = P^T \succ 0 \tag{1.45}$$

It is clear that the function is positive except at $x = 0$ where it is 0. Computing the time derivative of V along the trajectories solution of the system yields

$$\begin{aligned} \dot{V} &= \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) \\ &= x(t)^T \left(A^T P + PA\right) x(t) \end{aligned}$$

Since the derivative needs to be negative definite for every $x \neq 0$, then we must have

$$A^T P + PA \prec 0, \quad P = P^T \succ 0$$

Finally, if one can find $P = P^T \succ 0$ such that $A^T P + PA \prec 0$ then the system is asymptotically stable. An explicit solution to such an inequality is provided in Appendix D.3. For instance, if $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0$ then we have

$$A^T P + PA = \begin{bmatrix} 2(p_2 - p_1) & -2p_2 + p_3 \\ \star & -2p_3 \end{bmatrix} \prec 0$$

This is equivalent to satisfying the following nonlinear system of matrix inequalities (a symmetric matrix is positive definite if and only if all its principal minors are positive):

$$\begin{aligned} p_1 &> 0 \\ p_3 &> 0 \\ p_1 p_3 - p_2^2 &> 0 \\ p_1 - p_2 &> 0 \\ 4p_3(p_2 - p_1) - (p_3 - 2p_2)^2 &> 0 \end{aligned}$$

A suitable choice is given by

$$\begin{aligned} p_1 &= 3 \\ p_2 &= 2 \\ p_3 &= 2 \end{aligned}$$

In the framework of uncertain systems, the matrix A depends on uncertain terms δ and is then denoted by $A(\delta)$. These uncertain terms may be either constant or time-varying. Let us focus now, for simplicity, on constant uncertain parameters taking values in a compact set $\mathbf{\Delta} \subset \mathbb{R}^n$ and the uncertain system

$$\dot{x}(t) = A(\delta)x(t), \quad x(0) = x_0 \tag{1.46}$$

where x and x_0 are respectively the state and the initial condition.

Remark 1.3.2 *We also assume that $x = 0$ is an equilibrium point for all $\delta \in \mathbf{\Delta}$. This assumption is fundamental to apply Lyapunov theory and is responsible of many errors in published paper on stability of nonlinear uncertain systems. When the equilibrium point is nonzero and depends on the value of the uncertain parameters, the following change of variable*

$$\tilde{x}(t) = x(t) - x_e(\delta)$$

transfers the equilibrium point to 0. It is worth noting that this remark does not hold for linear systems which always have an equilibrium point at the origin [Vidyasagar, 1993].

It has been shown that the stability of the system can be determined in two different ways. Each one of them leads to a specific stability notion: the quadratic and robust stability.

Definition 1.3.3 *System (1.46) is said to be quadratically stable if there exists a Lyapunov function $V_q(x(t)) = x(t)^T P x(t) > 0$ for every $x \neq 0$ and $V(0) = 0$ such that*

$$\dot{V}_q(t, \delta) = x(t)^T (A(\delta)^T P + PA(\delta))x(t) < 0$$

for every $x \neq 0$ and $\dot{V}_q(0, \delta) = 0$ for all $\delta \in \mathbf{\Delta}$.

Definition 1.3.4 *System (1.46) is said to be robustly stable if there exists a parameter dependent Lyapunov function $V_r(x(t), \delta) = x(t)^T P(\delta) x(t) > 0$ for every $x \neq 0$ and $V(0) = 0$ such that*

$$\dot{V}_r(t, \delta) = x(t)^T (A(\delta)^T P(\delta) + P(\delta)A(\delta))x(t) < 0$$

for every $x \neq 0$ and $\dot{V}_r(0, \delta) = 0$ for all $\delta \in \mathbf{\Delta}$.

Since the Lyapunov function used to determine robust stability depends on the uncertain constant parameters, it is clear that the robust stability implies quadratic stability. The converse does not hold necessary, indeed it may be possible to find uncertain systems which are robustly stable but not quadratically. The following example illustrates this claim.

Example 1.3.5 *Let us consider the uncertain system with constant uncertainty* $\delta \in [-1, -1/2] \cup [1/2, 1]$*:*

$$\dot{x} = A(\delta, \tau)x \tag{1.47}$$

where $A(\delta, \tau) = \begin{bmatrix} 1 & \delta \\ -(\tau+2)/\delta & -(\tau-1) \end{bmatrix}$ *where* $\tau > 0$ *is a known system parameter.*

The characteristic polynomial of the system is given by $s^2+\tau s+1$ *and shows that the eigenvalues of the system do not depend on the uncertain parameter* δ*. Moreover, the eigenvalues have strictly negative real since* $\tau > 0$ *and proves that the system is robustly asymptotically stable for constant uncertainty* δ*. We aim at showing now that the system is not quadratically stable using reductio ad absurdum. Assume that the system is quadratically stable then there exists a matrix* $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ 0$ *such that the LMI*

$$\begin{array}{rcl} L_q(\delta) & := & A(\delta)^T P + PA(\delta) \prec 0 \\ & = & \begin{bmatrix} 2p_1 - 2\dfrac{\tau+2}{\delta}p_2 & p_2 - \dfrac{\tau+2}{\delta}p_3 + p_1\delta - p_2(\tau+1) \\ \star & 2\delta p_2 - 2p_3(\tau+1) \end{bmatrix} \prec 0 \end{array} \tag{1.48}$$

holds for all $\delta \in [-1, -1/2] \cup [1/2, 1]$*. Since the latter LMI is satisfied for every admissible value of* δ *then we have* $L_q(-\delta_0) \prec 0$ *and* $L_q(\delta_0) \prec 0$ *for every given* $\delta_0 \in [-1, -1/2] \cup [1/2, 1]$*. This implies that the inequality given by their sum* $L_q(-\delta_0)+L_q(\delta_0) \prec 0$ *also holds. Computing the sum explicitly yields*

$$\begin{array}{rcl} L_q(-\delta_0) + L_q(\delta_0) & = & [A(-\delta_0) + A(\delta_0)]^T P + P[A(-\delta_0) + A(\delta_0)] \\ & = & \begin{bmatrix} 4p_1 & -\tau p_2 \\ -\tau p_2 & -4(\tau+1)p_3 \end{bmatrix} \end{array} \tag{1.49}$$

The sum is not negative definite since $p_1 > 0$ *by definition, yielding a contradiction, showing that the system (1.47) is not quadratically stable.*

Let us consider now a parameter dependent Lyapunov matrix

$$P(\delta) = P_0 + P_1\delta + P_2\delta^2 = \begin{bmatrix} p_1(\delta) & p_2(\delta) \\ p_2(\delta) & p_3(\delta) \end{bmatrix}$$

It is relatively tough to show analytically that such a matrix allows to prove robust stability of system (1.47). However, we will show that the contradiction does not occur with such a Lyapunov matrix $P(\delta)$*. Let* $L_r(\delta) = A(\delta)^T P(\delta) + P(\delta)A(\delta)$ *and compute*

$$\begin{array}{rcl} L_r(\delta_0) + L_r(-\delta_0) & = & \begin{bmatrix} 2(p_1(\delta_0) + p_1(-\delta_0)) & * \\ * & * \end{bmatrix} \\ & = & \begin{bmatrix} 4(p_1^2\delta_0^2 + p_1^0) & * \\ * & * \end{bmatrix} \end{array} \tag{1.50}$$

with $p_i(\delta) = p_i^2\delta^2 + p_i^1\delta + p_i^0$*. This LMI is feasible since the only constraint is* $p_1(\delta) > 0$ *for all* $\delta \in [-1, -1/2] \cup [1/2, 1]$ *which allows* $p_1^2\delta_0^2 + p_1^0$ *to take negative values.*

Numerical experiment with $\tau = 2$ shows that a suitable choice for $P(\delta)$ is given by

$$P(\delta) = \begin{bmatrix} 2.9218 & -0.0017 \\ -0.0017 & 0.0293 \end{bmatrix} + \begin{bmatrix} 0.0157 & 1.1383 \\ 1.1383 & 0.0005 \end{bmatrix} \delta + \begin{bmatrix} 0.0857 & 0.0087 \\ 0.0087 & 0.7601 \end{bmatrix} \delta^2 \qquad (1.51)$$

Proposition 1.3.6 *Quadratic stability implies robust stability and quadratic stability is a sufficient condition to stability.*

Proof: *The see that quadratic stability implies robust stability, it suffices to let $P_i = \bar{P}$ and quadratic stability is then a particular case of robust stability where all the matrices P_i are identical. Quadratic stability is a sufficient condition for stability since the Lyapunov function $V(x) = x^T Px$ is the most simple one that can be used to determine stability. Thus is stability is ensured for a simple Lyapunov function then it will also ensured using more complex ones.* □

If the uncertainties were time-varying, the quadratic stability would check the stability for unbounded parameter variation rates while the robust stability would consider bounded parameter variation rates. Indeed, the Lyapunov function derivative becomes in this case

$$\dot{V} = x(t)^T \left(A(\delta)^T P(\delta) + P(\delta)A(\delta) + \sum_{i=1}^{N} \dot{\delta}_i \frac{\partial P(\delta)}{\partial \delta_i} \right) x(t)$$

This illustrates the fact that even if an uncertain system with time-varying uncertainties is stable for each frozen uncertainty, the derivative of the Lyapunov function may not be negative definite for some values of δ and $\dot{\delta}$. This shows the importance of the rate of variation of the uncertainties in the stability of the system. This will be detailed a bit further in the discussion but is convenient to introduce here the following remak.

Remark 1.3.7 *In the case of uncertainties with infinite variation rates, robust stability cannot be defined. Indeed, suppose that robust stability is sought for such systems through a parameter dependent Lyapunov function of the form $V(x, \delta) = x^T P(\delta)x$. Due to the affine dependence of the Lyapunov function derivative $V(x, \delta, \dot{\delta})$ on the term $\dot{\delta}$ (with a slight abuse of language since δ is not differentiable at some points), it is possible to consider only extremal values (the bounds) of the polytope in which $\dot{\rho}$ evolves. Since the uncertainties have unbounded parameter variation rate, then the polytope is the whole space $\mathbb{R}^N$ including infinity. This implies that the term $\frac{\partial P(\delta)}{\partial \delta}\dot{\delta}$ may reach an infinite value, making the stability condition unfeasible. The only way to make the matrix inequality feasible again is to fix $\frac{\partial P(\delta)}{\partial \delta} = 0$ but this means that the Lyapunov function is independent of δ and thus reduces to $P(\delta) = P_0$. Finally, the Lyapunov function becomes a Lyapunov function for quadratic stability. This shows that only quadratic stability can be verified for systems with arbitrarily fast parameter variation rate by considering a Lyapunov function smoothly scheduled by parameters. On the other hand, it seems possible to define the following Lyapunov function*

$$V(x) = \max_i \{x(t)^T P_i x(t)\} > 0$$

with $P_i = P_i^T$ in order to improve results of quadratic stability for such systems and obtain results similar to robust stability.

In the framework of stability analysis of LPV systems, the exact trajectories of the parameters are unknown in advance, only the set of values (and sometimes other properties such as their rate of variation) is known. Hence, in stability analysis framework, LPV and uncertain systems appear to be equivalent and thus, it seems correct to apply tools of robust stability to LPV systems. On the other hand, due to the specific nature of LPV systems, more specific notions of stability should be considered. They are introduced in what follows.

Let us consider now the LPV system

$$\dot{x}(t) = A(\rho)x(t) \tag{1.52}$$

where $x \in \mathcal{X} \in \mathbb{X}$ and $\rho \in U_\rho$ are respectively the system state and the parameters. It is assumed here that the parameters have bounded derivatives. Since the stability of LPV systems depends on the parameters values, a specific terminology is introduced.

Definition 1.3.8 *System (1.52) is said to be locally parametrically exponentially stable at $\rho_0 \in U_\rho$ if and only if it is exponentially stable at frozen parameter value $\rho_0 \in U_\rho$ (i.e. all the eigenvalues of $A(\rho_0)$ have strictly negative real part for all $\rho_0 \in U_\rho$).*

A common misconception is to say that if a LPV system is locally parametrically exponentially stable at every frozen $\rho_0 \in U_\rho$, then it is uniformly globally parametrically exponentially stable. But this statement contains imprecisions. First of all, the term stability should be defined precisely as in the framework of uncertain systems: are we talking about robust or quadratic stability ? Secondly, the time-varying nature of the system is not considered while dealing with frozen systems only.

Definition 1.3.9 *System (1.52) is said to be quadratically globally parametrically exponentially stable at $\rho_0 \in U_\rho$ if and only if it is locally parametrically exponentially stable at every frozen parameter $\rho_0 \in U_\rho$ (i.e. all the eigenvalues of $A(\rho_0)$ have strictly negative real part for all $\rho_0 \in U_\rho$).*

In this case no information of the rate of variation of the parameters is considered and thus arbitrarily fast variations of the parameters are allowed (unbounded derivatives). This obviously results in a conservative stability condition while considering systems with bounded parameter variation rates. This is completed by the following definition.

Definition 1.3.10 *System (1.52) is said to be robustly globally parametrically exponentially stable at $\rho_0 \in U_\rho$ if and only if it is locally parametrically exponentially stable at every frozen parameter value $\rho_0 \in U_\rho$ (i.e. all the eigenvalues of $A(\rho_0)$ have strictly negative real part for all $\rho_0 \in U_\rho$) and for all $\dot{\rho} \in U_\nu \subset \mathbb{R}^n$.*

The robust stability considers rate of variations of the parameters and thus reduces the conservatism of the quadratic stability.

Remark 1.3.11 *Note that uniform stability (in the Lyapunov sense) is implied by uniform global parametric stability (see Appendix B.36 for different notions of stability of time-invariant dynamical systems). Moreover, the asymptotic stability and exponential stability coincides for linear dynamical systems.*

We have shown that if the LPV system is stable for every frozen parameters, the system could be globally parametrically stable. However, the rate of variation of parameters plays an important role here and we aim at showing now, that the global parametric stability holds provided that the rate of variation is sufficiently small. This is illustrated in the following example:

Example 1.3.12 *Let us consider LPV system*

$$\dot{x} = A(\rho)x \tag{1.53}$$

where

$$A(\rho) = \begin{bmatrix} 7 & 12 & \cos(\rho) & \sin(\rho) \\ 6 & 10 & -\sin(\rho) & \cos(\rho) \\ \tau(\gamma+7)\cos(\rho) - 6\tau\sin(\rho) & 12\tau\cos(\rho) - (\gamma+10)\sin(\rho) & -\tau & 0 \\ \tau(\gamma+7)\sin(\rho) + 6\tau\cos(\rho) & 12\tau\sin(\rho) + \tau(10+\gamma)\cos(\rho) & 0 & -\tau \end{bmatrix}$$

For $\tau \geq 17.1169$ and $\gamma > 0$, the matrix $A(\rho)$ has negative eigenvalues for all $\rho \in [-\pi, \pi]$. For similar reasons as for system in Example 1.3.5, the system is not quadratically stable (i.e. the sum of the right-upper block for $\rho = -\pi$ and $\rho = \pi$ is a zero matrix and the matrix $\begin{bmatrix} 7 & 12 \\ 6 & 10 \end{bmatrix}$ is unstable). If ρ is constant then the system is robustly stable, while if ρ is allowed to vary arbitrarily fast, the system is not asymptotically stable (since quadratic stability is equivalent to stability with unbounded parameter variation rate). From these considerations it seems that the parameter variation rate plays a role in the stability of the LPV system.

Let us consider a parameter dependent Lyapunov function of the form $V(x, \rho) = x^T P(\rho)x$ where $P(\rho) = P_0 + P_1\cos(\rho) + P_2\sin(\rho) + P_3\cos(\rho)^2 + P_4\sin(\rho)^2$ and for which the parameter ρ satisfies $\rho \in [-\pi, \pi]$, $|\dot{\rho}| \leq \nu$. A LMI test is performed in order to find the admissible bound ν with respect to τ such that the system is asymptotically stable. The results are depicted on Figure 1.11.

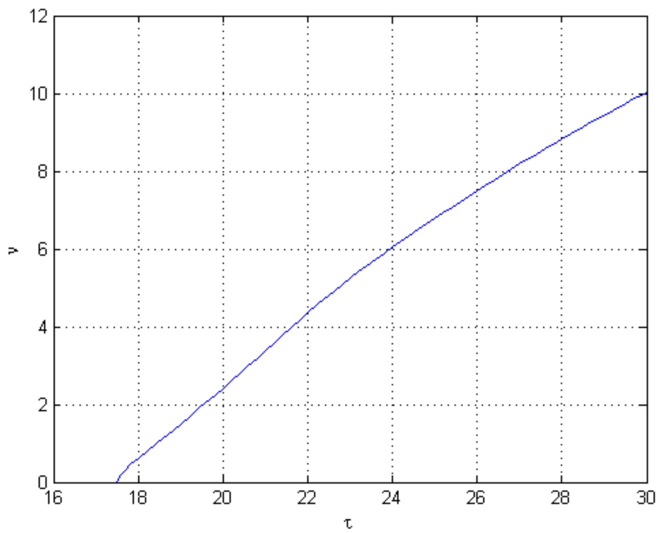

Figure 1.11: Evolution of the derivative bound ν with respect to τ that ensures stability

A best fit approach conjectures that $\nu \sim -0.0198\tau^2 + 1.7402\tau - 24.3626$. To interpret the latter result, first note that matrix $A(\rho)$ can be rewritten as:

$$A(\rho) = \begin{bmatrix} 7 & 12 & \cos(\rho) & \sin(\rho) \\ 6 & 10 & -\sin(\rho) & \cos(\rho) \\ 0 & 0 & -\tau & 0 \\ 0 & 0 & 0 & -\tau \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \tau & 0 \\ 0 & \tau \end{bmatrix} K(\rho)$$

with $K(\rho) = \begin{bmatrix} \tau(\gamma+7)\cos(\rho) - 6\tau\sin(\rho) & 12\tau\cos(\rho) - (\gamma+10)\sin(\rho) & 0 & 0 \\ \tau(\gamma+7)\sin(\rho) + 6\tau\cos(\rho) & 12\tau\sin(\rho) + \tau(10+\gamma)\cos(\rho) & 0 & 0 \end{bmatrix}$.

Above, the terms $K(\rho)$ and τ play respectively the role of a parameter-dependent state-feedback gain and the bandwidth of the actuators. By transposition of the preceding analysis to stabilization, it is clear that it is not possible to find $K(\rho)$ such that the closed-loop is quadratically stabilizable. On the other hand, it is possible to find $K(\rho)$ such that the system is asymptotically stable provided that ν is sufficiently small (robust stability). From Figure 1.11, we can see that the larger the bandwidth of the actuator τ is, the larger the allowed bound on parameter derivative ν is.

According to [Wu et al., 1996], the reason for which the system is not quadratically stable is the particular parameter trajectories that allow to the right-upper block to switch arbitrarily fast between values I and $-I$. So regardless of the bandwidth τ of the actuators, the rapidly varying parameter ρ do not allow for parameter-dependent quadratic stabilization.

The difference between stability for unbounded (quadratic stability) and bounded (robust stability) parameter variation rate has been emphasized in the preceding example. It is important to note that, for the moment, only values of the parameters and bounds on parameters derivative have been considered to study LPV systems stability. The remaining part extends the discussion when the system matrix $A(\rho)$ is unstable for some parameter values. We will show that under some (sometimes strong) assumptions, the LPV system may be globally parametrically asymptotically stable even in presence of local parametric instability. This is done by considering additional properties on trajectories of the parameters.

Definition 1.3.13 *If there exists a (possibly infinite) countable family of vectors ρ_i for which the system is parametrically locally unstable (i.e. at least one of the eigenvalues of $A(\rho_i)$ has zero real part), then system (1.52) is said to be exponentially stable almost everywhere.*

The latter definition has important consequences in the uniform global parametric stability. Indeed, the uniform global parametric stability is ensured if the parameter values belong to the set of $U_\rho^\circ := U_\rho - \bar{U}_\rho$ where $\bar{U}_\rho$ is a countable set of parameter vectors for which the system is unstable. It follows that the set U_ρ° is not convex anymore and thus results based on the convexity are not applicable (see for instance Section 1.2.2.1).

While considering these systems, it is clear that the exponential stability depends on the parameter trajectories (and not only their values). If the trajectories avoid parameters in $\bar{U}_\rho$ then the system would be uniformly exponentially parametrically stable over U_ρ°. On the other hand, if the trajectory stops on an unstable parameter values then the system becomes unstable. This is illustrates on Figure 1.12.

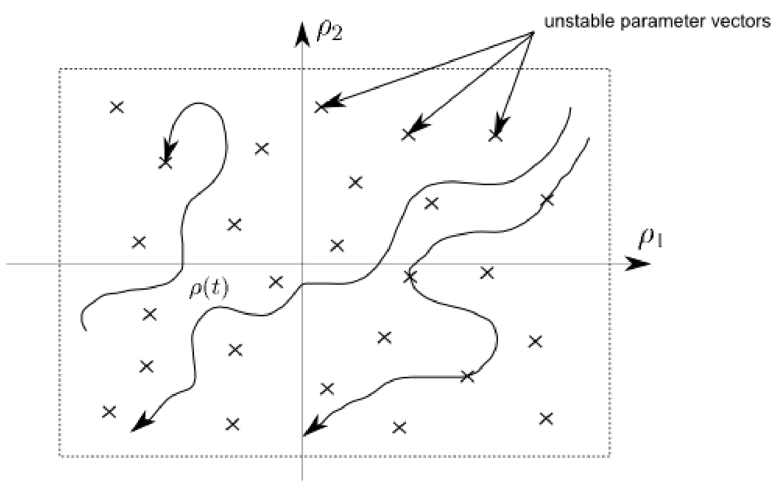

Figure 1.12: Example of trajectories for which the system is unstable (upper trajectory) and exponentially stable (lower trajectories) provided that the trajectories cross singular points sufficiently 'quick'

It is worth noting here that, if for a particular parameter vector ρ_0, the system is unstable (at least one of the eigenvalues of $A(\rho_0)$ has strictly positive real part) then the family contains an infinite, but not countable, number of parameter vectors for which the system is unstable (see Figure 1.13). This is a consequence of the smooth dependence of the system on the parameters; and the continuity of the eigenvalues of a parameter dependent matrix with respect to these parameters. A necessary condition to system stability would be that the parameter trajectories remain in stable regions but this contradicts the definition of the parameters which are assumed to evolve over the complete domain. Finally, we arrive at the conclusion that the stability of the system depends on the values of the parameters and on the behavior of the parameters. If the parameters just cross or avoid the singular (unstable) parameter values, then the system would be exponentially stable. But if one of the parameter remains at a singular value permanently, then the system would have a unstable behavior. This brought us to the following idea: if one can characterize the mean duration of the instability then it is possible to characterize global parametric stability of the system. This has been shown in [Hespanha and Morse, 1999] for switched system and generalized to LPV systems in [Hespanha et al., 2001, Mohammadpour and Grigoriadis, 2007b].

Let us denote now $\bar{U}_\rho$ the set (with nonzero measure) of parameter vectors for which the system is unstable. An example of such a set is depicted on Figure 1.13.

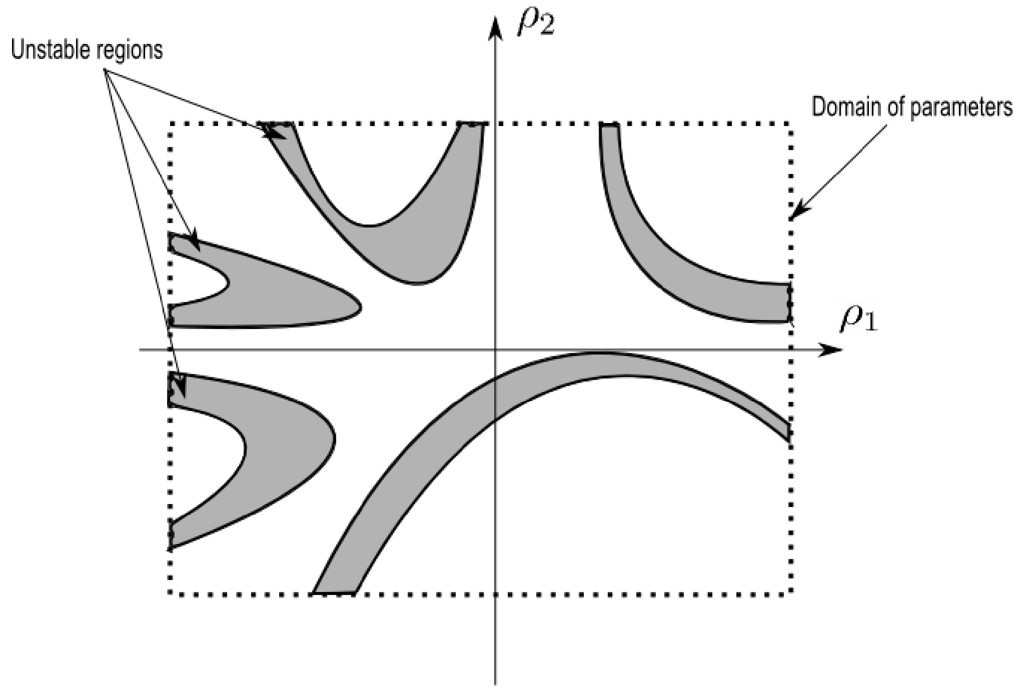

Figure 1.13: Example of stability map of a LPV system with two parameters; the grey regions are unstable regions

Introduce the characteristic measure $\delta(\alpha)$ of the set $\bar{U}_\rho$ such that

$$\delta(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \bar{U}_\rho \\ 0 & \text{if } \alpha \in U_\rho - \bar{U}_\rho \end{cases} \tag{1.54}$$

and the quantity

$$T_p(\tau, t) = \int_\tau^t \delta_u(\rho(t))dt \tag{1.55}$$

The quantity $T_p(\tau, t)$ measures the time spent by the system to be unstable over the interval $[\tau, t]$. This leads to the following definition (see [Hespanha et al., 2001, Hespanha and Morse, 1999]):

Definition 1.3.14 *It is said that system (1.52) has brief instabilities if*

$$T_p(\tau,t) \leq T_0 + \alpha(t-\tau),\ \forall\, t \geq \tau \geq 0 \tag{1.56}$$

for some $T_0 \geq 0$, $\alpha \in [0,1]$. In this case, the scalar T_0 is called the instability bound and α the exponential instability ratio.

The instability ratio α plays a central role in the stability analysis for LPV/swicthed systems with brief instabilities. It allows to give an upper bound on the measure, in an average fashion, of the time spent by the system being unstable.

This leads to the following result:

Theorem 1.3.15 *Let $\varpi > 0$ and $\beta > 0$ be decay rates of respectively the stable and unstable system:*

$$\begin{array}{lll} x(t)e^{\varpi t}x(0) \to 0 & \text{as} & t \to +\infty \text{ for each frozen } \rho \in U_\rho - \bar{U}_\rho \\ x(t)e^{-\beta t}x(0) \to 0 & \text{as} & t \to +\infty \text{ for each frozen } \rho \in \bar{U}_\rho \end{array} \tag{1.57}$$

In this case the system is said to be (ϖ,β)-uniformly globally parametrically exponentially stable if $\alpha < \alpha^* = \dfrac{\varpi}{\varpi+\beta}$ with decay rate $-\varpi + \alpha(\beta+\varpi)$.

Proof: *A sketch is given here and a more complete one can be found in [Hespanha et al., 2001, Hespanha and Morse, 1999] in the framework of switched systems. This has been extended to LPV systems in [Mohammadpour and Grigoriadis, 2007b] and the following proof is inspired by the latter paper.*

Let V be a Lyapunov function for such a system defined by $V(x(t)) = x(t)^T P e^{2\eta t} x(t)$ where $\eta = -\beta$ if $\rho \in \bar{U}_\rho$ and $\eta = \varpi$ if $\rho \in U_\rho - \bar{U}_\rho$.

Computing the derivative it is possible to show that under certain LMI conditions, we have

$$\dot{V} = \begin{cases} -2\varpi V & \text{if } \rho \in \bar{U}_\rho \\ 2\beta V & \text{if } \rho \in U_\rho - \bar{U}_\rho \end{cases} \tag{1.58}$$

Solving the linear differential inequality we get

$$V(x(t)) \leq e^{-2(t-\tau-T_p(\tau,t))\varpi + 2\beta T_p(\tau,t)} V(x(\tau)) \tag{1.59}$$

The latter inequality must be non increasing over $[\tau,t]$ hence the factor of the term $t-\tau$ in the argument of the exponential must be nonpositive and thus this leads to, for every $t > \tau \geq 0$:

$$\begin{array}{rlcl} & -2(t-\tau-T_p(\tau,t))\varpi + 2\beta T_p(\tau,t) & < & 0 \\ \Rightarrow & -\varpi(t-\tau) + (\beta+\varpi)T_p(\tau,t) & < & 0 \\ \Rightarrow & -\varpi(t-\tau) + (\beta+\varpi)(T_0 + \alpha(t-\tau)) & < & 0 \text{ using (1.56)} \end{array} \tag{1.60}$$

Note that the term T_0 does not factor the term $t-\tau$, hence it does not affect the decay rate of V over $[\tau,t]$. It only acts on an eventual overshoot of V. Finally the exponential convergence to 0 of V is guaranteed, under the assumption that (1.56) is satisfied, if

$$-\varpi + \alpha(\beta+\varpi) < 0 \tag{1.61}$$

which is equivalent to $\alpha < \dfrac{\varpi}{\beta+\varpi}$. This concludes the proof. □

This latter result, initially provided for switched systems and extended here for LPV systems, shows that while considering additional information on parameters behavior, a system which is locally parametrically unstable may be (ϖ, β)-uniformly exponentially stable.

After this brief presentation of different forms of stability of LPV systems, some results on their representation and associated tools are provided. If not stated otherwise, in the following, by 'stability' we tacitly means to $(\varpi, 0)$-uniform global parametric exponential stability. Moreover signals $x \in \mathcal{X},\ u \in \mathcal{U},\ w \in \mathcal{W},\ z \in \mathcal{Z}$ and $y \in \mathcal{Y}$ denote respectively the system state, the control input, the disturbances, the controlled/performances outputs and the measured outputs.

1.3.2 Stability of Polytopic Systems

This section is devoted to the stability analysis of LPV polytopic systems. Quadratic and Robust stability are discussed and compared in the polytopic systems framework. In what follows, the following polytopic LPV system is considered

$$\dot{x}(t) = \sum_{i=1}^{N} \lambda_i(t) A_i x(t), \qquad x(0) = x_0 \tag{1.62}$$

where x is the system state and $\lambda(t) \in \Gamma$ where

$$\Gamma : \left\{ \underset{i=1}{\overset{N}{\mathrm{col}}}(\lambda_i(t)) : \ \sum_{i=1}^{N} \lambda_i(t) = 1,\ \lambda_i(t) \geq 0 \right\} \tag{1.63}$$

A necessary and sufficient condition for robust stability is given below:

Proposition 1.3.16 *The LPV polytopic system (1.62) is quadratically stable if and only if there exists a matrix $P = P^T \succ 0$ such that*

$$A_i^T + PA_i \prec 0 \tag{1.64}$$

holds for all $i = 1, \ldots, N$.

Proof: *Define the Lyapunov function $V(x(t)) = x(t)^T P x(t)$ with $P = P^T \succ 0$. The time-derivative of the Lyapunov functions computed along trajectories of system (1.22) with $w \equiv 0$ leads to*

$$\dot{V}(x(t)) = x(t)^T (A(\lambda(t))^T P + PA(\lambda(t))) x(t)$$

The quadratic stability of the equilibrium point $x_{eq} = 0$ of system (1.22) is proved if $\dot{V}(x(t)) \prec 0$ for every $x \neq 0$. This yields the following parameter dependent LMI

$$\sum_{i=1}^{N} \lambda_i(t) \left(A_i^T P + PA_i \right) \prec 0 \tag{1.65}$$

for any $\lambda \in \Gamma$.

Sufficiency: Assume that $A_i^T P + PA_i \prec 0$ for all $i = 1, \ldots, N$. Then it is obvious that (1.65) holds.

Necessity: Since (1.65) must be true for every value of $\lambda(t) \in \Gamma$ then it must be true at every vertices $\mathcal{V}_i$ of the polytope and this implies

$$A_i^T P + PA_i \prec 0$$

for all $i = 1, \dots, N$. $\square$

An interesting fact of the previous result is the transformation of the parameter dependent LMI (1.65) into a set of N LMIs. In other words, a semi-infinite dimensional problem is reduced to a finite dimensional problem (sometimes huge) independent of the parameters vector $\lambda(t)$.

Example 1.3.17 *In Example 1.2.1, the multi-affine system*

$$\dot{x}(t) = (A_1\rho_1(t) + A_2\rho_2(t))x(t)$$

is turned in a polytopic formulation

$$\dot{x} = [A_1[(\lambda_1 + \lambda_3)\rho_1^- + (\lambda_2 + \lambda_4)\rho_1^+] + A_2[(\lambda_1 + \lambda_2)\rho_2^- + (\lambda_3 + \lambda_4)\rho_2^+]]x \tag{1.66}$$

with $\lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) = 1$, $\lambda_i(t) \geq 0$.

From this formulation, quadratic stability is ensured if and only if there exists a matrix $P = P^T \succ 0$ *such that the set of 4 LMIs is satisfied*

$$\begin{array}{rcl} (A_1\rho_1^- + A_2\rho_2^-)^T P + P(A_1\rho_1^- + A_2\rho_2^-) & \prec & 0 \\ (A_1\rho_1^+ + A_2\rho_2^-)^T P + P(A_1\rho_1^+ + A_2\rho_2^-) & \prec & 0 \\ (A_1\rho_1^- + A_2\rho_2^+)^T P + P(A_1\rho_1^- + A_2\rho_2^+) & \prec & 0 \\ (A_1\rho_1^+ + A_2\rho_2^+)^T P + P(A_1\rho_1^+ + A_2\rho_2^+) & \prec & 0 \end{array} \tag{1.67}$$

A necessary and sufficient condition to quadratic stability of the multi-affine system is ensured if the stability of the system at each vertex of the hypercube $[\rho_1^-, \rho_1^+] \times [\rho_2^-, \rho_2^+]$ *is ensured using an unique Lyapunov function. The exactness of the procedure is a consequence of the fact that an hypercube is also a convex polyhedral and every convex polyhedral can be parametrized over* Γ.

Example 1.3.18 *Let us consider again example 1.2.2 where a LPV system with quadratic dependence on a parameter is turned, in a nonequivalent polytopic description recalled below:*

$$\dot{x}(t) = [A_0 + A_1[(\lambda_1(t) + \lambda_3(t))\rho^- + (\lambda_2(t) + \lambda_4(t))\rho^+] + A_2[\lambda_3(t)(\rho^-)^2 + \lambda_4(t)(\rho^+)^2]]x(t)$$

with $\sum_{i=1}^4 \lambda_i(t) = 1, \quad \lambda_i(t) \geq 0$. *A sufficient condition to stability of such system is hence given by the feasibility of the set of 4 LMIs*

$$\begin{array}{lcl} (A_0 + A_1\rho^-)^T P + P(A_0 + A_1\rho^-) & \prec & 0 \\ (A_0 + A_1\rho^- + A_2(\rho^-)^2)^T P + P(A_0 + A_1\rho^- + A_2(\rho^-)^2) & \prec & 0 \\ (A_0 + A_1\rho^+)^T P + P(A_0 + A_1\rho^+) & \prec & 0 \\ (A_0 + A_1\rho^+ + A_2(\rho)^+)^T P + P(A_0 + A_1\rho^+ + A_2(\rho)^+) & \prec & 0 \end{array} \tag{1.68}$$

As suggested in the proof and illustrated in the examples above, a necessary and sufficient condition to quadratic stability (or sufficient condition to stability) of (1.65) is the stability of all A_i (A_i have eigenvalues with strictly negative real part for all $i = 1, \dots, N$). The main difficulty comes from the fact that, even if all the matrices A_i are Hurwitz, a matrix P satisfying the LMIs may not exist. The robust stability overcomes this problem.

Proposition 1.3.19 *The LPV polytopic system (1.62) is robustly stable if there exists matrices $P_i = P_i^T \succ 0$, a matrix X and a sufficiently large scalar $\sigma > 0$ such that*

$$\begin{bmatrix} -(X+X^T) & P_i + X^T A_i & X^T \\ \star & -\sigma P_i + \mathcal{P}\dot{\lambda}(t) & 0 \\ \star & \star & -P_i/\sigma \end{bmatrix} \prec 0 \tag{1.69}$$

holds for all $i = 1, \ldots, N$ and all $\dot{\lambda} \in \mathcal{S}$ where $\mathcal{P} := \dfrac{\partial P(\lambda)}{\partial \lambda} = \begin{bmatrix} P_1 & P_2 & \ldots & P_N \end{bmatrix}$.

For simplicity, the set $\mathcal{S}$ is not detailed here but represents the set in which evolves the derivative of $\lambda(t)$. More details are provided in Section 3.4.

Proof: *The proof is made in three steps, the first step is to provide a relevant parameter dependent Lyapunov function and differentiate it. The second part aims at showing the equivalence between two matrix inequalities in order to linearize the dependence on parameters. Finally, the last step turns a parameter dependent matrix inequality into a set of matrix inequalities independent of the parameters.*

Let us consider the parameter dependent Lyapunov function

$$V(x(t), \lambda(t)) = x(t)^T P(\lambda(t)) x(t)$$

where $P(\lambda(t)) = \sum_{i=1}^{N} \lambda_i(t) P_i$, $P_i = P_i^T \succ 0$. We also assume here that the parameter $\lambda(t)$ is differentiable and, in this case, the derivative of V along the trajectories solutions of system (1.62) is given by

$$\dot{V}(x(t), \lambda(t), \dot{\lambda}(t)) = x(t) \left(A(\lambda(t))^T P(\lambda(t)) + P(\lambda(t)) A(\lambda(t)) + \mathcal{P}\dot{\lambda}(t) \right) x(t) \tag{1.70}$$

where $A(\lambda(t)) = \sum_{i=1}^{N} \lambda_i(t) A_i$. Since $\dot{V}(\cdot,\cdot,\cdot)$ must be negative definite for all $x \in \mathbb{R}^n$, $\lambda \in \Gamma$ and $\dot{\lambda} \in \mathcal{S}$ we must have

$$A(\lambda(t))^T P(\lambda(t)) + P(\lambda(t)) A(\lambda(t)) + \mathcal{P}\dot{\lambda}(t) \prec 0 \tag{1.71}$$

The idea would be to use the same proof as for quadratic stability to provide a sufficient condition for robust stability. However, the arguments of the proof work if only if the dependence on the parameters is affine. Due to the product $A(\lambda(t))^T P(\lambda(t))$ in LMI (1.71) the dependence is not affine anymore but quadratic. The idea now is to turn LMI (1.71) into an equivalent formulation where these quadratic terms are removed.

Let us consider the following LMI where X is a constant full matrix of appropriate dimensions

$$\begin{bmatrix} -(X+X^T) & P(\lambda) + X^T A(\lambda) & \\ \star & -\sigma P(\lambda) + \mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix} \prec 0 \tag{1.72}$$

We aim now at showing that LMI (1.72) and (1.71) are equivalent. Note that (1.72) can be rewritten in the expanded form

$$\underbrace{\begin{bmatrix} 0 & P(\lambda) & 0 \\ \star & -\sigma P(\lambda) + \mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix}}_{\Psi} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X^T \begin{bmatrix} -I & A(\lambda) & I \end{bmatrix} + \begin{bmatrix} -I \\ A(\lambda)^T \\ I \end{bmatrix} X \begin{bmatrix} I & 0 & 0 \end{bmatrix} \prec 0 \tag{1.73}$$

Since the matrix X is unconstrained (free) then the projection lemma applies (see Appendix E.18). A basis of the null-space of $U_1 := \begin{bmatrix} -I & A(\lambda) & I \end{bmatrix}$ and $U_2 := \begin{bmatrix} I & 0 & 0 \end{bmatrix}$ are given respectively by

$$\mathrm{Ker}[U_1] = \begin{bmatrix} A(\lambda) & I \\ I & 0 \\ I & I \end{bmatrix} \qquad \mathrm{Ker}[U_2] = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$$

Finally the projection lemma yields the following two underlying LMIs

$$\begin{array}{rcl} \mathrm{Ker}[U_1]^T \Psi \mathrm{Ker}[U_1] & = & \mathrm{Ker}[U_1]^T \begin{bmatrix} 0 & P(\lambda) & 0 \\ \star & -\sigma P(\lambda) + \mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix} \mathrm{Ker}[U_1] \prec 0 \\ & = & \begin{bmatrix} A(\lambda)^T P(\lambda) + P(\lambda)A(\lambda) + \sigma P(\lambda) + \mathcal{P}\dot{\lambda} & P(\lambda) \\ \star & -P(\lambda)/\sigma \end{bmatrix} \prec 0 \\ \mathrm{Ker}[U_2]^T \Psi \mathrm{Ker}[U_2] & = & \mathrm{Ker}[U_2]^T \begin{bmatrix} 0 & P(\lambda) & 0 \\ \star & -\sigma P(\lambda) + \mathcal{P}\dot{\lambda} & 0 \\ \star & \star & -P(\lambda)/\sigma \end{bmatrix} \mathrm{Ker}[U_2] \prec 0 \\ & = & \begin{bmatrix} -\sigma P(\lambda) + \mathcal{P}\dot{\lambda} & 0 \\ \star & -P(\lambda)/\sigma \end{bmatrix} \prec 0 \end{array} \tag{1.74}$$

A Schur complement on the first LMI yields

$$A(\lambda)^T P(\lambda) + P(\lambda)A(\lambda) + \mathcal{P}\dot{\lambda} \prec 0$$

which is identical to (1.71). The second LMI is satisfied if and only if $-\sigma P(\lambda) + \mathcal{P}\dot{\lambda} \prec 0$ and this inequality is verified if σ is sufficiently large. This means that (1.72) and (1.71) are equivalent. The final part of the proof is the transformation of the parameter dependent matrix inequality (1.72) into a set of N matrix inequalities (1.69). This is done in the same way than for quadratic stability. □

It is worth noting here that condition (1.69) is not a LMI condition due to the unknown scalar term $\sigma > 0$. Nevertheless, if σ is fixed, the condition becomes a LMI. Moreover, σ is not completely unknown since it must be sufficiently large. In this case, it suffices to fix it to a very large value and then solve the LMIs.

Note also that in the case of constant λ, the term $\mathcal{P}\dot{\lambda}$ is 0 and hence a suitable and simple choice for σ is 1.

Remark 1.3.20 *The principle of the polytopic formulation is based on the fact that the system and stability conditions (here in a LMI form) have affine dependence on the parameters. If, for some reason, the affine dependence is lost the stability of the system is not equivalent (or even implied only) to the feasibility of the LMI at each vertex. In [Apkarian and Tuan, 1998, Jungers et al., 2007, Oliveira et al., 2007] some results are provided for which the dependence is lost and some sufficient conditions are expressed to relax the parameter dependent LMI conditions. This is also introduced in Section 3.2.*

In terms of computational complexity, let us consider that a system has p parameters, hence the number of LMIs to be solved simultaneously is then given by $\#(LMIs) = 2^p$. This can be very time and memory consuming for some applications.

1.3.3 Stability of Polynomially Parameter Dependent Systems

The most simple and intuitive description of polynomially parameter dependent systems or systems with polynomial of functions of parameters (e.g. $\cos(\rho), e^{\rho} \dots$) is to deal directly with a primal formulation:

$$\dot{x}(t) = A(\rho(t))x(t),\ x(0) = x_0,\ \rho \in U_\rho \subsetneq \mathbb{R}^N \tag{1.75}$$

as done in Section 1.3.1. In order to avoid repetition on stability of such systems, we will focus on how to express stability conditions and how solving them. The reader should refer to Section 1.3.1 to get preliminary results. We only recall here LMIs used to define quadratic (rate-independent) and robust (rate dependent) stability and then a discussion is provided on relaxation techniques.

Lemma 1.3.21 *System (1.75) is quadratically stable if and only if there exists a matrix $P = P^T \succ 0$ such that*

$$A(\rho)^T P + PA(\rho) \prec 0 \tag{1.76}$$

holds for all $\rho \in U_\rho$.

Proof: The proof is an application of the Lyapunov stability theory with $V(x) = x^T Px$ as a Lyapunov function. □

Lemma 1.3.22 *System (1.75) is robustly stable if and only if there exists a continuously differentiable matrix function $P(\rho) = P(\rho)^T \succ 0$ such that*

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) + \sum_{i=1}^{N} \nu_i \frac{\partial P(\rho)}{\partial \rho_i} \prec 0 \tag{1.77}$$

holds for all $\rho \in U_\rho$ and all $\nu = \mathrm{col}_{i=1}^N(\nu_i) \in U_\nu$ where U_ν is the set of vertices of the polytope in which the derivative of the parameters $\dot{\rho}$ evolves.

Proof: The proof is an application of the Lyapunov stability theory with $V(x, \rho) = x^T P(\rho)x$ as a Lyapunov function. After differentiation, the term $\dot{\rho}$ enters affinely in the LMI and hence a polytopic formulation is equivalent. Hence it suffices to considers the vertices of the polytope only to consider all the values of derivative of the parameters inside. □

The LMI for quadratic stability is technically called semi-infinite dimensional LMI due to the dependence on parameters. Indeed, a continuum of LMIs is parametrized by ρ. This means that it must be satisfied for all $\rho \in U_\rho$ and the verification of such a LMI constraint is a challenging problem due to the infinite number of values of ρ.

The LMIs for robust stability is technically called infinite dimensional semi-infinite LMI. The term 'infinite dimensional' comes from the fact that the unknown variable $P(\rho)$ to be determined is a function (and thus belong to an infinite dimensional space) and the term 'semi-infinite' comes from the fact that the LMI must be satisfied for all $(\rho, \dot{\rho}) \in U_\rho \times U_\nu$. Solving this LMI is also challenging to the matrix function $P(\rho)$.

The remaining of this section aims at showing different relaxations schemes allowing to turns these difficult LMI problems into more tractable LMI condition. Roughly speaking, primal LMIs are relaxed into a set of finite number of finite dimensional LMIs which is easier to solve with convex optimization tools. First of all, a method to relax the infinite-dimensional part into a finite dimensional problem is provided. It is based on a projection of a function on a particular basis of functions. Second, methods to relax the semi-infinite part of the LMIs are introduced. Some of these methods work for every parameter dependent LMIs independently of the type of LPV system (affine, polynomial or rational). However, these (more or less) recent results are rather complicated and remain technically difficult due to large theoretical background. Nevertheless, they will be explained in broad strokes with a sufficient number of references if precisions are wished. Three methods will be introduced: the relaxation by discretization (or commonly called 'gridding'), the 'Sum-of-Squares' approach and the global polynomial optimization. They will be illustrated through examples and a discussion on advantages and drawbacks will be provided.

1.3.3.1 Relaxation of matrix functions

The relaxation of the infinite dimensional part can be reduced to a finite dimensional problem by projecting the function on a finite basis of function; for instance let us consider a polynomial basis

$$f_{\alpha_i}(\rho) = \rho^{\alpha_i},\ i = 1, \ldots, N_b \tag{1.78}$$

and therefore the matrix $P(\rho)$ can be chosen as

$$P(\rho) = \sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho) \tag{1.79}$$

where the matrices $P_i = P_i^T$ have to be determined. Therefore the robust stability conditions becomes

Corollary 1.3.23 *System (1.75) is robustly stable if and only if there exist matrices $P_i = P_i^T$ such that such that LMIs*

$$\begin{array}{rcl} A(\rho)^T \left(\sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho)\right) + \left(\sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho)\right) A(\rho) + \sum_{i=1}^{N} \nu_i \left(\sum_{i=1}^{N_b} P_i \dfrac{\partial f_{\alpha_i}(\rho)}{\partial \rho_i}\right) & \prec & 0 \\ \sum_{i=1}^{N_b} P_i f_{\alpha_i}(\rho) & \succ & 0 \end{array}$$

holds for all $\rho \in U_\rho$ and all $\nu = \mathrm{col}_{i=1}^{N}(\nu_i) \in U_\nu$ where U_ν is the set of vertices of the polytope in which the derivative of the parameters $\dot{\rho}$ evolves.

We have explicitly turned an infinite dimensional problem into a finite dimensional problem where only N_b matrices are sought. The main difficulty of this relaxation stems from the difficulty of finding the 'good' type and number of basis functions. The central idea, generally admitted, is to mimic to behavior of the system and reproduce the same parameter dependence for $P(\rho)$ and even go a little bit further in the choice of the order (number of basis functions). An iterative procedure in the best technique but remains time consuming.

1.3.3.2 Relaxation of parametrized LMIs by discretization (gridding)

This LMI relaxation is applicable for any type of parametrized LMIs provided that it is well-defined for every value of the parameter in their admissible set. The discretization is the most intuitive and simple way to make the problem finite dimensional. It proposes to replace the initial semi-infinite problem into a discretized version involving a finite number of finite dimensional LMI. This is illustrated in the following example.

Example 1.3.24 *The following generic problem is considered. Let $L(M,\rho)$ be a real symmetric matrix in the unknown matrix variable $M \in \mathcal{M}$ where the parameter vector ρ belongs to some compact subset U_ρ of $\mathbb{R}^N$. The problem aimed to be solved is:*

$$\begin{array}{ll} \text{Solve} & L(M,\rho) \prec 0 \\ s.t. & M \in \mathcal{M} \\ & \text{for all } \rho \in U_\rho \end{array}$$

The gridding approach proposes to simplify the latter problem into a discretized version. Let $\bar{U}_\rho := \{\rho^1, \ldots, \rho^k\}$ be a set of distinct points belonging to U_ρ (i.e. $\rho^j \in U_\rho$ for all $j = 1, \ldots, k$. Hence the problem reduces to

$$\begin{array}{ll} \text{Solve} & L(M,\rho) \prec 0 \\ s.t. & M \in \mathcal{M} \\ & \text{for all } \rho \in \bar{U}_\rho \end{array}$$

This approach is based on the claim that, by discretizing the parameter space, there exists a density of the grid for which most of critical points are considered. By critical points, we mean, in the stability analysis, points for which the system is unstable. However, the density which has to be considered is unknown a priori and its determination remains a difficult problem. Indeed, if one wants to find a 'good' density, the location of unstable regions in the parameter domain is a crucial information. Unfortunately, this information is not accessible since the knowledge of unstable regions is equivalent to the knowledge of the (in)stability of the system which is actually sought. This paradox shows that probably no method to find a 'perfect' gridding would be developed someday.

Example 1.3.25 *For instance, let us consider the trivial LPV system*

$$\dot{x}(t) = (\rho^2 - 1)x(t) \tag{1.80}$$

and $\rho \in [-1, 1]$.

It is clear that the asymptotic stability is lost for $\rho = -1$ and $\rho = 1$ (we retrieve here an almost exponentially stable system). Hence if the discretization do not consider explicitly these two values, the system would be considered as asymptotically stable. It seems very difficult in this case to prove exactly (i.e. find a 'good' grid) that the system is asymptotically stable and moreover, this cannot be viewed in simulations since the parameters have to stay, for a long time, at critical values of the parameters to discern its instability.

Example 1.3.26 *In the case of systems which are locally parametrically unstable:*

$$\dot{x}(t) = (\rho^2 - 1 + \varepsilon)x(t) \qquad 0 \leq \varepsilon \leq 1 \tag{1.81}$$

with $\rho \in [-1, 1]$, there exist an infinite number of parameters for which the system is unstable: $\rho \in [-1, -\sqrt{1-\varepsilon}] \cup [\sqrt{1-\varepsilon}, 1]$.

The Lebesgue measure of the interval of values of ρ, for which the system is unstable, is $2(1 - \sqrt{1-\varepsilon})$ and taking a gridding of 5 equally spaced points suffices to prove instability of the system. The largest ε is, the easiest is the proof of instability (the measure of the interval grows up). On the contrary, the smallest ε is, the hardest is the proof of instability. When $\varepsilon > 0$, stability with average dwell-time should be considered rather than global parametric stability. When $\varepsilon \to 0$, the example tends to the first example.

A second drawback of the approach is the non-characterization of the behavior of the eigenvalues of the LMIs between gridding points and then it seems difficult to know when the grid is sufficiently thin.

It is worth noting that the discretization grid may be nonuniform over the whole parameter space. Indeed, in theory of interpolation, it has been shown, in many works, that an uniform discretization may be far from the best choice. For instance, in Lagrange polynomial interpolation, if the points are equally distant, the interpolated function oscillates above and below the real curve (Gibbs phenomenon) which can be a problem since, between points, the eigenvalues may change of sign. It has been shown that if the gridding points coincide with zero of some polynomial (Chebyshev polynomials are one of the most famous), the oscillations do not occur anymore. For interpolation with function and there derivatives, Hermite polynomials should be considered instead. Although these methods give ideas on the discretization scheme, they lead to complicated expression for unknown functions since the order of polynomials approximately equals to the number of discretization points. For more details about these topics, see for instance [Abramowitz and Stegun, 1972, Bartels et al., 1998, **?**].

In terms of computational complexity, let us consider by simplicity that the system has p parameters whose parameter space are discretized in $N+1$ points. This means that the total number of points is $(N+1)^p$. Hence, the number of LMI to be solved simultaneously is equal to the number of points, and thus we have $\#(LMIs) = (N+1)^p$. Generally, this number is quite large since the number of gridding points must be sufficiently large to be 'sure' to capture the behavior of the system.

1.3.3.3 Relaxation of Parametrized LMIs using methods based on Sum-of-Squares (SOS)

We show here, in a very simple way, what is the sum-of-squares relaxation; where does it come from and how to use it in the framework of parameter dependent LMIs. The interested reader should refer to [Gatermann and Parrilo, 2004, Helton, 2002, Parrilo, 2000, Prajna et al., 2004, Scherer and Hol, 2006] and references therein to get more details. This method applies only for polynomially parameter dependent LMIs (or possibly to some vary special cases of rationally parameter dependent LMIs).

The idea is to describe the set of parameter values by a set of polynomial inequalities. Then using an interesting variation of the $\mathcal{S}$-procedure (see Appendix E.9) constraints are injected in the LMIs. In such a method, the scalar variables introduced by the $\mathcal{S}$-procedure are no more constant but vary with respect to parameters, allowing a more thin relaxation. Finally, it is aimed to show that the latter LMI is sum-of-squares with respect to parameters. Indeed, if the LMI is sum-of-squares then it is positive definite. Moreover, testing if a polynomial is a sum-of-squares can be cast as a semidefinite programming problem (SDP problem), this is an important fact demonstrating the interest of such an approach.

Theorem 1.3.27 *Let $p(x)$ be a univariate polynomial of order N. $p(x)$ is nonnegative if and only if it is sum-of-squares, i.e. there exists N polynomials $h_i(x)$ such that $p(x) = \sum_{i=1}^{N} h_i(x)^2$. Moreover, the degree of $p(x)$ is even and the coefficient of the higher power is positive.*

Proof: *Necessity: The necessity is obvious. Suppose that $p(x)$ is SOS thus it writes $p(x) = \sum_i q_i(x)^2$ which is obviously nonnegative.*

Sufficiency: Since $p(x) = p_n x^n + \ldots + p_1 x + p_0 \geq 0$ is univariate then it can be factorized as

$$p(x) = p_n \prod_i (x - r_i)^{n_i} \prod_j (x - \alpha_k + j\beta_k)^{m_j} (x - \alpha_k - j\beta_k)^{m_j}$$

where r_i and $\alpha_k \pm j\beta_k$ are respectively all real and complex roots of $p(x)$ with respective order of multiplicity n_i and m_j. It is clear that a univariate polynomial is nonnegative if and only if $p_n > 0$ and the orders of multiplicity of real roots are even and let $n_i = 2n'_j$. Noting that

$$(x - \alpha_k + j\beta_k)(x - \alpha_k + j\beta_k) = (x - \alpha_k)^2 + \beta_k^2$$

then we have

$$p(x) = p_n \prod_i (x - r_i)^{2n'_i} \prod_j ((x - \alpha_k)^2 + \beta_k^2)^{m_j}$$

In virtue of the property that products of sums of squares are sums of squares (the set of SOS is closed under multiplication), and that all the expression above are SOS, it follows that $p(x)$ is SOS. $\square$

To illustrates the fact that the nonnegativity of a polynomial can be expressed as a SDP, let us consider a SOS nonnegative multivariate (n variables) polynomial $p(x)$ of degree $2d$. Then we have

$$\begin{aligned} p(x) &= \sum_i q_i(x)^2 \geq 0 \\ &= \sum_i z(x)^T L_i^T L_i z(x) \geq 0 \\ &= \sum_i z(x)^T Q_i z(x) \geq 0 \\ &= z(x)^T Q z(x) \geq 0 \end{aligned}$$

where $z(x)$ is a vector containing monomial of degree up to d whose number of components equals $\begin{pmatrix} n+d \\ d \end{pmatrix}$. Since $Q_i = Q_i^T \succeq 0$ then $Q = \sum_i Q_i \succeq 0$ and equivalently $Q = Q^T \succeq 0$. Moreover, the number of squares is equal to rank$[Q]$.

This can be easily transposed to the matrix case:

Theorem 1.3.28 *Let $P(x)$ be a matrix univariate polynomial of order N. $P(x)$ is nonnegative if and only if it is sum-of-squares, i.e. there exists N matrix polynomials $H_i(x)$ such that $P(x) = \sum_{i=1}^N H_i(x)^T H_i(x)$.*

In the univariate case, the positivity of the polynomial is equivalent to the existence of a SOS (sum-of-squares) decomposition. This is also true for quadratic polynomials and quartic polynomials in two variables. On the other hand, in the multivariate case, a positive definite polynomial is not necessarily SOS in general. Fortunately, the set of SOS multivariate polynomials is dense in the set of nonnegative polynomials and allows SOS approach to lead to good results and formulate equivalent tests to problems which are not SOS initially. The following example describes a nonnegative polynomial which is not SOS but whose nonnegativity can be expressed as a SOS decomposition problem through an equivalent test.

Example 1.3.29 *The Motzkin's polynomial*

$$m(x) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)$$

is globally nonnegative but cannot be written as a SOS. It is depicted on Figure 1.14 showing that it vanishes at $|x_1| = |x_2| = 1$

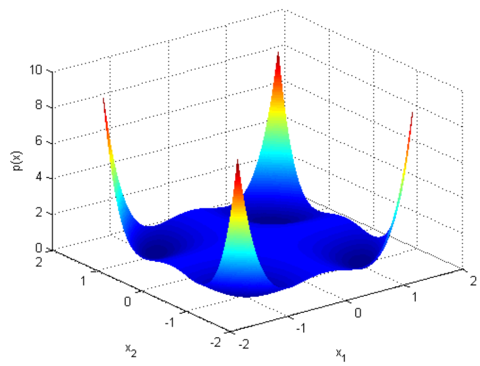

Figure 1.14: Motzkin's polynomial

To see that it is globally nonnegative, let us consider the triplet $(1, x_1^2 x_2^4, x_1^4 x_2^2)$ *and in virtue of the arithmetic-geometric mean inequality (i.e. the arithmetic mean is greater or equal to the geometric mean) then we have*

$$\begin{array}{rrcl} & \dfrac{1 + x_1^2 x_2^4 + x_1^4 x_2^2}{3} & \geq & \sqrt[3]{x_1^6 x_2^6} \\ \Rightarrow & 1 + x_1^2 x_2^4 + x_1^4 x_2^2 - 3x_1^2 x_2^2 & \geq & 0 \\ \Rightarrow & 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) & \geq & 0 \end{array} \tag{1.82}$$

It is relatively tough to show that the Motzkin's polynomial is not SOS. On the other hand, we will show that its nonnegativity can be cast as a SDP problem anyway by turning the nonnegativity analysis of $m(x)$ *into an equivalent problem involving another polynomial which is SOS. First multiply the Motzkin's polynomial* $m(x)$ *by the positive polynomial* $x_1^2 + x_2^2 + 1$ *and we get*

$$m'(x) = (x_1^2 + x_2^2 + 1) m(x) \tag{1.83}$$

It is clear that nonnegativity of $m'(x)$ *and* $m(x)$ *are equivalent. Hence by solving a SDP the following SOS decomposition of* $m'(x)$ *is obtained:*

$$\begin{array}{rcl} m'(x) & = & (x_1^2 + x_2^2 + 1)(1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)) \\ & = & (x_1^2 x_2 - x_2)^2 + (x_1 x_2^2 - x)^2 + (x_1^2 x_2^2 - 1)^2 + \dfrac{1}{4}(x_1 x_2^3 - x_1^3 x_2)^2 \\ & & + \dfrac{3}{4}(x_1 x_2^3 + x_1^3 x_2 - 2x_1 x_2)^2 \end{array} \tag{1.84}$$

The SOS approach is explained in the remaining of the section. Let $\mathcal{M}(\rho) \succ 0$ be a parameter dependent LMI to be satisfied for every value of ρ in a hyperrectangle $\mathcal{I}$ explicitly given by

$$\mathcal{I} := [\rho_1^-, \rho_1^+] \times \ldots \times [\rho_p^-, \rho_p^+]$$

This hyperrectangle can defined through a set of polynomial inequalities (a semi-algebraic set):

$$\mathcal{I} = \{\rho : \; g_i(\rho) \geq 0, \; i = 1, \ldots, p\}$$

The following example describes the construction of such polynomials.

Example 1.3.30 *For example, let* $(\rho_1, \rho_2) \in \mathcal{I}_2 := [-1, 1] \times [2, 3]$ *hence we have*

$$\begin{aligned} g_1(\rho_1) &= -\rho_1^2 + 1 \\ g_2(\rho_2) &= -\rho_2^2 + 5\rho_2 - 6 \end{aligned}$$

The expression of $\mathcal{I}$ through polynomial inequalities is not unique. In the example, above we have chosen to define one polynomial of degree 2 for each parameter. It would also be possible to define 4 polynomials of degree 1.

Supplementary constraints can be added in order to specify other relations between parameters. All these constraints can be combined into a more general semi-algebraic set, say $\mathcal{I}'$. Hence, by invoking the classical version of the $\mathcal{S}$-procedure we claim that

$$\mathcal{M}'(\rho) = \mathcal{M}(\rho) - \sum_{i=1}^{N} g_i(\rho) Z_i \succ 0 \tag{1.85}$$

where matrices $Z_i = Z_i^T \succ 0$ are sought.

The idea is to show that if $\mathcal{M}'(\rho)$ is a sum-of-squares (i.e. $\mathcal{M}'(\rho) \succ 0$) for all $\rho \in \mathcal{I}$ (or $\mathcal{I}'$) and in this case we should have

$$\begin{aligned} \mathcal{M}'(\rho) \succ 0 \quad &\Leftrightarrow \quad \mathcal{M}(\rho) - \sum_{i=1}^{N} g_i(\rho) Z_i \succ 0 \\ &\Rightarrow \quad \mathcal{M}(\rho) \succ \sum_{i=1}^{N} g_i(\rho) Z_i \succeq 0 \end{aligned}$$

The second step of the reasoning is based on the expression of the parameter dependent LMI in a quadratic form. Let $\mathcal{B}(\rho)$ be a basis of the multivariate matrix valued polynomial $\mathcal{M}'(\rho)$ such that we have

$$\mathcal{M}'(\rho) = \mathcal{B}(\rho)^T \mathcal{Q} \mathcal{B}(\rho) \tag{1.86}$$

where $\mathcal{Q}$ is a constant symmetric matrix. This is called the *spectral factorization*. Now by stating that $\mathcal{Q} \succ 0$ then this implies that $\mathcal{M}'(\rho)$ is sum-of-squares. Therefore, the goal is to find matrices $Z_i = Z_i^T \succ 0$ such that $\mathcal{Q} \succ 0$.

With this formulation, it may happen that no solutions is found even though the $\mathcal{M}(\rho) \succ 0$ for all $\rho \in \mathcal{I}$. So, the next idea is to replace the positive definite matrices Z_i by a matrix multivariate polynomials $Z_i(\rho)$ which are sum-of-squares. Changing constant to parameter dependent matrices adds flexibility (as when dealing with robust stability rather than quadratic stability) and allows to give less conservative stability conditions. Moreover, it has been shown that by taking greater and greater degrees for matrix polynomials $Z_i(\rho)$, the condition provides less and less conservative results until reach a nonconservative result.

Finally let

$$\mathcal{B}_2(\rho)^T \mathcal{Q}' \mathcal{B}_2(\rho)$$

be the spectral factorization of

$$\mathcal{M}(\rho) - \sum_{i=1}^{N} g_i(\rho) Z_i(\rho) \tag{1.87}$$

where $\mathcal{B}_2(\rho)$ is a quadratic basis for (1.87) and $\mathcal{Q}'$ is constant symmetric matrix.

It is also possible to add other degrees of freedom based on the kernel of quadratic forms, indeed there exist matrices $\mathcal{K}$ such that

$$\mathcal{B}_2(\rho)^T \mathcal{K} \mathcal{B}_2(\rho) = 0 \tag{1.88}$$

where $\mathcal{K}$ is constant symmetric matrix. This constraint allows to take into account relations between monomials in the basis $\mathcal{B}_2(\rho)$. The basis is not necessarily $\mathcal{B}(\rho)$ since with the adjunction of $Z(\rho)$ the degree of polynomial may raise.

Thus determining that

$$\mathcal{Q}' + \mathcal{K} \succ 0 \tag{1.89}$$

we have

$$\begin{aligned} \mathcal{Q}' + \mathcal{K} \succ 0 \quad &\Rightarrow \quad \mathcal{B}_2(\rho)^T (\mathcal{Q}' + \mathcal{K}) \mathcal{B}_2(\rho) \succ 0, \quad \text{for all } \rho \in \mathcal{I} \\ &\Leftrightarrow \quad \mathcal{B}_2(\rho)^T \mathcal{Q}' \mathcal{B}_2(\rho) \succ 0, \quad \text{for all } \rho \in \mathcal{I} \\ &\Leftrightarrow \quad \mathcal{M} - \sum_{i=1}^{N} g_i(\rho) Z_i(\rho) \succ 0, \quad \text{for all } \rho \in \mathcal{I} \\ &\Leftrightarrow \quad \mathcal{M} \succ \sum_{i=1}^{N} g_i(\rho) Z_i(\rho) \succ 0 \succeq 0, \quad \text{for all } \rho \in \mathcal{I} \end{aligned}$$

This method leads to interestingly good results and by growing up the degree of the matrix valued polynomials $Z_i(\rho)$, it asymptotically converges to a necessary and sufficient condition (non conservative condition). Fortunately, the nonconservative condition is generally attained for reasonable degree values.

To conclude on the computational complexity, on the one hand, the number of variables grows up very quickly while raising the degree of SOS polynomials. On the second hand, the size grows up quickly with respect to the order of polynomials involved in the problem formulation. See for instance [Dietz et al., 2006] for a brief analysis of the increase of the number of decision variables on a particular case. This is a common fact that good relaxations for parameter dependent LMIs lead to expensive test from a computational point of view.

The following example ends the part of SOS relaxation.

Example 1.3.31 *Let us consider the matrix*

$$\mathcal{M}(\rho) := \begin{bmatrix} -(\rho^2 - 4) & 1 \\ 1 & -(\rho^2 - 4) \end{bmatrix} \tag{1.90}$$

and $\rho \in [-1, 1]$. The goal is to prove, using SOS, that $\mathcal{M}(\rho) \succ 0$ for all $\rho \in [-1, 1]$. It is clear that $\mathcal{M}(\rho)$ is not globally positive definite (i.e. for all $\rho \in \mathbb{R}$). To see this, remember that for a univariate polynomial positive definiteness is equivalent to the existence of SOS *decomposition. Hence, if we show that $\mathcal{M}(\rho)$ is not SOS then it is not positive definite on $\mathbb{R}$. A spectral decomposition on $\mathcal{M}(\rho)$ yields*

$$\mathcal{M}(\rho) = \mathcal{B}(\rho)^T \left[\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \mathcal{B}(\rho) \tag{1.91}$$

where $\mathcal{B}(\rho) = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \rho & 0 \\ 0 & \rho \end{array}\right]$.

In the univariate case, we have the equivalence

$$\mathcal{M}(\rho) \succ 0 \text{ for all } \rho \in \mathbb{R} \Leftrightarrow \mathcal{Q} \succ 0$$

The latter matrix is not globally positive definite since the $(2,2)$ right-lower block is negative definite. Now define the set $\mathcal{I} := [-1, 1]$. A second definition of $\mathcal{I}$ is given by

$$\mathcal{I} = \left\{x \in \mathbb{R} : \ g(x) := -x^2 + 1 \geq 0\right\}$$

in terms of a polynomial inequality. Introduce

$$\mathcal{M}(\rho) - g(\rho)Z = \mathcal{B}(\rho)^T \mathcal{Q}\mathcal{B}(\rho)$$

where $\mathcal{Q} = \left[\begin{array}{cc|cc} 4-z_1 & 1-z_2 & 0 & 0 \\ 1-z_2 & 4-z_3 & 0 & 0 \\ \hline 0 & 0 & -1+z_1 & z_2 \\ 0 & 0 & z_2 & -1+z_3 \end{array}\right]$ *and* $Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \succ 0$. *We can see that the positive definite matrix Z appears positively in the right-lower block and could make it positive definite. Now we seek $Z = Z^T \succ 0$ such that $\mathcal{Q} \succ 0$. Hence, $\mathcal{Q}$ is positive definite if and only if*

$$\begin{aligned} \begin{bmatrix} 4-z_1 & 1-z_2 \\ 1-z_2 & 4-z_3 \end{bmatrix} &\succ 0 \\ \begin{bmatrix} -1+z_1 & z_2 \\ z_2 & -1+z_3 \end{bmatrix} &\succ 0 \end{aligned}$$

Note that the problem is affine in the variable Z and hence can be solved using SDP. From these inequalities we get

$$\begin{aligned} Z &\succ I \\ \begin{bmatrix} -1+z_1 & z_2 \\ z_2 & -1+z_3 \end{bmatrix} &\succ 0 \end{aligned}$$

Choosing $Z = 2I$ we obtain

$$\begin{bmatrix} 4-z_1 & 1-z_2 \\ 1-z_2 & 4-z_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The eigenvalues of the latter matrix are respectively $\{1, 3\}$ showing that $\mathcal{Q} \succ 0$. Hence we have $\mathcal{M}(\rho) - g(\rho)Z \succ 0$ and finally

$$\mathcal{M}(\rho) \succ g(\rho)Z \succeq 0$$

If $\mathcal{Q}$ was not found positive definite, then Z would have been chosen as a function of ρ, and the procedure applied again.

This shows that the SOS approach allows to elaborate LMI conditions for the positive (negative) definiteness of parameter dependent matrices in which parameters evolve in a bounded compact set. The great interest is that the resulting conditions are independent of the parameters and nonconservative provided that the order of relaxation (i.e. the degrees of the polynomials) is sufficiently large.

1.3.3.4 Global Polynomial Optimization and the Problem of Moments

This approach is dual to the sum-of-squares relaxation. Since the matrix case can be straightforwardly turned into the scalar case, we will focus here on the scalar case only for illustration purpose. The reader should refer to [Henrion and Lasserre, 2004, 2006, Lasserre, 2001, 2007] and references therein to get more details. This method is based on measure theory and aims at turning the initial optimization problem over $\mathbb{R}^n$ into another optimization problem over a measure space. Although, the optimization over measure spaces is a rather complicated problem, such a reformulation is very general and allows to solve a wide type of optimization problems, including polynomial optimization problems, using SDP.

Consider the optimization problem

$$\begin{array}{l} \inf c(x) \text{ s.t.} \\ x \in \mathbb{R}^n \\ g_i(x) \geq 0 \end{array} \tag{1.92}$$

where $c(x) = \sum_{i=1}^{N} \beta_i x^{\alpha_i}$ and $g_i(x)$ are scalar multivariate polynomials with $\alpha_i = \begin{bmatrix} \alpha_i^1 & \dots & \alpha_i^n \end{bmatrix}$ and $x^{\alpha_i} = x_1^{\alpha_i^1} x_2^{\alpha_i^2} \dots x_n^{\alpha_i^n}$.

Assuming that the set

$$\mathcal{X} := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \text{ for all } i = 1, \dots, N\} \tag{1.93}$$

is non empty, then the optimization problem (1.92) is equivalent to the following optimization problem

$$\begin{array}{l} \inf\limits_{\mu} \displaystyle\int_{\mathcal{X}} c(x) d\mu(x) \text{ s.t.} \\ \displaystyle\int_{\mathcal{X}} d\mu(x) = 1 \end{array} \tag{1.94}$$

where μ is a probability measure over $\mathcal{X}$.

To see the equivalence, note that

$$\begin{array}{rcl} \displaystyle\int_{\mathcal{X}} c(x) d\mu(x) & \geq & \inf\limits_{x \in \mathcal{X}} c(x) \displaystyle\int_{\mathcal{X}} d\mu(x) \\ & \geq & \inf\limits_{x \in \mathcal{X}} c(x) \end{array} \tag{1.95}$$

Then suppose that x^* is a global minimizer of $c(x)$ over $x \in \mathcal{X}$ then the corresponding measure is

$$\mu^*(x) = \delta(x - x^*) \tag{1.96}$$

where δ is the Dirac measure.

This shows that the global minimum of problem (1.92) coincides with the global minimum of problem (1.94). Now, the aim is to explain how the measure μ is found since an optimization problem over a measure space is not trivial. First note, that a measure is uniquely characterized by what we call its moments:

$$m_{\alpha_i}(\mu) = \int_{\mathcal{X}} x^{\alpha_i} d\mu(x) \tag{1.97}$$

where $\alpha_i = \begin{bmatrix} \alpha_i^1 & \dots & \alpha_i^n \end{bmatrix}$ and $x^{\alpha_i} = x_1^{\alpha_i^1} x_2^{\alpha_i^2} \dots x_n^{\alpha_i^n}$.

The modified cost writes:

$$\int_{\mathcal{X}} c(x) d\mu(x) = \sum_{i=1}^{N} \beta_i m_{\alpha_i}(\mu) \tag{1.98}$$

and then the optimization problem becomes

$$\min \sum_{i=1}^{N} \beta_i m_{\alpha_i} \tag{1.99}$$

such that

$$\begin{array}{lcl} m_{[0\ 0\ \dots\ 0]} & = & 1 \\ M_k(m) & \succeq & 0 \\ M_{k-d_i}(g_i m) & \succeq & 0 \end{array} \tag{1.100}$$

where $2d_i$ or $2d_i - 1$ is the degree of polynomial $g_i(x)$. $M_k(m) \succ 0$ and $M_{k-d_i}(g_i m) \prec 0$ are LMIs constraints in m (the moments) corresponding to respective truncations of moment and localizing matrices (matrices defining the constraints corresponding to the $g_i(x)$ in terms of moments).

The following example should make the above reformulation clearer.

Example 1.3.32 *Let us consider the following polynomial optimization problem*

$$\begin{array}{ll} \inf\limits_{x \in \mathbb{R}^2} & 2x_1 + 2x_1^2 - x_1 x_2 \quad \text{s.t.} \\ & g_1(x) := 2x_1^2 - x_2 \geq 0 \\ & g_2(x) := -x_1^2 - x_2^2 + 4 \geq 0 \end{array} \tag{1.101}$$

It is clear that the semi-algebraic set

$$\left\{ x \in \mathbb{R}^2 : \ g_1(x) \geq 0, \ g_2(x) \geq 0 \right\}$$

is non convex since it consists in the closed-interior of a ball minus the epigraph of a parabola crossing through the origin. This is illustrated in Figure 1.15.

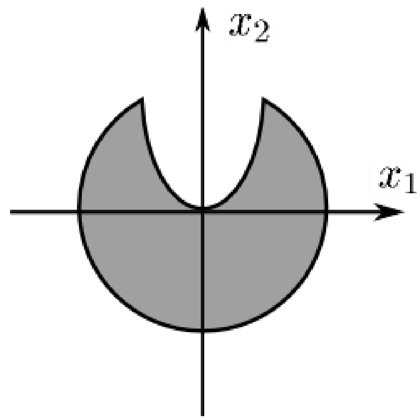

Figure 1.15: Representation of the nonconvex set $\{x \in \mathbb{R}^2 : \ g_1(x) \geq 0, \ g_2(x) \geq 0\}$ *considered in the polynomial optimization problem (1.101)*

Turning the optimization into the measure formulation, we get

$$\begin{array}{l} \inf 2m_{10} + 2m_{20} - m_{12} \text{ s.t.} \\ 2m_{20} - m_{01} \geq 0 \\ -m_{20} - m_{02} + 4 \geq 0 \\ m_{00} = 1 \end{array} \tag{1.102}$$

where $m_{ij} = \int_{\mathcal{X}} x_1^i x_2^j d\mu$. *Moreover, let us define the following rank-one matrix:*

$$N_1(x) := \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1 x_2 \\ x_2 & x_1 x_2 & x_2^2 \end{bmatrix} \succeq 0 \tag{1.103}$$

Computing the integral of $N_1(x)$ *over* $\mathcal{X}$ *with measure* $d\mu(x)$ *we get*

$$\int_{\mathcal{X}} N_1(x) d\mu(x) = M_1(m) \succeq 0 \tag{1.104}$$

where $M_1(m) = \left[\begin{array}{c|cc} 1 & m_{10} & m_{01} \\ \hline m_{10} & m_{20} & m_{11} \\ m_{01} & m_{11} & m_{02} \end{array}\right] \succeq 0$

This leads to the first approximation of the polynomial optimization problem

$$\begin{array}{l} \inf 2m_{10} + 2m_{20} - m_{12} \text{ s.t.} \\ 2m_{20} - m_{01} \geq 0 \\ -m_{20} - m_{02} + 4 \geq 0 \\ m_{00} = 1 \\ M_1(m) \succeq 0 \end{array} \tag{1.105}$$

In order to derive tighter relaxations, note that the matrices $g_1(x)N_1(x)$ *and* $g_2(x)N_1(x)$ *are positive semidefinite since* $N_1(x) \succeq 0$ *and* $g_1(x), g_2(x) \geq 0$. *Hence we obtain,*

$$\begin{aligned}
g_1(x)N_1(x) &= (2x_1^2 - x_2)\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1^2 - x_2 & 2x_1^3 - x_2x_1 & 2x_1^2x_2 - x_2^2 \\ 2x_1^3 - x_2x_1 & 2x_1^4 - x_2x_1^2 & 2x_1^3x_2 - x_2^2x_1 \\ 2x_1^4 - x_2x_1^2 & 2x_1^3x_2 - x_2^2x_1 & 2x_1^2x_2^2 - x_2^3 \end{bmatrix} \succeq 0 \\
g_2(x)N_1(x) &= (-x_1^2 - x_2^2 + 4)\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{bmatrix} \\
&= \begin{bmatrix} -x_1^2 - x_2^2 + 4 & -x_1^3 - x_2^2x_1 + 4x_1 & -x_1^4 - x_2^2x_1^2 + 4x_1^2 \\ -x_1^3 - x_2^2x_1 + 4x_1 & -x_1^4 - x_2^2x_1^2 + 4x_1^2 & -x_1^3x_2 - x_2^3x_1 + 4x_1x_2 \\ -x_1^4 - x_2^2x_1^2 + 4x_1^2 & -x_1^3x_2 - x_2^3x_1 + 4x_1x_2 & -x_1^2x_2^2 - x_2^4 + 4x_2^2 \end{bmatrix} \succeq 0
\end{aligned} \tag{1.106}$$

Computing the integral of $g_1(x)N_1(x)$ *and* $g_2(x)N_1(x)$ *over* $\mathcal{X}$ *with measure* $d\mu(x)$ *leads respectively to matrices* $M_1(g_1m)$ *and* $M_1(g_2m)$ *writing*

$$\begin{aligned}
M_1(g_1m) &= \left[\begin{array}{c|cc} 2m_{20} - m_{01} & 2m_{30} - m_{11} & 2m_{21} - m_{02} \\ \hline 2m_{30} - m_{11} & 2m_{40} - m_{21} & 2m_{31} - m_{12} \\ 2m_{21} - m_{02} & 2m_{31} - m_{12} & 2m_{22} - m_{03} \end{array}\right] \succeq 0 \\
M_1(g_2m) &= \begin{bmatrix} -m_{20} - m_{02} + 4 & -m_{30} - m_{12} + 4m_{10} & -m_{40} - m_{22} + 4m_{20} \\ -m_{30} - m_{12} + 4m_{10} & -m_{40} - m_{22} + m_{20} & -m_{31} - m_{13} + 4m_{11} \\ -m_{40} - m_{22} + 4m_{20} & -m_{31} - m_{13} + 4m_{11} & -m_{22} - m_{04} + 4m_{02} \end{bmatrix} \succeq 0
\end{aligned} \tag{1.107}$$

Since higher order moments are present (up to order 4), we construct the higher order relaxation matrix $M_2(m)$

$$M_2(m) = \left[\begin{array}{c|cc|ccc} 1 & m_{10} & m_{01} & m_{20} & m_{11} & m_{02} \\ \hline m_{10} & m_{20} & m_{11} & m_{30} & m_{21} & m_{12} \\ m_{01} & m_{11} & m_{02} & m_{21} & m_{12} & m_{03} \\ \hline m_{20} & m_{30} & m_{21} & m_{40} & m_{31} & m_{22} \\ m_{11} & m_{21} & m_{12} & m_{31} & m_{22} & m_{13} \\ m_{02} & m_{12} & m_{03} & m_{22} & m_{13} & m_{33} \end{array}\right] \succeq 0$$

Finally, the optimization problem becomes

$$\begin{aligned}
&\inf 2m_{10} + 2m_{20} - m_{12} \text{ s.t.} \\
&2m_{20} - m_{01} \geq 0 \\
&-m_{20} - m_{02} + 4 \geq 0 \\
&m_{00} = 1 \\
&M_1(g_1m) \succeq 0 \\
&M_1(g_2m) \succeq 0 \\
&M_2(m) \succeq 0
\end{aligned} \tag{1.108}$$

With a similar procedure, it is possible to construct higher order relaxations until obtain satisfying results.

Finally, it has been shown that the global minimum found using the relaxation asymptotically converges to the actual global minimum when the order of relaxation k tends to $+\infty$. Fortunately, as in the sum-of-squares approach, the global minimizer is found for small values of k. In order to point out the duality between these two methods, just memorize that raising k corresponds to raise the degree of sum-of-squares polynomials.

The generalization to parameter dependent LMIs is obtained by noticing that a parameter dependent symmetric matrix $\mathcal{M}(\rho)$ is negative definite if and only if all its principal minors are strictly negative. This brings back the matrix problem to a multiple polynomial scalar problem. Indeed, for a polynomially parameter dependent symmetric matrix of dimension k, there are k principal minors taking the form of polynomials, which is exactly the form presented in this section.

However, the formulation of LMI problem is not trivial and this is illustrated in the following example.

Example 1.3.33 *Let $L(\rho, M) \prec 0$ be a parameter dependent LMI aimed to be satisfied where $M \in \mathcal{M}$ represents decision matrices and the parameter vector ρ belongs to a compact set U_ρ. We define the following optimization problem:*

$$\begin{array}{ll} \inf & -t \\ & f_i(\rho, M, t) > 0 \\ & M \in \mathcal{M} \\ & \rho \in U_\rho \end{array}$$

where $f_i(\rho, M, t))$ are all minors of $L(\rho, M) - tI \succ 0$. The scalar t allows to determine whether the LMI is satisfied or not. If $t < 0$ then the problem is feasible and there exists $M \in \mathcal{M}$ such that $L(\rho, M) \prec 0$ for all $\rho \in U_\rho$. Moreover, the parameter vector for which the minimum is attained is also returned by the optimization procedure. On the other hand, if $t > 0$ then this means that there exists a parameter vector for which $L(\rho, M) \not\prec 0$ and the parameter vector for which maximal eigenvalue of $L(\rho, M)$ is attained is returned.

Consider the scalar inequality $f(\rho) = \rho^2 - 4$ where $\rho \in [-1, 1]$. It is clear that $f(\rho) < 0$ over that domain. Now consider the optimization problem:

$$\begin{array}{ll} \inf & -t \\ & \rho^2 - 4 - t > 0 \\ & \rho \in [-1, 1] \end{array}$$

It is simple to show that $t_{opt} = -3$ for $\rho = \pm 1$. Therefore inequality $\rho^2 - 4 < 0$ is satisfied for all $\rho \in [-1, 1]$.

Now consider the second optimization problem:

$$\begin{array}{ll} \inf & -t \\ & \rho^2 - 4 - t > 0 \\ & \rho \in [0, 3] \end{array}$$

In this case, $t_{opt} = 5$ for $\rho = 3$. Finally consider the problem of finding k such that $\rho^2 - 4 + k < 0$. In this case, the constraint $t < 0$ must be added in order to obtain coherent results. The optimization problem is thus defined by

$$\begin{array}{ll} \inf & -t \\ & t < 0 \\ & \rho^2 - 4 + k - t > 0 \\ & \rho \in [0, 3] \end{array}$$

We obtain $k = -5 - \varepsilon$ and $t = -\varepsilon$ for sufficiently small $\varepsilon > 0$.

This example illustrates that the moment approach can be used in order to prove stability of LPV systems and find suboptimal stabilizing controllers.

This approach is well-dedicated for small to medium size problems. Indeed, the dimension of LMIs grows quickly, slowing dramatically the resolution by classical SDP solvers. Hence the computational complexity is globally the same as of sum-of-squares relaxations.

It is worth noting that, as a by-product, such method can be used to find solutions to BMIs using either scalarization or directly by considering Polynomial Matrix Inequalities (PMI) [Henrion and Lasserre, 2006]. Nevertheless, although the theory for matrix valued problem is ready, it is still experimentally at a very preliminary level [Henrion, 2008].

1.3.4 Stability of 'LFT' systems

The stability of 'LFT' systems is still an active research topic. Indeed, 'LFT' systems provide an unified way to model LPV systems with every type of parameter dependence: affine, polynomial and rational. By rewriting LPV systems in LFT form, the initial system is split in two interconnected subsystems: a constant and time-varying one. The stability of the LPV systems is then determined using results on the stability interconnected systems. Most of these results are summarized in this section.

Let us recall the LPV system is a 'LFT' formulation:

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \\ w(t) & = & \Theta(\rho)z(t) \end{array} \qquad (1.109)$$

The parameter matrix $\Theta(\rho)$ is not detailed here since its structure is not fixed a priori and depend on stability analysis methods. It is important to note that all the tools provided in that section have been initially developed for robust stability analysis of linear systems. Due to the genericity of the LFT procedure, all these tools apply naturally to the LPV case.

1.3.4.1 Passivity

The passivity is a very strong result for LPV systems. It can only be used with positive $\Theta(\rho)$ and the LTI system must satisfy a very constraining inequality. Let $H(s) = C(sI - \tilde{A})^{-1}B + D$ be the transfer function from w to z corresponding to system (1.109) and assume that $\Theta(\rho)$ is diagonal with bounded nonnegative components. We have the following definition (see for instance [Scherer and Wieland, 2005])

Definition 1.3.34 *System $H(s)$ is (strictly) passive if and only if*

$$H(j\omega) + H(j\omega)^*(\succ 0) \succeq 0, \qquad \text{for all } \omega \in \mathbb{R} \qquad (1.110)$$

This means that, in the SISO case, that the Nyquist plot of $H(j\omega)$ must lie within the complex open right half-plane (which is very constraining for system of order greater than 1). We need the following result:

Proposition 1.3.35 *If a strictly passive system is interconnected with a passive system, then the resulting system is passive.*

As any passive system is asymptotically stable, then the stability of the interconnection is proved. Then System (1.109) is asymptotically stable if $\Theta(\rho)$ is a passive operator and $H(s)$ a strictly passive one. $\Theta(\rho)$ is passive since it is diagonal and has nonnegative elements and $H(s)$ is strictly passive if strict inequality (1.110) holds. The following examples illustrates the approach.

Example 1.3.36 *Let us consider the SISO LPV system*

$$\dot{x} = -(2-\rho)x \tag{1.111}$$

where $\rho \in [0,1]$. It is clear that the system is quadratically stable since there exists $p > 0$ such that $-(2+\rho)p < 0$ for all $\rho \in [0,1]$. The LPV system is then rewritten into the 'LFT' form

$$\begin{array}{rcl} \dot{x} & = & -2x + w \\ z & = & x \\ w & = & \rho x \end{array} \tag{1.112}$$

The transfer function $H(s)$ corresponding to the LTI system is then given by $H(s) = \dfrac{1}{s+2}$ and is strictly passive if and only if

$$\begin{array}{rcl} H(j\omega) + H(j\omega)^* & \succ & 0, \qquad \text{for all } \omega \in \mathbb{R} \\ & = & \dfrac{1}{j\omega+2} + \dfrac{1}{-j\omega+2}, \qquad \text{for all } \omega \in \mathbb{R} \\ & = & \dfrac{4}{\omega^2+4} \succ 0, \qquad \text{for all } \omega \in \mathbb{R} \end{array} \tag{1.113}$$

Hence the LPV system (1.111) is asymptotically stable.

Example 1.3.37 *Let us consider the SISO LPV system*

$$\dot{x} = -(2+\rho)x \tag{1.114}$$

where $\rho \in [0,1]$. This system is also quadratically stable and the 'LFT' formulation is then given by

$$\begin{array}{rcl} \dot{x} & = & -2x - w \\ z & = & x \\ w & = & \rho x \end{array} \tag{1.115}$$

The transfer function $H(s)$ corresponding to the LTI system is then given by $H(s) = \dfrac{-1}{s+2}$ and is strictly passive if and only if $H(j\omega) + H(j\omega)^* \succ 0, \qquad$ for all $\omega \in \mathbb{R}$. However

$$\begin{array}{rcl} H(j\omega) + H(j\omega)^* & = & \dfrac{-1}{j\omega+2} + \dfrac{-1}{-j\omega+2}, \qquad \text{for all } \omega \in \mathbb{R} \\ & = & \dfrac{-4}{\omega^2+4} \not\succ 0, \qquad \text{for all } \omega \in \mathbb{R} \end{array} \tag{1.116}$$

Since $H(s)$ is not strictly passive then asymptotic stability of system (1.114) cannot be proved by passivity.

In the above examples, the sign analysis of the sum $H(j\omega) + H(j\omega)^*$ is performed analytically due to its simple expression. However, if the transfer function is more complex (i.e. higher order systems and/or MIMO systems), an analytical analysis is far tougher. Fortunately, a LMI test has been provided, for instance, in [Scherer and Wieland, 2005] allowing to an easy test for MIMO systems.

Theorem 1.3.38 *A system (A_s, B_s, C_s, D_s) is passive if and only if there exists a matrix $P = P^T \succ 0$ such the LMI*

$$\begin{bmatrix} A_s^T P + P A_s & P B_s - C_s^T \\ \star & -(D_s + D_s^T) \end{bmatrix} \prec 0 \tag{1.117}$$

is feasible.

The origin of this LMI is detailed in Appendix E.4.

Example 1.3.37 shows that if a system has non minimum phase, the passivity may fail even in the more simple case of a 1^{st} order system. The fact that a very few systems are (strictly) passive implies that the stability analysis of LPV systems in 'LFT' form is very restrictive and should not be considered.

Nevertheless, in many applications, passivity is very important since it shows that no energy is added by the loop while interconnecting two systems. For instance, teleoperation [Hokayem and Spong, 2006, Niemeyer, 1996], drive-by-wire and more generally network controlled systems may be tackled using the passivity theory.

1.3.4.2 Small-Gain Theorem

The small-gain theorem is an enhancement of the passivity based stability analysis of interconnections since it takes into account variations of energy between inputs and outputs of dynamical systems involved in the interconnection. A simple energy analysis of loop-signals suggests that asymptotic stability of the interconnection is equivalent to finiteness of the energy of the loop-signals involved in the interconnection. Hence the problem remains to determine whether the energy of these signals is finite or not.

It is convenient to introduce first the following definitions:

Definition 1.3.39 *The energy gain (or $\mathcal{L}_2$-gain or $\mathcal{L}_2$-induced norm) of a time-invariant operator T is defined by the relation*

$$\gamma_{\mathcal{L}_2} = ||T||_{\mathcal{L}_2-\mathcal{L}_2} := \sup_{w\in\mathcal{L}_2,\ w\neq 0} \frac{||Tw||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} \tag{1.118}$$

where $\mathcal{L}_2$ is set of bounded energy signal (see appendix C for more details). For instance, unitary energy inputs give at most $\gamma_{\mathcal{L}_2}$ energy outputs.

If the operator is not asymptotically stable, then by definition we have $\gamma_{\mathcal{L}_2} = +\infty$.

Definition 1.3.40 *The $\mathcal{H}_\infty$-norm of a linear time-invariant operator T is given by*

$$\gamma_{\mathcal{H}_\infty} = ||T||_{\mathcal{H}_\infty} := \sup_{w\in\mathbb{R}} \bar{\sigma}(T(j\omega)) \tag{1.119}$$

where $\bar{\sigma}(T)$ is the maximal singular value of the transfer matrix $T(s)$ (see appendix A.6 for more details on singular values and singular values decomposition).

In the LTI case, the $\mathcal{H}_\infty$-norm of a time-invariant operator coincides with the $\mathcal{L}_2$-induced norm (see for instance [Doyle et al., 1990]). As suggested by the definitions, if a LTI system is asymptotically stable then it has finite $\mathcal{H}_\infty$-norm.

As an illustration of the approach, let us consider for simplicity, the interconnection of two SISO transfer functions $H_1(s)$ and $H_2(s)$ as shown in Figure 1.16.

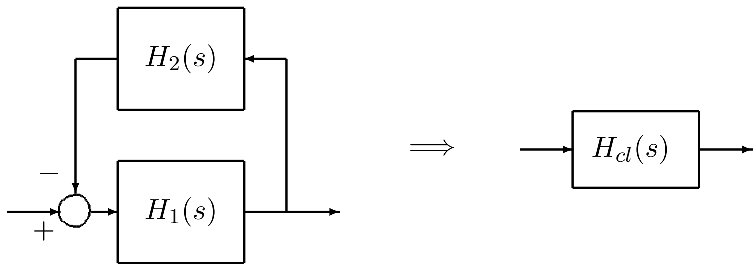

Figure 1.16: Interconnection of two SISO transfer functions

The closed-loop transfer function is then given by the expression

$$H_{cl}(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \tag{1.120}$$

It is clear that the closed-loop system is asymptotically stable if and only if $H_1(s)H_2(s) \neq -1$ for all $s \in \mathbb{C}^+$. From this consideration, by imposing the condition $\sup_{s\in\mathbb{C}^+} |H_1(s)H_2(s)| < 1$ it is ensured that $H_1(s)H_2(s) \neq -1$ for all $s \in \mathbb{C}^+$. Finally, noting that $\sup_{s\in\mathbb{C}^+} |H_1(s)H_2(s)| < 1$ is equivalent to

$$||H_1H_2||_{\mathcal{H}_\infty} < 1$$

we get a sufficient condition for stability in term of $\mathcal{H}_\infty$-norm analysis.

We introduce now the small-gain theorem, the main result of the section (see for instance [Zhou et al., 1996]).

Theorem 1.3.41 *The LPV system (1.109) is asymptotically stable if the inequality*

$$\frac{||\Theta(\rho)Hw||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} < 1 \tag{1.121}$$

holds where $\Theta(\rho)$ is a full-matrix depending on the parameters such that $\Theta(\rho)^T\Theta(\rho) \preceq I$ and H is the LTI operator mapping w to z

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{array} \tag{1.122}$$

from system (1.109).

It is clear that the sufficient condition $\dfrac{||\Theta(\rho)Hw||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} < 1$ may be tough to verify due to the time-varying nature of the matrix $\Theta(\rho)$. Hence, in virtue of the submultiplicative property of the $\mathcal{H}_\infty$ norm, i.e.

$$||H_1H_2||_{\mathcal{H}_\infty} < ||H_1||_{\mathcal{H}_\infty} \cdot ||H_2||_{\mathcal{H}_\infty} \tag{1.123}$$

then a more conservative sufficient condition is given by

$$||\Theta(\rho)||_{\mathcal{L}_2} \cdot ||H||_{\mathcal{H}_\infty} < 1 \tag{1.124}$$

It is clear that the condition is sufficient only since, by considering the norm, we loose all information on the phase of $H_1(s)H_2(s)$. Indeed, the sup constraint restricts the bode magnitude plot of $H_1(j\omega)H_2(j\omega)$ to evolve inside the unit disk ignoring the value of the phase. This results evidently in a conservative (hence sufficient) stability condition. The following examples illustrate non-equivalence between these results on asymptotic stability.

Example 1.3.42 *Let us consider two asymptotically stable LTI SISO system $H_1(s)$ and $H_2(s)$ interconnected as depicted on Figure 1.16 and defined by*

$$\begin{array}{rcl} H_1(s) & = & \dfrac{10}{(s+1)(s+2)} \\ H_2(s) & = & \dfrac{10}{(s+3)(s+4)} \end{array} \tag{1.125}$$

Since both $H_1(s)$ and $H_2(s)$ are asymptotically stable then $H_1(s)H_2(s)$ is asymptotically stable too. Then we have

$$\begin{array}{rcl} ||H_1H_2||_{\mathcal{H}_\infty} & = & \sup_{s\in\mathbb{C}^+} |H_1(s)H_2(s)| \\ & = & \sup_{\omega\in\mathbb{R}} |H_1(j\omega)H_2(j\omega)| \text{ by the maximum modulus principle (see Appendix F.5)} \\ & = & H_1(0)H_2(0) \\ & = & 100/24 > 1 \end{array} \tag{1.126}$$

Hence according to the small-gain theorem the interconnection is not asymptotically stable even though we have

$$H_{bf}(s) = \frac{1}{1 + H_1(s)H_2(s)} = \frac{s^4 + 10s^3 + 35s^2 + 50s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 124} \tag{1.127}$$

which has poles $\{-4.8747+2.0950i, -4.8747-2.0950i, -0.1253+2.0950i, -0.1253-2.0950i\}$ with negative real part, showing that the interconnection is asymptotically stable.

In the latter example, the equality $||H_1H_2||_{\mathcal{H}_\infty} = ||H_1||_{\mathcal{H}_\infty} \cdot ||H_2||_{\mathcal{H}_\infty}$ holds since the transfer functions $H_1(s)$ and $H_2(s)$ reach their maximum modulus value at the same argument $s = 0$. The following example presents a case for which this equality does not hold:

Example 1.3.43 *Let us consider two asymptotically stable LTI SISO system $H_1(s)$ and $H_2(s)$ interconnected as depicted on Figure 1.16 and defined by*

$$\begin{array}{rcl} H_1(s) & = & \dfrac{1}{s^2 + 0.1s + 10} \\ H_2(s) & = & \dfrac{10}{(s+3)(s+4)} \end{array} \tag{1.128}$$

In this case we have

$$\begin{array}{rcl} ||H_1||_{\mathcal{H}_\infty} & = & \dfrac{10^3}{\sqrt{99975}} \text{ at } \omega = \dfrac{\sqrt{39.98}}{2} \\ ||H_2||_{\mathcal{H}_\infty} & = & \dfrac{10}{12} \text{ at } \omega = 0 \\ ||H_1H_2||_{\mathcal{H}_\infty} & = & 0.7084 \text{ at } \omega = 3.1608 \end{array} \tag{1.129}$$

This shows that while the nyquist plot of $H_1(j\omega)H_2(j\omega)$ remains within the unit disk (asymptotic stability of the interconnection). On the other hand, the inequality based on the sub-multiplicative property of the $\mathcal{H}_\infty$-norm gives $\dfrac{10^4}{12\sqrt{99975}}$, which is approximately 2.6356, and does not allow to conclude on the stability of the interconnection.

It is important to emphasize similarities with open loop analysis for closed-loop systems, which historically has led to the development of stability margins. As an example the module margin is defined for a SISO system depicted on Figure 1.16 by

$$\begin{aligned} \inf_{\omega\in\mathbb{R}} 1 + H_1(j\omega)H_2(j\omega) &= \sup_{j\omega\in\mathbb{R}} \frac{1}{1 + H_1(j\omega)H_2(j\omega)} \\ &= \left|\left|\frac{1}{1 + H_1(j\omega)H_2(j\omega)}\right|\right|_{\mathcal{H}_\infty} \end{aligned} \tag{1.130}$$

Let us now come back to LPV system (1.109). Since, by definition $\Theta(\rho)^T\Theta(\rho) \leq I$, we have $||\Theta(\rho)||_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$. To see this, let $\tilde{z}(t) = \Theta(\rho)\bar{z}(t)$ and then the energy of $\tilde{z}(t)$ writes

$$\begin{aligned} \int_0^{+\infty} \tilde{z}(s)^T\tilde{z}(s)ds &= \int_0^{+\infty} \bar{z}(s)^T\Theta(\rho(s))^T\Theta(\rho(s))\bar{z}(s)ds \\ &\leq \int_0^{+\infty} \bar{z}(s)^T\bar{z}(s)ds \end{aligned} \tag{1.131}$$

Finally the stability condition reduces to

$$||H||_{\mathcal{H}_\infty} < 1$$

and the verification of the latter inequality can be computed by semidefinite programming through a LMI feasibility test. Indeed, instead of the initial $\mathcal{H}_\infty$-norm computation using bisection algorithm [Zhou et al., 1996] or Hamiltonian matrix [Doyle et al., 1990], the bounded real lemma (see Appendix E.7 and [Scherer and Wieland, 2005, Scherer et al., 1997, Skelton et al., 1997]) allows to compute the $\mathcal{L}_2$-induced norm of linear (possibly time/parameter varying) systems.

Lemma 1.3.44 *The interconnected system (1.109) is asymptotically stable if there exist a matrix $P = P^T \succ 0$ and a scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} \tilde{A}^TP + P\tilde{A} & PB & C^T \\ \star & -(1-\varepsilon)I & D^T \\ \star & \star & -(1-\varepsilon)I \end{bmatrix} \prec 0 \tag{1.132}$$

It is important to point out that, while condition $||\Theta(\rho)||_{\mathcal{L}_2} \cdot ||H||_{\mathcal{H}_\infty} < 1$ implies $||\Theta(\rho)H||_{\mathcal{L}_2} < 1$, the converse generally does not hold (see Examples 1.3.42 and 1.3.43). Indeed, the equality in (1.123) rarely holds, except for very special cases (see Example 1.3.42).

The small-gain condition is a simple stability test but is however rather conservative. First of all, it does not consider the phase and secondly no information is looked out on how the elements interconnect, the shape of the intersecting elements but their maximal value only (their norm). Indeed, as illustrated in Example 1.3.42, if the maximum values do not occur at the same frequency, the submultiplicative inequality is conservative. This is far more complicated when dealing with nonlinear or non-stationary elements. The last example illustrates this.

Example 1.3.45 *As an example note that*

$$\begin{array}{rcl} ||2\sin(t)||_{\mathcal{L}_\infty} & < & 2 \\ \left|\left|\dfrac{1}{2+\cos(t)}\right|\right|_{\mathcal{L}_\infty} & < & 1 \end{array}$$

Hence we have

$$||2\sin(t)||_{\mathcal{L}_\infty} \cdot \left|\left|\frac{1}{2+\cos(t)}\right|\right|_{\mathcal{L}_\infty} < 2$$

but actually we have $\left|\left|\dfrac{2\sin(t)}{2+\cos(t)}\right|\right|_{\mathcal{L}_\infty} < 2/3$ *which shows that the application of the submultiplicative property may result in very conservative bounds (conditions).*

The latter example shows that how the interconnection holds, is primordial is the stability analysis. Figure 1.17 provides a geometric representation of the conservatism of the small-gain theorem.

Figure 1.17: Illustration of the conservatism induced by the use of the $\mathcal{H}_\infty$-norm. Although, the pieces of puzzle fit together, the consideration of the $\mathcal{H}_\infty$ norm says the contrary.

In order to explain Figure 1.17 assume that the free piece is an operator P and let O be the center of the piece. Since the piece is two-dimensional we assume that it belongs to a two-dimensional normed vector space. In what follows we will consider that the free piece is an operator and the remaining of the puzzle constitutes the other operator. The interconnection of systems is substituted to an interconnection of pieces of puzzle. Moreover, the asymptotic stability of the interconnection is replaced by the possibility of placing the piece at this place. The image shows that the piece interlocks perfectly. We show hereafter that by ignoring the shape of the piece and reducing it to a single value (a norm), the piece and the remaining cannot be shown to fit together.

The operator P affects any vector $v(\theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ to a new vector $v'(\theta)$ whose 2-norm equals the length between the center O and the boundary of the piece in the orientation $\theta \in [0, 2\pi]$ (orientation 0 points to the right).

Therefore for every $v(\theta)$ we have

$$Pv(\theta) = v'(\theta) = \lambda(\theta)v(\theta) \tag{1.133}$$

since only the norm of the vector is changed. As a comparison the $\mathcal{H}_\infty$ of an operator is the largest energy gain that is applied to an input signal entering this operator. It is the modification of the norm of the input signal where the norm is the energy.

It is clear that by considering the norm

$$||P|| = \sup_{\theta\in[0,2\pi]} \frac{||Pv(\theta)||_2}{||v(\theta)||_2} \tag{1.134}$$

the farthest point on the boundary from 0 is taken into account and the piece is considered as circle shaped with radius $||P||$. In this condition no puzzle can be finished and hence more sophisticated techniques should be employed to determine if some pieces correspond.

The puzzle analogy shows that the shape (the structure) of interconnected elements should play a crucial role in the stability analysis. It is clear that if the matrix $\Theta(\rho)$ is full, then a priori only norm-information can be extracted. On the other hand, if this matrix has a specific and known form it is possible to refine the small-gain theorem in a new version.

1.3.4.3 Scaled-Small Gain Theorem

The aim of the scaled-small gain lemma is to reduce the conservatism of the small-gain theorem by considering the structure of the parameter-varying matrix gain $\Theta(\rho)$. It is generally assumed, in the LPV framework, that the matrix has a diagonal structure

$$\Theta(\rho) = \mathrm{diag}(I_{n_1} \otimes \rho_1, \dots, I_{n_p} \otimes \rho_p) \tag{1.135}$$

Let us introduce the set of D-scalings is defined by

$$\mathcal{S}_D(\Theta) := \left\{ L \in \mathbb{S}^{\bar{n}}_{++} : \ \Theta(\rho)L^{1/2} = L^{1/2}\Theta(\rho) \text{ for all } \rho \in [-1,1]^p \right\} \tag{1.136}$$

where $\bar{n} = ||col(n_1, \dots, n_p)||_1$ and $L^{1/2}$ denotes the positive square-root of L. For more details on the scalings the reader should refer to [Apkarian and Gahinet, 1995] or Appendix E.11.

The key idea is to define a matrix $L \in \mathcal{S}(\Theta)$ to embed information on the structure of the parameter matrix, through a commutation property. This additional matrix will then be introduced in the bounded-real lemma and provides extra degree of freedom and thus, a reduction of conservatism is expected [Packard and Doyle, 1993].

Example 1.3.46 *Consider the following parameter matrix* $\Theta(\rho) = \begin{bmatrix} \rho_1 I_5 & 0 \\ 0 & \rho_2 I_2 \end{bmatrix}$*, then a suitable matrix* $L \in \mathcal{S}(\Theta)$ *is given by*

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$$

where $L_1 \in \mathbb{S}^5_{++}$ *and* $L_2 \in \mathbb{S}^2_{++}$*.*

Since $L \in \mathcal{S}_D(\Theta)$ is positive definite, let us define a dual parameter matrix

$$\tilde{\Theta}(\rho) = L^{1/2}\Theta(\rho)L^{-1/2}$$

It is clear that, in virtue of the definition of set $\mathcal{S}_D(\Theta)$, the following identity holds

$$\tilde{\Theta}(\rho) = \Theta(\rho)$$

In what follows, we aim at showing that the feasibility of the scaled-bounded real lemma implies asymptotic stability of the interconnection. To this aim, let w_2 and z_2 be $\mathcal{L}_2$ signals satisfying $w_2(t) = \tilde{\Theta}(\rho)z_2(t)$. First, let us show that operator $\tilde{\Theta}(\rho)$ has unitary energy gain. Suppose that it is has energy gain of $\gamma_\theta > 0$, then the following integral quadratic function must be nonnegative.

$$\begin{array}{rcl} \displaystyle\int_0^{+\infty} \begin{bmatrix} w_2(s) \\ z_2(s) \end{bmatrix}^T \begin{bmatrix} \gamma_\theta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w_2(s) \\ z_2(s) \end{bmatrix} ds & = & \displaystyle\int_0^{+\infty} z(s)^T \Xi(\rho, L) z(s) ds \\ \Xi(\rho, L) & = & \gamma_\theta^2 I - L^{-1/2}\Theta(\rho)^T L \Theta(\rho) L^{-1/2} \end{array} \quad (1.137)$$

where $z_2(t)$ and $w_2(t)$ are respectively the input and output of operator $\tilde{\Theta}(\rho)$. The latter integral quadratic form is nonnegative for all z if and only if

$$\gamma_\theta^2 I - L^{-1/2}\Theta(\rho)^T L \Theta(\rho) L^{-1/2} \succeq 0 \quad (1.138)$$

and, according to the definition of the set $\mathcal{S}_D(\Theta)$ by (1.136), if and only if

$$\gamma_\theta^2 I - \Theta(\rho)^T \Theta(\rho) \succeq 0 \quad (1.139)$$

Since $\Theta(\rho)^T\Theta(\rho) \preceq I$ then $\gamma_\theta = 1$ is the minimal value such that (1.139) holds. This shows that rather than considering $\Theta(\rho)$ it is not aberrant to consider $\tilde{\Theta}(\rho)$ instead.

Finally, in virtue of these considerations, if the interconnection of the LTI system and dual parameter matrix $\tilde{\Theta}(\rho)$

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + Ew_2(t) \\ z_2(t) & = & Cx(t) + Dw_2(t) \\ w_2(t) & = & \tilde{\Theta}(\rho)z_2(t) \end{array} \quad (1.140)$$

is asymptotically stable, then (1.109) is asymptotically stable. It is worth noting that, by introducing notations $L^{-1/2}z_2(t) = z(t)$ and $w_2(t) = L^{1/2}w(t)$ we get the following system:

$$\begin{array}{rcl} \dot{x}(t) & = & \tilde{A}x(t) + BL^{1/2}w(t) \\ z_2(t) & = & L^{1/2}(Cx(t) + DL^{1/2}w(t)) \\ w(t) & = & \Theta(\rho)z(t) \end{array} \quad (1.141)$$

Finally, applying the bounded-real lemma on scaled system (1.141) we get the matrix inequality in P and $L^{1/2}$:

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & PBL^{1/2} & C^T L^{1/2} \\ \star & -I & L^{1/2}D^T L^{1/2} \\ \star & \star & -I \end{bmatrix} \prec 0$$

Performing a congruence transformation with respect to matrix $\text{diag}(I, L^{1/2}, L^{1/2})$ yields the following result:

Lemma 1.3.47 *System (1.109) is asymptotically stable if there exist* $P = P^T \succ 0$ *and* $L \in \mathcal{S}_D(\Theta)$ *such that*

$$\begin{bmatrix} A^TP + PA & PB & C^TL \\ \star & -L & D^TL \\ \star & \star & -L \end{bmatrix} \prec 0 \tag{1.142}$$

Note that, if $L = I$, the condition of the small-gain theorem is retrieved.

Another vision of the scaled-small gain lemma, is the problem of finding a bijective and sign preserving ($L \succ 0$) change of variable for signals involved in the interconnection, such that the system behavior is unchanged (role of the commutation). A suitable change of variable is then given by matrices $L^{1/2}$ and $L^{-1/2}$. Several different approaches can be used to establish such a result, for instance using the $\mathcal{S}$-procedure (see the next section and Appendix E.12), or the bounding lemma (see appendix E.14).

The scaled-small gain theorem leads to less conservative result than the small-gain but only considers the structure of the parameter varying matrix $\Theta(\rho)$. This is the reason why, for small uncertainties, the result is necessary and sufficient [Packard and Doyle, 1993] (actually if the sum of the number repeated scalar blocks and the unrepeated full-blocks is lower than 3). For larger uncertainties, too low information is taken into account on how the two subsystems are interconnected (the shape of the operators). Indeed, this information is destroyed again by the use of norms which gathers multiple data (each entry of matrices) into one unique positive scalar value (see the Examples 1.3.42, 1.3.43 and 1.3.45).

The next idea would be to find a better framework in which the shape of the operators can be better characterized and considered, avoiding then the use of coarse norms (e.g. the $\mathcal{H}_\infty$). The next section on the full-block $\mathcal{S}$-procedure and the notion of well-posedness of feedback systems, partially solves this problem.

1.3.4.4 Full-Block $\mathcal{S}$-procedure and Well-Posedness of Feedback Systems

Both recent results have brought many improvements in the field of LPV system analysis and LPV control. We have chosen to present them simultaneously since they are two facets of the same theory but are proved using different fundamental theories.

Full-Block $\mathcal{S}$-procedure

The full-block $\mathcal{S}$-procedure has been developed in several research papers [Scherer, 1996, 1997, 1999, 2001, Scherer and Hol, 2006] and applied to several topics [Münz et al., 2008, Wu, 2003]. In [Briat et al., 2008c], we have developed a delay-dependent stabilization test using this approach and is an extension of the results of [Wu, 2003] which considers delay-independent stability.

In Section **??**, this approach will be used in order to derive a delay-dependent robust parameter dependent state-feedback control law for time-delay systems .

This approach is based on the theory of dissipativity (see appendix E.1 and [Scherer and Wieland, 2005] for more details on dissipativity) but to avoid too much (and sometimes tough) explanations, the fundamental result of the full-block $\mathcal{S}$-procedure will be retrieved here through a simple application of the $\mathcal{S}$-procedure (see appendix E.9 and [Boyd et al., 1994]).

Let us consider system (1.109) where $\Theta(\rho)$ is neither diagonal and possibly has a full structure. We also relax the image set of the parameters to a more general compact set, we hence assume that

$$\rho \in \times_{i=1}^{p}[\rho_i^-,\ \rho_i^+] \tag{1.143}$$

The key idea of the full-block $\mathcal{S}$-procedure is to characterize the parameter matrix $\Theta(\rho)$ in a more complex way allowing for a tighter approximation of the set in which the parameter matrix $\Theta(\rho)$ evolves. This is performed using an integral quadratic constraint (IQC). Indeed assume that there exists a bounded matrix function of time $M_\Theta(t) = M_\Theta(t)^T$ such that

$$\int_0^t \begin{bmatrix} z(s) \\ w(s) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} z(s) \\ w(s) \end{bmatrix} ds \succ 0 \tag{1.144}$$

for all $t > 0$ and $w(t) = \Theta(\rho)z(t)$. The latter IQC is equivalent to

$$\int_0^t z(s)^T \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix} z(s) ds \succ 0 \tag{1.145}$$

It is clear that the matrix M_Θ has, a priori, no imposed inertia. It will be shown at the end of this section that it is possible to define specific matrices M_Θ for which previous results are retrieved (passivity, small-gain and scaled small-gain results). Hence, the current framework should provide less conservative results.

Hence, the following Lyapunov function is considered

$$V(x(t)) = x(t)^T P x(t) > 0 \tag{1.146}$$

with constraint on input and output signals w and z taking the form of the IQC (1.144).

Thus, by invoking the $\mathcal{S}$-procedure (see Appendix E.9) or the theory of dissipative systems (see [Scherer and Wieland, 2005] or E.1), the following function is constructed

$$\begin{aligned} H(x(t), w) &= x(t)^T P x(t) - \int_0^t \begin{bmatrix} z(s) \\ w(s) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} z(s) \\ w(s) \end{bmatrix} ds \\ &= x(t)^T P x(t) - \int_0^t z(s)^T \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix}^T M_\Theta(s) \begin{bmatrix} I \\ \Theta(\rho(s)) \end{bmatrix} z(s) ds \end{aligned} \tag{1.147}$$

Since, the integrand of (1.145) is a quadratic form and $z \in \mathcal{L}_2$ in a general signal, then inequality (1.145) holds if and only if

$$\begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix}^T M_\Theta(t) \begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix} \succ 0 \tag{1.148}$$

for any trajectories tracked by $\rho(t) \in \times_{i=1}^{p}[\rho_i^-,\ \rho_i^+]$ and all $t > 0$.

Finally, computing the time-derivative of H leads to the result:

Theorem 1.3.48 *System (1.109) is asymptotically stable if and only if there exist a matrix $P = P^T \succ 0$ and a bounded matrix function $M_\Theta : \mathbb{R}^+ \to \mathbb{S}^{n_w+n_z}$ such that the LMIs*

$$\begin{bmatrix} A^T P + PA & PE \\ \star & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T M_\Theta(t) \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0 \tag{1.149}$$

$$\begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix}^T M_\Theta(t) \begin{bmatrix} I \\ \Theta(\rho(t)) \end{bmatrix} \succ 0 \tag{1.150}$$

hold for all $t > 0$.

Proof: *A complete proof with meaningful discussions can be found in [Scherer, 1997, 1999, 2001].* □

The main difficulty in such a result resides the computation of LMI (1.150). Even if M_Θ is chosen constant, we are faced to a problem involving infinitely many inequalities since the inequality should be satisfied for any parameter trajectories. Methods for dealing with such parameter dependent LMIs have been introduced in Sections 1.3.3.2, 1.3.3.3 and 1.3.3.4 where gridding, SOS and global polynomial optimization approaches are introduced.

It is important to point out that, due to the losslessness of the $\mathcal{S}$-procedure for 1-constrained quadratic functions (see appendix E.9 and [Boyd et al., 1994]), the conservativeness of the approach stems from the choice of $M_\Theta(s)$ satisfying LMI (1.150). Moreover, by simplicity in most of the cases this matrix is chosen to be constant.

Well-Posedness Approach

The well-posedness approach is now compared to the full-block $\mathcal{S}$-procedure . This approach has been initially introduced in [Iwasaki and Hara, 1998] and deployed in many frameworks [Gouaisbaut and Peaucelle, 2006a, 2007, Iwasaki, 2000, 1998, Langbort et al., 2004, Peaucelle and Arzelier, 2005, Peaucelle et al., 2007]. The key idea behind well-posedness is the notion of topological separation [Safonov, 2000, Teel, 1996, Zames, 1966] and is explained in what follows.

Consider two interconnected systems H_1 and H_2 such that

$$\begin{array}{rcl} z & = & H_1(w+u_1) \\ w & = & H_2(z+u_2) \end{array} \tag{1.151}$$

where u_1, u_2 are exogenous signals as described in Figure 1.18. Note that here, systems H_1 and H_2 are not dynamical systems but only general applications from a vector space to another.

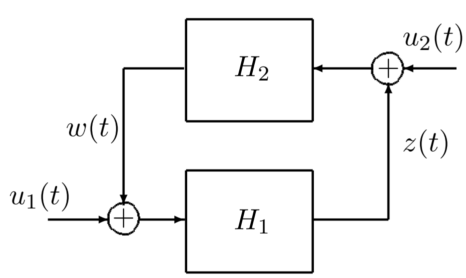

Figure 1.18: Setup of the well-posedness framework

It is convenient to introduce here the definition of well-posedness:

Definition 1.3.49 *The interconnection depicted in Figure 1.18 is said to be well-posed if and only if the loop-signals z, w are uniquely defined by the input signals u_1, u_2 and initial values of the loop-signals. In other terms, it is equivalent to the existence of a positive scalar $\gamma > 0$ such that the energy of loop signals is bounded by a function of the energy of input signals, i.e.*

$$\left|\left|\begin{pmatrix} z \\ w \end{pmatrix}\right|\right|_{\mathcal{L}_2} \leq \gamma \left|\left|\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right|\right|_{\mathcal{L}_2}$$

The latter definition says that if for finite energy input signals, we get finite energy loop-signals then the system is well-posed. Recall that the notion of stability is not defined yet since operators H_1 and H_2 are general and not necessarily dynamic.

The idea behind well-posedness is to prove well-posedness of the interconnection when the interconnection describes a dynamical system. In this case, well-posedness is equivalent to asymptotic stability or equivalently $\mathcal{L}_2 - \mathcal{L}_2$ stability. The following example shows how a dynamical system can be represented in an interconnection as of Figure 1.18.

Example 1.3.50 *Let us consider the trivial linear dynamical system described by $\dot{x} = A(x(t)+v(t))$ where $v(t)$ is an external input.. First, note that by imposing $H_1 = A$ and $H_2 = s^{-1}$ where s is the Laplace variable, then the interconnection depicted in Figure 1.19 is equivalent to system $\dot{x} = A(x(t) + v(t))$.*

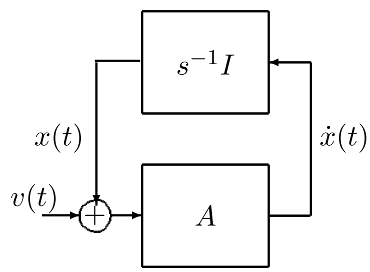

Figure 1.19: Representation of a linear time invariant dynamical system in the well-posedness framework

Now suppose that the interconnection is well-posed: the future evolution of $x(t)$ for $t > t_0$ is uniquely defined by $x(t_0)$ and signal $v(t)$ for all $s \in \mathbb{C}^+$. We aim now to illustrate that well-posedness in this case is equivalent to asymptotic (and even exponential) stability.

From Figure 1.19 we have

$$\dot{x} = A(x + v)$$

which is equivalent to

$$sx = A(x + v) + x(t_0)$$

and thus

$$(sI - A)x = Av + x(t_0)$$

Therefore if $x(t)$ is uniquely defined by $v(t)$ and $x(t_0)$ for $t > t_0$, this means that $(sI - A)$ is nonsingular for all $s \in \mathbb{C}^+$. This condition is equivalent to saying that A has no eigenvalues in the complex right-half plane and that A is a stable matrix.

On the other hand, suppose that A is Hurwitz then this means that $sI - A$ is non singular for all $s \in \mathbb{C}^+$ and hence x is uniquely defined by $v(t)$ and $x(t_0)$. Thus the interconnection is well-posed.

We aim now at introducing how well-posedness can be proved efficiently (using numerical tools), at least for linear dynamical systems. This is performed trough nice geometrical arguments.

Coming back to the setup depicted in Figure 1.18, let $\mathcal{G}_1$ and $\mathcal{G}_2^-$ be respectively the graph of H_1 and the inverse graph of H_2 defined as:

$$\begin{aligned}\mathcal{G}_1 &:= \left\{\begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{R}^{n_w+n_z} :\ z = H_1 w\right\} = \operatorname{Im}\begin{pmatrix} H_1 \\ I \end{pmatrix} \\ \mathcal{G}_2^- &:= \left\{\begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{R}^{n_w+n_z} :\ w = H_2 z\right\} = \operatorname{Im}\begin{pmatrix} I \\ H_2 \end{pmatrix}\end{aligned} \tag{1.152}$$

where n_w and n_z denote respectively the dimension of w and z. We have the following important result:

Proposition 1.3.51 *Interconnection (1.151) is well-posed if and only if the following relation holds:*

$$\mathcal{G}_1 \cap \mathcal{G}_2^- = \{0\} \tag{1.153}$$

In order to visualize this important result, let us consider the case where $z = H_1(w + u_1)$ and $w = H_2(z + u_2)$, $H_1 \in \mathbb{R}^{n_z \times n_w}$ and $H_2 \in \mathbb{R}^{n_w \times n_z}$. The graphs are then given by

$$\begin{aligned}\mathcal{G}_1 &= \operatorname{Im}\begin{pmatrix} H_1 \\ 1 \end{pmatrix} \\ \mathcal{G}_2^- &= \operatorname{Im}\begin{pmatrix} 1 \\ H_2 \end{pmatrix}\end{aligned} \tag{1.154}$$

We aim now to find the intersection of these sets and we get the following system of linear equations

$$\begin{aligned} H_1 w - z &= 0 \\ w - H_2 z &= 0 \end{aligned} \tag{1.155}$$

Converted to a matrix form, it becomes

$$H\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.156}$$

where $H = \begin{bmatrix} -I_{n_z} & H_1 \\ -H_2 & I_{n_w} \end{bmatrix}$.

If $\det(H) = 0$ then there exists a infinite number of vectors $\begin{bmatrix} z \\ w \end{bmatrix}$ such that (1.156) is satisfied and thus the interconnection is not well-posed. Moreover, in this case we have $\det(I - H_2 H_1) = 0$ and and the null-space is spanned by $\begin{pmatrix} H_1 \\ I \end{pmatrix}$.

If the matrix H is non singular, then the null-space reduces to the singleton $\{0\}$ and the system is well-posed since the intersection of the graphs is $\{0\}$. It is important to point out that the following relation holds in any case

$$\mathcal{G}_1 \cap \mathcal{G}_2^- = \operatorname{Null}\begin{pmatrix} -I & H_1 \\ -H_2 & I \end{pmatrix} \tag{1.157}$$

Therefore, the problem of determining if an interconnection of systems is well-posed is crucial in the framework of interconnected dynamical systems and reduces to the analysis of the intersection of graphs (or equivalently to matrix algebra for linear mappings H_1 and H_2).

The idea now is to find a simple way to prove that the graphs do not intersect except at 0. In what follows, the framework is restricted to linear mappings and in this case, the graphs become convex sets, which is an interesting property. First, recall a fundamental result on convex analysis called the *Separating Hyperplane Theorem*; [Boyd and Vandenbergue, 2004, p. 46].

Theorem 1.3.52 *Suppose C_1 and C_2 are two convex sets that do not intersect (i.e. $C_1 \cap C_2 = \emptyset$). Then there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C_1$ and $a^T x \geq b$ for all $x \in C_2$. In other words, the affine function $a^T x - b$ is nonpositive on C_1 and nonnegative on C_2. The hyperplane $\{x: \ a^T x = b\}$ is called a separating hyperplane for the sets C_1 and C_2, or is said to separate the sets C_1 and C_2.*

This results says that two convex sets are disjoints if and only if one can find a function which is positive on one set and negative on the others. The latter result applied to the separation of graphs $\mathcal{G}_1$ and $\mathcal{G}_2^-$ leads to the following theorem proved in [Iwasaki, 2000, Iwasaki and Hara, 1998]:

Theorem 1.3.53 *The following statements are equivalent:*

1. *The interconnection of system is well-posed*
2. $\det(I - H_2 H_1) \neq 0$
3. *There exist $M = M^T$ such that*
   a) $\begin{bmatrix} I & H_1 \end{bmatrix} M \begin{bmatrix} I \\ H_1^* \end{bmatrix} \prec 0$
   b) $\begin{bmatrix} H_2 & I \end{bmatrix} M \begin{bmatrix} H_2^* \\ I \end{bmatrix} \succeq 0$

The following example illustrates the method.

Example 1.3.54 *Let us consider again the LTI system of Example 1.3.50 and define $H_1 := A$ and $H_2 := s^{-1} I$. From Theorem 1.3.53, the system is asymptotically (exponentially) stable if and only if the LMIs hold*

$$\begin{bmatrix} I & A \end{bmatrix} M \begin{bmatrix} I \\ A^T \end{bmatrix} \prec 0 \qquad \text{and} \qquad \begin{bmatrix} s^{-1} I & I \end{bmatrix} M \begin{bmatrix} s^{-*} I \\ I \end{bmatrix} \succeq 0$$

As in Example 1.3.50, the well-posedness of the interconnection is sought for all $s \in \mathbb{C}^+$. Indeed, this would mean that A has no eigenvalues in $\mathbb{C}^+$ implying that the system is asymptotically stable.

It is clear that if M is chosen to be $M := \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$ with $X = X^T \succ 0$, then

$$\begin{bmatrix} s^{-1} I & I \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} s^{-*} I \\ I \end{bmatrix} = (s^{-1} + s^{-*}) X = 2\Re[s^{-1}] X \succeq 0 \qquad \text{since } s \in \mathbb{C}^+$$

Therefore, the stability of system $\dot{x} = Ax$ is ensured if and only if

$$\begin{bmatrix} I & A \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} I \\ A^T \end{bmatrix} \prec 0$$

which is equivalent to

$$AX + XA^T \prec 0 \Leftrightarrow PA + A^T P \prec 0 \tag{1.158}$$

where $P = X^{-1}$. The well-known LMI condition obtained from Lyapunov theorem is retrieved.

It may be thought that the choice of $M = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$ would be conservative but in fact it is not. This explained in what follows.

List of scalings

We aim now at presenting several scalings/separators that may be used in both full-block $\mathcal{S}$-procedure and well-posedness approaches; [Iwasaki and Hara, 1998].

First of all, let us introduce the P-separator. Suppose that H_2 is block diagonal and satisfies

$$\begin{bmatrix} H_2 & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ \star & M_{22} \end{bmatrix} \begin{bmatrix} H_2^* \\ I \end{bmatrix} \succeq 0$$

with fixed matrices M_{ij}. Then under the assumption that $M_{22} \succ 0$ and $M_{11} - M^{12} M_{22}^{-1} M_{12}^T \prec 0$, the P-separator defined as ($\otimes$ denotes the Kronecker product):

$$P \otimes M = \begin{bmatrix} P \otimes M_{11} & P \otimes M_{12} \\ \star & P \otimes M_{22} \end{bmatrix} \tag{1.159}$$

provides a nonconservative condition if $2c + f \leq 3$ where c is the number of repeated scalar blocks in H_2 and f is the number of unrepeated full-blocks in H_2 (see [Iwasaki and Hara, 1998, Packard and Doyle, 1993] for more details). For instance, in example 1.3.54, $s = 1$ (only s^{-1} is repeated) and $f = 0$ (no full-blocks). Hence condition (1.158) is a necessary and sufficient condition to stability of system $\dot{x} = Ax$ (which is actually a well-known result).

Example 1.3.55 *Let us consider the latter example. The set of values of $s \in \mathbb{C}^+$ can be defined in an implicit way*

$$\mathbb{C}^+ := \left\{ s \in \mathbb{C} : \begin{bmatrix} s^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s^{-*} \\ 1 \end{bmatrix} \geq 0 \right\}$$

Using the P-separator we get the matrix $M = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}$ and moreover since we have one repeated scalar block then the P-separator provides a nonconservative stability condition.

The following (non-exhaustive) list enumerates specific scalings/separators for different types of operators H_2:

1. The constant scaling $M = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ for positive operators results in a passivity based test.

2. The constant scaling $M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ for unitary norm bounded operators results in a test based on the bounded real lemma.

3. The constant D-scalings $M = \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}$ for $D = D^T > 0$ (for unitary norm bounded operators) are the most simple ones and the result of the scaled-bounded real lemma are retrieved.

4. The constant D-G scalings $M = \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix}$ for $D = D^T > 0$ and $G + G^* = 0$ (for unitary norm bounded operators) are a generalization of the constant D-scalings to a more general case.

5. LFT scalings: $M = \begin{bmatrix} R & S \\ S^* & Q \end{bmatrix}$ with $R \prec 0$ and $\begin{bmatrix} \Theta_k & I \end{bmatrix} M \begin{bmatrix} \Theta_k^T \\ I \end{bmatrix} \succeq 0$ for $k = 1, \ldots, 2^\alpha$ (time-invariant and time-varying parameters).

6. Vertex separators $M = \begin{bmatrix} R & S \\ S^* & Q \end{bmatrix}$ with $R_{ii} \preceq 0$ and $\begin{bmatrix} \Theta_k & I \end{bmatrix} M \begin{bmatrix} \Theta_k^T \\ I \end{bmatrix} \succeq 0$ for $k = 1, \ldots, 2^\alpha$ (time-invariant and time-varying parameters).

These separators lead to less and less conservative results despite of increasing the computational complexity.

After this brief description of well-posedness, we wish now to supply a LPV system description in the well-posedness framework. Let us rewrite the LPV system (1.109) into the form:

$$\begin{bmatrix} z(t) \\ \dot{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} D & C \\ B & A \end{bmatrix}}_{H_1} \begin{bmatrix} w(t) \\ x(t) \end{bmatrix} \tag{1.160}$$

with $\begin{bmatrix} w(t) \\ x(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \Theta(\rho) & 0 \\ 0 & s^{-1}I \end{bmatrix}}_{H_2} \begin{bmatrix} z(t) \\ \dot{x}(t) \end{bmatrix}$ for $s \in \mathbb{C}^+$.

Following previous results, well-posedness of the latter system is equivalent to the asymptotic (exponential) stability of system (1.109). In this case, M can be chosen as

$$M := \left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0 \\ 0 & 0 & 0 & P \\ \hline M_{12}^T & 0 & M_{22} & 0 \\ 0 & P & 0 & 0 \end{array}\right] \tag{1.161}$$

where $P = P^T \succ 0$, $M_{11} = M_{11}^T$, M_{12} and $M_{22} = M_{22}^T$ are free matrices to be determined. Note that the matrix M contains both a P-separator involving the matrix X (for the stability condition) and a full-separator $M = [M_{ij}]_{i,j}$ (for the parameter consideration).

Now applying Theorem 1.3.53, we get

$$\begin{array}{l}\left[\begin{array}{cc|cc} I & 0 & D & C \\ 0 & I & B & A\end{array}\right]\left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0\\ 0 & 0 & 0 & P\\ \hline M_{12}^T & 0 & M_{22} & 0\\ 0 & P & 0 & 0\end{array}\right]\left[\begin{array}{cc} I & 0\\ 0 & I\\ \hline D^T & B^T\\ C^T & A^T\end{array}\right] \prec 0\\ \left[\begin{array}{cc|cc} \Theta(\rho) & 0 & I & 0 \\ 0 & s^{-1}I & 0 & I\end{array}\right]\left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0\\ 0 & 0 & 0 & P\\ \hline M_{12}^T & 0 & M_{22} & 0\\ 0 & P & 0 & 0\end{array}\right]\left[\begin{array}{cc} \Theta(\rho) & 0\\ 0 & s^{-1}I\\ \hline I & 0\\ 0 & I\end{array}\right] \succeq 0\end{array} \tag{1.162}$$

Expanding the relations, we get

$$\Re[s^{-1}]P \succeq 0 \tag{1.163}$$

$$\begin{bmatrix} \Theta(\rho) & I \end{bmatrix}\begin{bmatrix} M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix} \Theta(\rho)^T\\ I\end{bmatrix} \succeq 0 \tag{1.164}$$

$$\begin{bmatrix} 0 & CP\\ \star & AP+PA^T\end{bmatrix} + \begin{bmatrix} I & D\\ 0 & B\end{bmatrix}\begin{bmatrix} M_{11} & M_{12}\\ \star & M_{22}\end{bmatrix}\begin{bmatrix} I & 0\\ D^T & B^T\end{bmatrix} \prec 0 \tag{1.165}$$

Inequality (1.163) is satisfied by assumption therefore only inequalities (1.164) and (1.165) have to be (numerically) checked. In order to bridge results from full-block $\mathcal{S}$-procedure and well-posedness, we will show that inequalities (1.164) and (1.165) are equivalent to (1.149) and (1.150).

Note that (1.165) can be rewritten into the form

$$\left[\begin{array}{cc} 0 & C\\ I & A\\ \hline I & D\\ 0 & B\end{array}\right]^T\left[\begin{array}{cc|cc} 0 & P & 0 & 0\\ P & 0 & 0 & 0\\ \hline 0 & 0 & M_{11} & M_{12}\\ 0 & 0 & M_{12}^T & M_{22}\end{array}\right]\left[\begin{array}{cc} 0 & C\\ I & A\\ \hline I & D\\ 0 & B\end{array}\right] \prec 0 \tag{1.166}$$

It is possible to show that the dualization lemma applies (see Appendix E.13 or [Iwasaki and Hara, 1998, Scherer and Wieland, 2005, Wu, 2003]) and then LMI (1.166) is equivalent to

$$\left[\begin{array}{cc} -A & -B\\ I & 0\\ \hline -C & -D\\ 0 & I\end{array}\right]^T\left[\begin{array}{cc|cc} 0 & \tilde{P} & 0 & 0\\ \tilde{P} & 0 & 0 & 0\\ \hline 0 & 0 & \tilde{M}_{11} & \tilde{M}_{12}\\ 0 & 0 & \tilde{M}_{12}^T & \tilde{M}_{22}\end{array}\right]\left[\begin{array}{cc} -A & -B\\ I & 0\\ \hline -C & -D\\ 0 & I\end{array}\right] \succ 0 \tag{1.167}$$

where $\tilde{P} = P^{-1}$ and $\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix} = \begin{bmatrix} M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}^{-1}$.

By expanding the latter inequality we get

$$\begin{bmatrix} -A^T\tilde{P}-\tilde{P}A & -\tilde{P}B\\ \star & 0\end{bmatrix} + \begin{bmatrix} -C & -D\\ 0 & I\end{bmatrix}^T\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12}\\ \tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}\begin{bmatrix} -C & -D\\ 0 & I\end{bmatrix} \succ 0 \tag{1.168}$$

or equivalently

$$\begin{bmatrix} -A^T\tilde{P}-\tilde{P}A & -\tilde{P}B\\ \star & 0\end{bmatrix} + \begin{bmatrix} C & D\\ 0 & I\end{bmatrix}^T\begin{bmatrix} \tilde{M}_{11} & -\tilde{M}_{12}\\ -\tilde{M}_{12}^T & \tilde{M}_{22}\end{bmatrix}\begin{bmatrix} C & D\\ 0 & I\end{bmatrix} \succ 0 \tag{1.169}$$

Moreover, in virtue of the dualization lemma again, we have

$$\begin{bmatrix} \Theta(\rho) & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \begin{bmatrix} \Theta(\rho)^T \\ I \end{bmatrix} \succeq 0 \Longleftrightarrow \begin{bmatrix} -I & \Theta(\rho)^T \end{bmatrix} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} -I \\ \Theta(\rho) \end{bmatrix} \prec 0 \tag{1.170}$$

and equivalently

$$\begin{bmatrix} I & \Theta(\rho)^T \end{bmatrix} \begin{bmatrix} \tilde{M}_{11} & -\tilde{M}_{12} \\ -\tilde{M}_{12}^T & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} I \\ \Theta(\rho) \end{bmatrix} \prec 0 \tag{1.171}$$

Finally, by multiplying inequalities (1.169) and (1.171) by -1, we get

$$\begin{array}{rcl} \begin{bmatrix} A^T\tilde{P} + \tilde{P}A & \tilde{P}B \\ \star & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -\tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & -\tilde{M}_{22} \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} & \succ & 0 \\ \begin{bmatrix} I & \Theta(\rho)^T \end{bmatrix} \begin{bmatrix} -\tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & -\tilde{M}_{22} \end{bmatrix} \begin{bmatrix} I \\ \Theta(\rho) \end{bmatrix} & \prec & 0 \end{array} \tag{1.172}$$

By identification these latter relations are identical to (1.149) and (1.150) obtained by application of the full-block $\mathcal{S}$-procedure where $M_\Theta = \begin{bmatrix} -\tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & -\tilde{M}_{22} \end{bmatrix}$ and $\tilde{P}$ plays the role of the Lyapunov matrix used to define the quadratic Lyapunov function.

This emphasizes the similarities between the results obtained from the full-block $\mathcal{S}$-procedure and the well-posedness approach. It is important to point out that in every methods presented up to here, only quadratic stability was considered and may result in conservative stability conditions. Robust stability is addressed here in the framework of well-posedness of feedback systems. The procedure used here can be applied to any approach presented in preceding sections. The main reasons for presenting robust stability for LFT systems at this stage only now is the simplicity of the well-posedness approach. Moreover, as we shall see later, it is possible to connect these results to parameter dependent Lyapunov functions results introduced in Section 1.3.3.

LPV systems as implicit systems

The following method has been developed in [Iwasaki, 1998] but some other methods have been developed in order to define a LPV system as an implicit system [Masubuchi and Suzuki, 2008, Scherer, 2001]. It is convenient to introduce the following result on well-posedness of implicit systems. This is the generalization of well-posedness theory for dynamical system governed by expressions of the form

$$\begin{bmatrix} \mathcal{A} - sI & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \zeta = 0 \tag{1.173}$$

where s is the Laplace variable and ζ are signals involved in the system. Such an expression describes a linear-time invariant dynamical system coupled with a static equality between signals. This type of systems is not without recalling singular systems in which static relations are captured in a matrix E factoring the time-derivative of x (e.g. $E\dot{x} = Ax$). As an illustration of the formalism, system (1.173) represents the wide class of systems governed by equations:

$$\begin{array}{rcl} \dot{x} & = & \mathcal{A}x + \mathcal{B}w \\ 0 & = & \mathcal{C}x + \mathcal{D}w \end{array} \tag{1.174}$$

when $\zeta = \text{col}(x, w)$ which in turn can be rewritten in the singular system form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{1.175}$$

Some fundamental definitions on implicit systems are recalled here for informational purpose.

Definition 1.3.56 *The implicit system (1.173) is said to be regular if the following conditions hold:*

1. *There is no impulsive solution, i.e. the system is impulse free [Verghese et al., 1981];*
2. *for each $x(0^-)$, the solution, if any, is unique.*

One of the particularities of such systems is that under certain circumstances (i.e. according system matrices), the state evolution may contain impulsive terms (i.e. Dirac pulses) of theoretically infinite amplitude. Moreover, it is also possible that no solutions exist or may be non unique. It is then important to characterize the regularity of implicit systems. The following lemma, proved in [Iwasaki, 1998], gives a necessary and sufficient condition for the system to be regular.

Lemma 1.3.57 *Implicit system (1.173) is regular if and only if $\mathcal{D}$ has full column rank.*

Definition 1.3.58 *System (1.173) is said to be stable if it is regular and for each $x(0^-)$ the solution, if any, coneverges to zero as t tends to $+\infty$.*

The latter definition generalizes the notion of stability of linear differential systems to linear differential systems with static equalities constraints.

The following lemmas [Iwasaki, 1998] provide necessary and sufficient conditions for stability and robust stability of (uncertain) implicit systems.

Lemma 1.3.59 *Consider implicit system (1.173). The following statements are equivalent:*

1. *The system is stable;*
2. *The matrix $\begin{bmatrix} \mathcal{A} - sI & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ has full-column rank for all $s \in \mathbb{C}^+ \cup \{\infty\}$;*
3. *$\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ has full-column rank for all $s \in \mathbb{C}^+ \cup \{\infty\}$.*

Let us consider now the uncertain implicit system governed by

$$\begin{bmatrix} \mathcal{A} - sI & \mathcal{B} & 0 \\ \mathcal{C} & \mathcal{D} & \Delta \end{bmatrix} \zeta = 0 \tag{1.176}$$

where $\Delta \in \mathbf{\Delta}$ is an unknown but constant matrix and ζ contains all signals involved the uncertain system. Under some technical assumptions and results which are not detailed here [Iwasaki, 1998], we have the following result on robust stability.

Lemma 1.3.60 *Consider the implicit system (1.176) where $\Delta \in \mathbf{\Delta}$. The following statements are equivalent:*

1. *Implicit system (1.176) is stable for all* $\Delta \in \mathbf{\Delta}$.

2. *for each* $\omega \in \mathbb{R} \cup \{\infty\}$, *there exists a Hermitian matrix* $\Pi(j\omega)$ *such that*

$$\begin{array}{rcl} [\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}]^T \Pi(j\omega) [\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}] & \prec & 0 \\ \Delta^T \Pi(j\omega) \Delta & \succeq & 0, \qquad \text{for all } \Delta \in \mathbf{\Delta} \end{array}$$

These inequalities have to be satisfied for all $\omega \in \mathbb{R} \cup \{\infty\}$ and may be difficult to solve. The reader should refer to Sections 1.3.3.1, 1.3.3 and 1.3.3.3 for more details on such parameter dependent LMIs. On the other hand if a constant matrix $\Pi(j\omega)$ is sought then the variable ω can be eliminated using an extension of the Kalman-Yakubovicth-Popov lemma [Iwasaki, 1998]. Finally, the following sufficient condition, given in terms of quadratic separation, is obtained for stability analysis of uncertain implicit systems.

Lemma 1.3.61 *Consider the uncertain implicit system (1.176). If there exist* $P = P^T$ *and* $\Pi \in \mathbf{\Pi}$ *such that*

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & P \\ 0 & \Pi & 0 \\ P & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ I & 0 \end{bmatrix} \prec 0 \tag{1.177}$$

where $\mathbf{\Pi} := \left\{\Pi : \ \Delta^T \Pi \Delta \succeq 0, \ \forall \ \Delta \in \mathbf{\Delta}\right\}$ *then the system is stable for all constant* $\Delta \in \mathbf{\Delta}$ *provided that there exists at least one* Δ_0 *such that (1.176) is stable. Moreover, if* $P \succ 0$ *then the system is stable for all time-varying* $\Delta(t) \in \mathbf{\Delta}$ *even if there is no* Δ_0 *such that (1.176) is stable.*

From these results we are able to provide a robust stability test for LPV systems. First of all, let us show how parameter variations can be taken into account. Differentiating z and w channels in (1.109) yields

$$\begin{array}{rcl} \dot{z} & = & C\dot{x} + D\dot{w} \\ & = & C\tilde{A}x + CBw + D\dot{w} \\ \dot{w} & = & \dot{\Theta}z + \Theta\dot{z} \end{array} \tag{1.178}$$

Finally defining $\phi = \dot{\Theta}z$ we have

$$\left[\begin{array}{c} \dot{x} \\ \dot{w} \\ \hline z \\ \dot{z} \\ z \\ \Theta z \\ \Theta \dot{z} \\ \dot{\Theta} z \end{array}\right] = \left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} x \\ w \\ \hline \dot{w} \\ \phi \end{array}\right] \tag{1.179}$$

Hence letting $\bar{\Theta} = \mathrm{diag}(\Theta, \Theta, \dot{\Theta}) \in \bar{\mathbf{\Theta}}$, $\Delta = \begin{bmatrix} I \\ \bar{\Theta} \end{bmatrix}$ and $\zeta = \mathrm{col}(x, w, \dot{w}, \phi, -z, -\dot{z}, -z)$ then form (1.176) is retrieved; i.e.

$$\left[\begin{array}{cc|cc|ccc} \tilde{A}-sI & B & 0 & 0 & 0 & 0 & 0 \\ 0 & -sI & I & 0 & 0 & 0 & 0 \\ \hline C & D & 0 & 0 & I & 0 & 0 \\ C\tilde{A} & CB & D & 0 & 0 & I & 0 \\ C & D & 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & \Theta & 0 & 0 \\ 0 & 0 & I & -I & 0 & \Theta & 0 \\ 0 & 0 & 0 & I & 0 & 0 & \dot{\Theta} \end{array}\right] \left[\begin{array}{c} x \\ w \\ \hline \dot{w} \\ \phi \\ \hline -z \\ -\dot{z} \\ -z \end{array}\right] = 0 \tag{1.180}$$

This leads to the following theorem [Iwasaki, 1998].

Theorem 1.3.62 *LPV system (1.109) is robustly stable for all* $\bar{\Theta} = \mathrm{diag}(\Theta, \Theta, \dot{\Theta}) \in \bar{\boldsymbol{\Theta}}$ *if there exist real symmetric matrices* P *and* $\bar{\Pi}$ *that*

$$\left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]^T \left[\begin{array}{c|c|c} 0 & 0 & P \\ 0 & \bar{\Pi} & 0 \\ P & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right] \prec 0$$
$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta(\rho) & 0 & 0 \\ 0 & \Theta(\rho) & 0 \\ 0 & 0 & \Theta(\dot{\rho}) \end{bmatrix}^T \bar{\Pi} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta(\rho) & 0 & 0 \\ 0 & \Theta(\rho) & 0 \\ 0 & 0 & \Theta(\dot{\rho}) \end{bmatrix} \succeq 0 \tag{1.181}$$

provided that $\tilde{A}$ *is Hurwitz.*

The interest of the following theorem is that the symmetric matrix P is not necessarily positive definite and provided less conservative conditions than by considering usual positivity requirement.

We aim at showing now that this stability condition can be interpreted in terms of a parameter dependent Lyapunov function depending on $\Theta(\rho)$. First of all, the LPV system (1.109) is rewritten in the compact form

$$\dot{x} = A_\Theta x := [\tilde{A} + B(I - \Theta(\rho)D)^{-1}\Theta(\rho)C]x$$

It is well-known that this system is stable if $(I - \Theta(\rho)D)$ is invertible for all $\Theta(\rho) \in \boldsymbol{\Theta}$ and there exists a parameter dependent Lyapunov function $V(x, \Theta) = x^T P_\Theta x$ such that $P_\Theta = P_\Theta^T \succ 0$ and

$$\dot{P}_\Theta + P_\Theta A_\Theta + A_\Theta^T P_\Theta \prec 0$$

for all $\Theta \in \boldsymbol{\Theta}$.

Let N_Θ be the matrix $N_\Theta := (I - \Theta D)^{-1}\Theta C$ such that we have $w = N_\Theta x$. Differentiating w yields

$$\dot{w} = \dot{N}_\Theta x + N_\Theta \dot{x} \tag{1.182}$$

where $\dot{N}_\Theta = (I - \Theta D)^{-1}\dot{\Theta}(I - \Theta D)^{-1}C$. Now construct a parameter dependent Lyapunov function $V(x) = x^T P_\Theta x$ with

$$P_\Theta = \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} \tag{1.183}$$

Then we have

$$\dot{P}_\Theta = \begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix}$$

and hence the Lyapunov inequality is given by

$$\dot{V}(x) = x^T(\dot{P}_\Theta + A_\Theta^T P_\Theta + P_\Theta A_\Theta)x < 0$$

for all $x \neq 0$. Moreover note that

$$\begin{aligned} \dot{x} &= A_\Theta x \\ &= \tilde{A}x + Bw \\ &= (\tilde{A} + BN_\Theta)x \\ \dot{w} &= \dot{N}_\Theta x + N_\Theta A_\Theta x \\ &= \dot{N}_\Theta x + N_\Theta(\tilde{A}x + Bw) \\ &= \left(\dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta)\right) x \end{aligned}$$

Hence $\dot{V}$ becomes

$$\begin{aligned} \dot{V}(t) &= x^T \left(\begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} 0 \\ \dot{N}_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} (\tilde{A} + BN_\Theta) \right. \\ &\qquad \left. + (\tilde{A} + BN_\Theta)^T \begin{bmatrix} I \\ N_\Theta \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} \right) x < 0 \\ &= x^T \left(\begin{bmatrix} \tilde{A} + BN_\Theta \\ \dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta) \end{bmatrix}^T P \begin{bmatrix} I \\ N_\Theta \end{bmatrix} + \begin{bmatrix} I \\ N_\Theta \end{bmatrix} P \begin{bmatrix} \tilde{A} + BN_\Theta \\ \dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta) \end{bmatrix} \right) x < 0 \end{aligned}$$

The following equalities hold

$$\begin{aligned} \begin{bmatrix} \tilde{A} + BN_\Theta \\ \dot{N}_\Theta + N_\Theta(\tilde{A} + BN_\Theta) \end{bmatrix} x &= \begin{bmatrix} \tilde{A} & B & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix} \\ \begin{bmatrix} I \\ N_\Theta \end{bmatrix} x &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix} \end{aligned}$$

And finally we get

$$\dot{V}(t) = \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{A}^T & 0 \\ B^T & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & B & 0 \\ 0 & 0 & I \end{bmatrix} \right) \begin{bmatrix} x \\ w \\ \dot{w} \end{bmatrix} < 0 \tag{1.184}$$

It is worth noting that in the latter condition, no information is taken into account about the parameters and their derivative. This is captured by the following static relations:

$$\begin{array}{rcl} & w & = & N_\Theta x \\ & & = & (I-\Theta D)^{-1}\Theta C x \\ \Rightarrow & 0 & = & \Theta C x + (\Theta D - I) w \\ & w & = & \Theta z = \Theta(Cx + Dw) \\ \Rightarrow & \dot{w} & = & \dot{\Theta} z + \Theta C \dot{x} + \Theta D \dot{w} \\ & & = & \eta + \Theta C \tilde{A} x + \Theta C B w + \Theta D \dot{w} \\ \Rightarrow & 0 & = & \Theta C \tilde{A} x + \Theta C B w + (\Theta D - I)\dot{w} \\ & \eta & = & \dot{\Theta}(Cx + Dw) \\ \Rightarrow & 0 & = & \dot{\Theta} C x + \dot{\Theta} D w - \eta \end{array}$$

where $\eta = \dot{\Theta} z$. Gathering these relations into a compact matrix form yields

$$\begin{bmatrix} \Theta C & \Theta D - I & 0 & 0 \\ \Theta C \tilde{A} & \Theta C B & \Theta D - I & I \\ \dot{\Theta} C & \dot{\Theta} D & 0 & -I \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix} = 0 \tag{1.185}$$

Then it follows that the Lyapunov inequality becomes

$$\begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix}^T \left(\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} + \begin{bmatrix} \tilde{A}^T & 0 \\ B^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix} < 0 \tag{1.186}$$

for all signals $\mathrm{col}(x, w, \dot{w}, \eta) \neq 0$ such that (1.185) holds.

Now rewrite (1.185) as a matrix product

$$\begin{bmatrix} \Theta C & \Theta D - I & 0 & 0 \\ \Theta C \tilde{A} & \Theta C B & \Theta D - I & I \\ \dot{\Theta} C & \dot{\Theta} D & 0 & -I \end{bmatrix} = \begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \tag{1.187}$$

It follows that the Lyapunov inequality is equivalent to (1.186) for all non zero vector $\mathrm{col}(x, w, \dot{w}, \eta)$ such that

$$\begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ \dot{w} \\ \eta \end{bmatrix} = 0$$

holds for some $\bar{\Theta} \in \bar{\boldsymbol{\Theta}}$ with $\bar{\Theta} = \mathrm{diag}(\Theta, \Theta, \dot{\Theta})$. This problem falls into the framework of the generalized Finsler's lemma (see Appendix E.17). It follows that the Lyapunov inequality feasibility is equivalent to the existence of symmetric matrices P and $\bar{\Pi}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} + \begin{bmatrix} \tilde{A}^T & 0 \\ B^T & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} P \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix}^T \bar{\Pi} \begin{bmatrix} C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \end{bmatrix} \prec 0$$
$$\mathrm{Ker}\left(\begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix}\right)^T \bar{\Pi} \mathrm{Ker}\left(\begin{bmatrix} \Theta & 0 & 0 & -I & 0 & 0 \\ 0 & \Theta & 0 & 0 & -I & 0 \\ 0 & 0 & \dot{\Theta} & 0 & 0 & -I \end{bmatrix}\right) \succeq 0$$

hold.

The first LMI is identical to

$$\left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]^T \left[\begin{array}{c|c|c} 0 & 0 & P \\ 0 & \bar{\Pi} & 0 \\ P & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cc} \tilde{A} & B & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline C & D & 0 & 0 \\ C\tilde{A} & CB & D & 0 \\ C & D & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & -I \\ 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right] \prec 0 \tag{1.188}$$

while the second one is identical to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \dot{\Theta} \end{bmatrix}^T \left[\begin{array}{c|c|c} 0 & 0 & P \\ 0 & \bar{\Pi} & 0 \\ P & 0 & 0 \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Theta & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \dot{\Theta} \end{bmatrix} \succeq 0 \tag{1.189}$$

These LMIs (1.188) and (1.189) are equivalent to LMIs provided in Theorem 1.3.62. This shows that the feasibility of the Lyapunov inequality implies the feasibility of LMIs of Theorem 1.3.62. By following the development backaward, this shows that feasibility of LMIs of Theorem 1.3.62 implies the existence of a parameter Lyapunov function $V(x) = x^T P_\Theta x$ where P_Θ is defined in (1.183).

We have shown in this section that full-block $\mathcal{S}$-procedure and well-posedness approach are equivalent. Moreover, they embed previously presented methods such as passivity and small-gain results. Using a variation of the well-posedness results extended to implicit systems it has been shown that it applies in both quadratic and robust stability. Moreover, the well-posedness allows for an explicit construction of a (parameter dependent) Lyapunov function proving the stability of the LPV system. It is worth noting that this Lyapunov function has the same dependence on parameters with than the system.

In Lemma 1.3.60, the separator $\Pi(j\omega)$ depends on the frequency variable ω and is relaxed to a constant matrix in order to provide tractable conditions. However, this simplification introduces some conservatism in the approach and it would be interesting to keep this dependence on ω in order to characterize, in the frequency domain, additional information on the parameters. Next sections are devoted to the introduction of methods in which constraints in the frequency domain are allowed: the first one to be presented is the extension of the full-block $\mathcal{S}$-procedure to allow for frequency dependent scaling while the second one uses Integral Quadratic Constraints (IQC) which try to confine the stability conditions into a least conservatism domain.

1.3.4.5 Frequency-Dependent Scalings

The use of frequency-dependent scalings with full-block $\mathcal{S}$-procedure is very recent and has been proposed in [Scherer and Köse, 2007a,b]. The idea is to replace the constant D-scalings by frequency-dependent scalings playing the role of dynamic filters, which will characterize the uncertainties/parameters in the frequency domain. Indeed, constant D-scalings allow to characterize the $\mathcal{H}_\infty$ (or induced $\mathcal{L}_2$ norm) over the whole frequency domain and results in conservative conditions if the parameters belong to a specific frequency domain (note that in certain cases, D-scalings are lossless as emphasized in [Iwasaki and Hara, 1998, Packard and Doyle, 1993] and Section 1.3.4.4 in the list of scalings).

Let us consider system (1.109) and suppose that

$$\Theta = \mathrm{diag}(\Theta_1(\rho), \dots, \Theta_q(\rho))$$

and $||\Theta(\rho)||_{\mathcal{H}_\infty} \leq 1$.

Frequency dependent D-scalings will consider the set $\mathcal{Q}$ of matrices structured as

$$Q(s) = \mathrm{diag}(q_1(s)I, \dots, q_m(s)I) \tag{1.190}$$

in correspondence with the structure of $\Theta(\rho)$ where the components q_i are SISO transfer functions, real valued and bounded on the imaginary axis $\mathbb{C}_0$. The stability of the LPV system is then guaranteed if there exists some multiplier $Q \in \mathcal{Q}$ for which

$$\begin{bmatrix} H(s) \\ I \end{bmatrix}^* \begin{bmatrix} Q(s) & 0 \\ 0 & -Q(s) \end{bmatrix} \begin{bmatrix} H(s) \\ I \end{bmatrix} \prec 0, \qquad Q(s) \succ 0 \text{ on } \mathbb{C}_0 \tag{1.191}$$

The key idea is to approximate any filter by a finite basis of elementary filters of the form

$$\begin{array}{rcl} f_{1,\kappa}(s) & = & \begin{bmatrix} 1 & f_1(s) & f_2(s) & \dots & f_\kappa(s) \end{bmatrix} \\ f_{2,\kappa}(s) & = & \begin{bmatrix} 1 & f_1(s)^* & f_2(s)^* & \dots & f^\kappa(s)^* \end{bmatrix} \end{array} \tag{1.192}$$

where $f_{1,\kappa}(s)$ and $f_{2,\kappa}(s)$ are respectively stable and anti-stable rows with $f(s)^* = f(-s)^T$. Let us recall that an anti-stable transfer function has all its poles in $\mathbb{C}^+$. Hence for sufficiently large κ any filter stable (anti-stable) can be uniformly approximated on $\mathbb{C}_0$ by $f_{1,\kappa}(s)l_1$ ($f_{2,\kappa}(s)l_2$) for suitable real-valued columns vectors l_1 (l_2) (see [Boyd and Barrat, 1991, Pinkus, 1985, Scherer, 1995]). This implies that $Q(s)$ can be approximated by

$$\Psi_1(s)^* M \Psi_1(s) = \Psi_2(s)^* M \Psi_2(s) \tag{1.193}$$

where $\Psi_j := \text{diag}(I \otimes f_{j,\kappa}^T, \ldots, I \otimes f_{j,\kappa}^T)$ and M is a symmetric matrix such that $M := \text{diag}(I \otimes M^1, \ldots, I \otimes M_m)$ in which the M^i's have to be determined.

We give here the main stability result which has been initially introduced in [Scherer and Köse, 2007a,b]:

Theorem 1.3.63 *A is stable and (1.191) holds for Q represented as (1.193) if and only if the following LMIs are feasible:*

$$\begin{bmatrix} I & 0 \\ A_p & B_p \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} 0 & X & 0 \\ \star & 0 & 0 \\ \star & \star & \text{diag}(M, -M) \end{bmatrix} \begin{bmatrix} I & 0 \\ A_p & B_p \\ C_p & D_p \end{bmatrix} \prec 0 \tag{1.194}$$

$$\begin{bmatrix} I & 0 \\ A_{\Psi_1} & B_{\Psi_1} \\ C_{\Psi_1} & D_{\Psi_1} \end{bmatrix}^T \begin{bmatrix} 0 & Y & 0 \\ \star & 0 & 0 \\ \star & \star & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A_p & B_p \\ C_p & D_p \end{bmatrix} \succ 0 \tag{1.195}$$

$$\begin{bmatrix} X_{11} - Y & X_{13} \\ \star & X_{33} \end{bmatrix} \succ 0 \tag{1.196}$$

where $\left[\begin{array}{c|c} A_{\Psi_1} & B_{\Psi_1} \\ \hline C_{\Psi_1} & D_{\Psi_1} \end{array}\right]$ *is a minimal realization of* Ψ_1 *and*

$$\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] := \left[\begin{array}{ccc|c} A_{\Psi_1} & 0 & B_{\Psi_1}C & D \\ 0 & A_{\Psi_2} & 0 & B_{\Psi_2} \\ 0 & 0 & A & B \\ \hline C_{\Psi_1} & 0 & D_{\Psi_1} & D_{\Psi_1}D \\ 0 & C_{\Psi_2} & 0 & D_{\Psi_2} \end{array}\right] \tag{1.197}$$

is a minimal realization of $\begin{bmatrix} \Psi_1 G \\ \Psi_2 \end{bmatrix}$.

The idea in this approach is to choose a basis of transfer functions on which the matrix $Q(s)$ will be approximated. The conservatism of the approach thus depends on the complexity (the completeness) of the basis and on the type of scalings used (here D scalings). This result is very near results based on integral quadratic constraints which are presented in the next section.

1.3.4.6 Analysis via Integral Quadratic Constraints (IQC)

This section is devoted to IQC analysis and is provided for informative purposes only [Rantzer and Megretski, 1997]. The key ideas, which are very similar to the well-posedness and full-block $\mathcal{S}$-procedure, are briefly explained hereafter.

The central idea of the IQC framework is identical to the well-posedness: the loop signals must be uniquely defined by the inputs and for bounded energy inputs, we have bounded energy loop signals ($\mathcal{L}_2$ internal stability). The first step is to define any blocks and signals involved in the interconnection by means of integral quadratic constraints of the form

$$\int_{-\infty}^{+\infty} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^T \left(M_q \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \right) dt \geq 0 \tag{1.198}$$

where M_q is a bounded self-adjoint operator on the $\mathcal{L}_2$ space.

With such an IQC, it is possible to capture and characterize many behaviors of operators and signals (see [Rantzer and Megretski, 1997] for a nonexhaustive list of such IQCs). Using Parseval equality (see Appendix E.22), the latter IQC has a frequency dependent counterpart

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* M_q(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{1.199}$$

where $\hat{z}$ and $\hat{w}$ denote respectively the Fourier transform of z and w.

The aim of the IQC is to study stability of interconnected systems by constraining all signals and operators involved in the interconnection using IQCs, expressed as well in the frequency domain as in the time-domain. These IQCs include extra degree of freedom and this is the reason why the larger the number of IQCs is, the smaller the conservatism is. Moreover, it is worth noting that a wide class of operators and signals can be characterized using IQC: periodic signals, constant signals, norm-bounded operators, constant and time-varying uncertainties, static nonlinearities or even operators with memory (such as delay operators)...

Example 1.3.64 *Let us consider the LPV system under 'LFT' form:*

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bw(t) \\ z(t) &=& Cx(t) + Dw(t) \\ w(t) &=& \Theta(t)z(t) \end{array}$$

where $\Theta(t)$ is a diagonal matrix gathering the parameters involved in the system. According to the type of set where the parameters evolve, it is possible to define an IQC to define such sets. For instance, if the parameters evolve within the internal $[-\alpha, \alpha]$, then the signals w and z satisfy the following IQC

$$\int_{-\infty}^{+\infty} \begin{bmatrix} z(\theta) \\ w(\theta) \end{bmatrix}^T \begin{bmatrix} \alpha^2 Q & 0 \\ 0 & -Q \end{bmatrix} \begin{bmatrix} z(\theta) \\ w(\theta) \end{bmatrix} d\theta$$

for some $Q = Q^T \prec 0$. More generally, we can retrieve the results of the full-block $\mathcal{S}$-procedure by considering that the values of parameter matrix $\Theta(\rho)$ evolve within an ellipsoid, i.e. if we have

$$\begin{bmatrix} I \\ \Theta(\rho) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ \Theta(\rho) \end{bmatrix} \prec 0$$

By pre and post multiplying the latter inequality by $z(t)^T$ and $z(t)$ and noting that $w(t) = \Theta(\rho)z(t)$ we have

$$\begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \prec 0$$

Taking the integral from $-\infty$ to $+\infty$ we get

$$\int_{-\infty}^{+\infty} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} dt \prec 0$$

which is an IQC corresponding to the supply-rate of full-block $\mathcal{S}$-procedure approach or the multiplier used in the well-posedness approach.

Once all signals and operators have been defined through IQCs, then by invoking the Kalmna-Yakubovicth-Popov lemma (see Appendix E.3), it is possible to obtain a LMI where the sum of all IQC's are present. The methodology is illustrated hereafter by considering the stability analysis of a LPV system.

Let us consider system (1.109) with transfer function H mapping w to z. We assume that signals z and w satisfy all the following IQCs:

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi_q(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{1.200}$$

for all $q = 1, \ldots, N$ when $\Pi_q(j\omega)$ are Hermitian frequency dependent matrices defining the IQC's. In this case, there exist matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and a set of symmetric real matrices $M_1, \ldots, M_N$ to be determined such that

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \Pi_q(j\omega) \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} = \begin{bmatrix} \tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B} \\ I \end{bmatrix}^* M_q \begin{bmatrix} \tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B} \\ I \end{bmatrix} \tag{1.201}$$

for all $q = 1, \ldots, N$. By application of the Kalman-Yakubovitch-Popov Lemma (see appendix E.3 and references [Rantzer, 1996, Scherer and Wieland, 2005, Willems, 1971, Yakubovitch]), it follows that there exists a matrix $P = P^T \succ 0$ such that

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \star & 0 \end{bmatrix} + \sum_{i=1}^{N} \begin{bmatrix} \tilde{C}^T & 0 \\ 0 & I \end{bmatrix} M_q \begin{bmatrix} \tilde{C} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{1.202}$$

For instance, let $N = 1$ and define $\Pi_1 = \begin{bmatrix} -M & 0 \\ 0 & M \end{bmatrix}$ where $M = M^T \succ 0$ is a matrix to be determined. If all the parameters ρ take values in the interval $[-1, 1]$ and signals w and z are defined such that $w = \Theta(\rho)z$. Then it is clear that

$$\int_{-\infty}^{+\infty} \begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix}^* \Pi_1 \begin{bmatrix} z(j\omega) \\ w(j\omega) \end{bmatrix} d\omega \geq 0 \tag{1.203}$$

Thus Π_1 defines an IQC for the loop signals z and w. Since Π_1 does not depend on the frequency then $(\tilde{A}, \tilde{B}, \tilde{C}) = (A, B, C)$ and hence we get the LMI

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{1.204}$$

which is equivalent to the existence of $P = P^T \succ 0$ and $\tilde{M} = \tilde{M} \succ 0$ such that

$$\begin{bmatrix} A^T P + PA & PB & C^T M \\ B^T P & -M & 0 \\ MC & 0 & -M \end{bmatrix} \prec 0 \tag{1.205}$$

We recognize above the scaled-bounded real lemma and this points out that results obtained from full-block multipliers and well-posedness can be retrieved with an appropriate choice of the multipliers $\Pi_q(j\omega)$ (see for instance [Rantzer and Megretski, 1997]). The main difference between the full-block $\mathcal{S}$-procedure extended to frequency dependent D-scalings and the IQC approach resides in the choice of the filters: $Q(s)$ and $\Pi(j\omega)$ for respectively the full-block $\mathcal{S}$-procedure and the IQC analysis. In the full-block $\mathcal{S}$-procedure a basis is chosen and a suitable filter is computed by SDP while in the IQC framework the filter is computed by hand and then degrees of freedom are inserted in the IQCs. It is worth mentioning that, at this stage, only D-scalings have been extended to depend on the frequency but one can easily imagine to extend a more general case of scalings leading then to a framework, closer to IQC analysis. It is worth noting that filters initially computed for IQC can be used in the full-block $\mathcal{S}$-procedure and each one of these would be an element of the basis, and in this case, both methods would be equivalent.

This ends the part on stability analysis of LPV systems in 'LFT' formulation. Several methods have been presented and relations between results emphasized. The methods provide less and less conservative results and the latter ones are promising. It is important to note that while IQC analysis is currently one of the most powerful for stability analysis, it is generally difficult to derive stabilization conditions in terms of LMIs without restricting too much the type of IQC. On the other hand, the full-block $\mathcal{S}$-procedure is well-dedicated to LPV control of LPV systems as emphasized in [Scherer, 2001, Scherer and Köse, 2007a] and always results in LMI conditions.

1.4 Chapter Conclusion

This chapter has provided an overview of LPV systems. First of all, a precise definition of parameters is given and several classes of parameters have been isolated: discontinuous, continuous and smooth continuous parameters. It is highlighted that some classes enjoy nice properties which can be exploited to provide more precise stability and synthesis tools, leading, for instance, to different notions of stability.

Secondly, three types of LPV systems are presented: polytopic LPV systems, polynomial LPV systems and 'LFT' systems. While the first one is particularly adapted for systems with a linear dependence on the parameters, it leads generally to a conservative representation of systems with non-affine parameter dependence. Examples are given to show the interest of such a representation. On the second hand, polynomial systems are better suited to deal with more general representation excluding rational dependence. Finally, 'LFT' systems are the most powerful representation since they allow to consider any type of parameter dependence, including rational relations.

Thirdly, stability analysis techniques for each type of LPV systems are presented. It is shown that LMIs have a crucial role in stability analysis of LPV systems. Indeed, they provide an efficient and simple way to deal with the stability of LPV systems as well as for LTI systems. However, due to the time-varying nature of LPV systems the LMIs are also parameter-varying and hence more difficult to verify.

It has been shown that in the polytopic framework this infinite set of LMIs can be equivalently characterized by a finite set by considering the LMI at the vertices of the polytope only. This is a powerful property that makes the polytopic approach widely used in the literature.

On the other hand, when considering polynomial systems LMIs are far more complicated: they include infinite-dimensional decision variables (decision variables which are functions) and we are confronted to parameter dependent LMIs. In this case, relaxations play a central role in order to reduce this computationally untractable problem into a tractable one. First of all, infinite-dimensional variables are projected over a chosen basis (generally polynomial) in order to bring back the problem to a finite-dimensional one. Since the parameter dependence is nonlinear, it is not sufficient in this case to consider the LMI at the vertices of the set of the parameters, except for very special cases. This is why several methods have been developed to relax this part, namely the gridding, Sum-of-Squares and polynomial optimization approaches. The first one proposes to grid the space of parameters and to consider the LMI at these points only. Although simple, this method is shown to be computationally very expensive and very imprecise: it is possible to find systems for which the instability cannot be proved but these systems are not dense in the set of all unstable systems. The second one, is based on recent results on Sum-of-Squares polynomials and is very efficient but may be very computationally expensive. The third one is based on the application of a recent result on polynomial optimization problems solved by a sequence of LMI relaxations. The two latter methods are in fact equivalent but considers different frameworks.

Finally, stability analysis of LPV systems under 'LFT' form is developed. Several approaches are presented and the different results are linked between each others using underlying theories and by emphasizing that same results can be obtained using them. Passivity, small-gain and scaled-small gain results are described and generalized through the full-block $\mathcal{S}$-procedure and the dissipativity framework. The well-posedness approach based on topological separation is then shown to be equivalent to the full-block $\mathcal{S}$-procedure . A generalization of the full-block $\mathcal{S}$-procedure involving frequency-dependent scalings is then provided and is put in contrast with the IQC approach which consider Integral Quadratic Constraints in order to specify the types of signals involved in the interconnection.

Chapter 2

Overview of Time-Delay Systems

TIME-DELAY SYSTEMS are a particular case of infinite dimensional systems in which the current evolution of the state is affected not only by current signal values but also by past values, which justifies the denomination of hereditary systems. Such systems have suggested more and more attention these past years due to their applicability to communication networks and many large scale systems. The other interest of time-delay resides in their ability to model transport, diffusion, propagation phenomena [Niculescu, 2001]. These systems can be viewed as an approximation of distributed systems governed by partial differential equations. For instance, it has been shown that the Dirichlet's control problem (boundary control of systems governed by partial differential equations) can be approximated by a delay differential systems; see for instance Hayami [1951], Moussa [1996], Niculescu [2001].

This chapter provides some background on time-delay systems, mainly on stability analysis. The chapter will focus especially on the stability analysis of delay differential equations using several modern techniques such as Lyapunov-Krasovskii functionals, the interconnection of systems and Integral Quadratic Constraints (IQC).

Section 2.1 will provide different models of representation for time-delay systems where especially functional differential equations are detailed. Among this representation, several types of time-delay systems are isolated, depending on the type of delays (constant or time-varying) and how they act on system signals.

Section 2.2 is devoted to stability analysis of time-delay systems. Indeed, this greater and greater attention has led to a large arsenal of techniques for modeling, stability analysis and control design that need to be introduced to give a wide (but incomplete) panorama of the whole field. Only key and original results will be explained due to space limitations and redundant approaches will be avoided. Indeed, many results, although formulated differently, are completely equivalent and in this case a single version will be provided with references to equivalent approaches.

For completeness, Appendix I is devoted to determine properties of controllability/stabilizability and observability/detectability of time-delay systems. Here also, as for LPV systems, the notions of controllability and observability may be defined in different ways.

The readers discovering the field of time-delay systems are heavily encouraged to read this chapter carefully to get the necessary background to read this thesis. The interested or thirsted of knowledge readers will find several references in each section to deepen their understanding of the domain.

Most of this chapter is based on the books [Gu et al., 2003, Kolmanovskii and Myshkis, 1999, Niculescu, 2001] and several published papers which will be cited when needed.

2.1 Representation of Time-Delay Systems

Three different representations are commonly used for modeling time-delay systems:

1. Differential equation with coefficients in a ring of operators: This framework has been developed early to study time-delay systems in Conte and Perdon [1995, 1996], Conte et al. [1997], Kamen [1978], Morse [1976], Perdon and Conte [1999], Picard and Lafay [1996], Sename et al. [1995].

   A linear time-delay system is governed by a following linear differential equation with coefficient in a module, e.g.

   $$\dot{x}(t) = A(\nabla)x(t) \tag{2.1}$$

   where in the general case $\nabla = \mathrm{col}_i(\nabla_i)$ is the vector of delay operators such that $x(t-h_i) = \nabla_i x(t)$. In this case, the coefficient of the A matrix is a multivariate polynomial in the variable ∇.

   Since the inverse of ∇ (the predictive operator $x(t+h_i) = \nabla_i^{-1}x(t)$) is undefined from a causality point of view, the operators ∇ of the matrix A belong, indeed, to a ring.

2. Differential equation on an infinite dimensional abstract linear space: This type of representation stems from the application of infinite dimensional systems theory to the case of time-delay system.

   Let us consider system

   $$\dot{x}(t) = Ax(t) + A_h x(t-h) \tag{2.2}$$

   where h is the delay and x is the system state.

   This system is completely characterized by the state

   $$\tilde{x} = \begin{bmatrix} x(t) \\ x_t(s) \end{bmatrix}$$

   for all $s \in [-h, 0]$ and $x_t(s) = x(t+s)$. The state-space is then the Hilbert space

   $$\mathbb{R}^n \times \mathcal{L}_2([-h, 0], \mathbb{R}^n)$$

   One can easily see that the state of the system contains a point in an Euclidian space $x(t)$ and a function of bounded energy , $x_t(s)$, the latter belonging to an infinite dimensional linear space. This motivates the denomination of 'Infinite Dimensional Abstract Linear Space' [Bensoussan et al., 2006, Curtain et al., 1994, Iftime et al., 2005, Meinsma and Zwart, 2000].

   In that state space, the system rewrites

   $$\frac{d}{dt}\begin{bmatrix} y(t) \\ x_t(\cdot) \end{bmatrix} = \mathcal{A}\begin{bmatrix} y(t) \\ x_t(\cdot) \end{bmatrix} \tag{2.3}$$

   where the operator $\mathcal{A}$ is given by

   $$\mathcal{A}\begin{bmatrix} y(t) \\ x_t(\cdot) \end{bmatrix} = \begin{bmatrix} Ay(t) + A_h x_t(-h) \\ \dfrac{dx_t(\theta)}{d\theta} \end{bmatrix} \tag{2.4}$$

The operator $\mathcal{A}$ is the infinite dimensional counterpart of the finite dimensional operator A in linear systems described by $\dot{x} = Ax$, and many tools involved in the theory of finite dimensional systems have been extended to infinite dimensional systems (e.g. the exponential of matrix, the fundamental matrix or also the explicit solution). The readers should refer to [Bensoussan et al., 2006] to get more details on infinite dimensional systems and a complete characterization of time-delay systems as systems in a Banach functional space.

3. Functional Differential equation: evolution in a finite Euclidian space or in a functional space.

   Since only functional differential equations will be used throughout this thesis, only this one will be deeper explained.

2.1.1 Functional Differential Equations

The most spread representation is by the mean of functional differential equations [Bellman and Cooke, 1963, Gu et al., 2003, K.Hale and Lunel, 1991, Kolmanovskii and Myshkis, 1962, Niculescu, 2001]: several types of time-delay systems can be considered according to the worldly accepted denomination introduced by Kamenskii [Kolmanovskii and Myshkis, 1999]:

1. System with discrete delay acting on the state x, inputs u or/and outputs y, e.g.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h_x) + Bu(t) + B_h u(t-h_u) \\ y(t) &= C_h x(t-h_y) \end{aligned}$$

   where h_x, h_u and h_y are respectively the delay state, the input delay and the measurement delay.

2. Distributed delay systems where the delay acts on state x or inputs u in a distributed fashion, e.g.

$$\dot{x}(t) = Ax(t) + \int_{-h_x}^{0} A_h(\theta)x(t+\theta)d\theta + Bu(t) + \int_{-h_u}^{0} B_h(\theta)x(t+\theta)d\theta$$

3. Neutral delay systems where the delay acts on the higher-order state-derivative, e.g.

$$\dot{x}(t) - F\dot{x}(t-h) = Ax(t)$$

The following paragraphs are devoted to a brief emphasis of the difference between these classes of systems through illustrative application examples. These examples are borrowed from [Briat and Verriest, 2008, Kolmanovskii and Myshkis, 1999, Niculescu, 2001, Verriest and Pepe, 2007].

Systems with discrete delays

Systems with discrete delays are systems which locally remember past signals values, at some specific past time instants. An interesting example presented in [Niculescu, 2001] considers an irreversible chemical reaction producing a material B from a material A. Such a reaction is never instantaneous and never complete and in order to resolve enhance the quantity of reacted products, a classical technique is to use a recycle stream. The streaming process does not take place instantaneously and the whole process (i.e. reaction + streaming) can be modeled by a system of nonlinear delay differential equations with discrete delay:

$$\begin{array}{rcl} \dot{A}(t) & = & \frac{q}{V}[\lambda A_0 + (1-\lambda)A(t-\tau) - A(t)] - K_0 e^{-\frac{Q}{T}} A(t) \\ \dot{T}(t) & = & \frac{1}{V}[\lambda T_0 + (1-\lambda)T(t-\tau) - T(t)]\frac{\Delta H}{C\rho} - K_0 e^{-\frac{Q}{T}} A(t) - \frac{1}{VC\rho}U(T(t) - T_w) \end{array} \quad (2.5)$$

where $A(t)$ is the concentration of the component A, $T(t)$ is the temperature (A_0, T_0 correspond to these values at initial time $t = 0$) and $\lambda \in [0, 1]$ is the recycle coefficient, $(1 - \lambda)q$ is the recycle flow rate of the unreacted A and τ is the transport delay. The others terms are constant of the system.

Economic behaviors are other applications of functional differential equations [Belair and Mackey, 1989, Kolmanovskii and Myshkis, 1999, 1962, Niculescu, 2001]. For instance, the following discrete delay model has been used for describing interactions between consumer memory and price fluctuations on commodity market:

$$\ddot{x}(t) + \frac{1}{R}\dot{x}(t) + \dot{x}(t-\tau) + \frac{Q}{R}x(t) + \frac{1}{R}x(t-\tau) = 0 \quad (2.6)$$

where x denotes the relative variation of the market price of the commodity in question and Q, R, τ are parameters of the model. In particular, τ is the time that must elapse before a decision to alter production is translated into an actual change in supply.

Actually, this model is obtained by differentiating the following dynamical model involving a distributed delay:

$$\dot{x}(t) + \frac{Q}{R}\int_{-\infty}^{0} e^{-\theta/R} x(t+\theta)d\theta + x(t-\tau) = 0$$

Note that this operation cannot be always performed, more details on this procedure can be found for instance in [Verriest, 1999].

Other applications of time-delay systems with discrete delays arise in heat exchanger dynamics, traffic modeling, teleoperation systems, networks such as internet, modeling of rivers,population dynamics... Delays also appear in neural networks, any systems with delayed measurement, system controlled by delayed feedback and in this case, delays are a consequence of technological constraints.

The reader should refer to the following papers/books and references therein to get more details on pointwise delay systems:

Stability analysis: [Bliman, 2001, Chiasson and Loiseau, 2007, Fridman and Shaked, 2001, Gouaisbaut and Peaucelle, 2006b, Goubet-Batholoméus et al., 1997, Gu et al., 2003, Han, 2005a, 2008, Han and Gu, 2001, He et al., 2004, Jun and Safonov, 2001, Kao and Rantzer, 2007, K.Hale and Lunel, 1991, Kharitonov and Melchor-Aguila, 2003, Kharitonov and Niculescu, 2003, Kolmanovskii and Myshkis, 1962, Michiels and Niculescu, 2007, Moon et al., 2001, Niculescu, 2001, Park et al., 1998, Richard, 2000, Sipahi and Olgac, 2006, Xu and Lam, 2005, Zhang et al., 1999, 2001]

Control Design: [Ivanescu et al., 2000, Meinsma and Mirkin, 2005, Michiels and Niculescu, 2007, Michiels et al., 2005, Mirkhin, 2003, Mondié and Michiels, 2003, Niculescu, 2001, Seuret et al., 2008, Suplin et al., 2006, Verriest, 2000, Verriest et al., 2002, Verriest and Pepe, 2007, Verriest et al., 2004, Witrant et al., 2005, Wu, 2003, Xie et al., 1992, Xu et al., 2006]

Observers: [Darouach, 2001, Fattouh, 2000, Fattouh et al., 1999, 2000a, Germani et al., 2001, Sename, 2001]

Distributed delay Systems

Distributed delay systems are systems where the delay has not a local effect as in pointwise delay systems but in a distributed fashion over a whole interval. For instance, consider the following SIR-model [Anderson and May, 2002, 1982, Hethcote, 2002, Van den Driessche, 1999, Wickwire, 1977] used in epidemiology [Briat and Verriest, 2008]

$$\begin{array}{rcl}\dot{S}(t) & = & -\beta S(t)I(t)\\ \dot{I}(t) & = & \beta S(t)I(t) - \beta \displaystyle\int_h^{\infty} \gamma(\tau)S(t-\tau)I(t-\tau)d\tau\\ \dot{R}(t) & = & \beta \displaystyle\int_h^{\infty} \gamma(\tau)S(t-\tau)I(t-\tau)d\tau\end{array} \tag{2.7}$$

where S is the number of susceptible people, I the number of infectious people and R the number of recovered people. The distributed delay here taking value over $[h, +\infty]$ is the time spent by infectious people before recovering from the disease. This delay me be different from a individual to another but obeys a probability density represented by $\gamma(\tau)$ which tends to 0 at infinity and whose integral over $[-\infty, -h]$ equals 1. It is assumed here that once recovered from the disease, people become resistant and therefore remain within the set of recovered people. It can be easily shown that $\dot{S} + \dot{R} + \dot{I} = 0$, showing that the system is Hamiltonian (energy preserving) and hence stable.

Another example of systems governed by distributed delay differential equations are combustion models [Crocco, 1951, Fiagbedzi and Pearson, 1987, Fleifil et al., 1974, 2000, Niculescu, 2001, Zheng and Frank, 2002] involved in propulsion and power-generation. Delay in such models can have destabilizing effects but it has been shown these recent years that this delay can be used in advantageous manner. The following example is taken from [Niculescu, 2001, Niculescu et al., 2000].

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega_i^2 x \quad = \quad c_1 x(t-\tau_c) - c_2 \int_0^{t-\tau_c} x(\xi)d\xi \tag{2.8}$$

For more details and some results on systems with distributed delays, the readers should refer to [Chiasson and Loiseau, 2007, Fattouh et al., 2000b, Fiagbedzi and Pearson, 1987, Fridman and Shaked, 2001, Gu et al., 1999, 2003, Ivanescu et al., 1999, K.Hale and Lunel, 1991, Kolmanovskii and Richard, 1997, Niculescu, 2001, Richard, 2000, Tchangani et al., 1997, Verriest, 1995, 1999, Zheng and Frank, 2002] and references therein.

Neutral Delay Systems

Finally, neutral delay systems, arising for instance in the analysis of the coupling between transmission lines and population dynamics, are systems where discrete delays act on the higher order derivative of the dynamical system. (See [Brayton, 1966, K.Hale and Lunel, 1991, Kuang, 1993]. The origin of the term 'neutral' is unclear while the other terms are easy to understand.

An example of dynamical system governed by neutral delay equation is the evolution of forests. The model is based on a refinement of the delay-free logistic (or Pearl-Verhulst equation) where effects as soil depletion and erosion have been introduced

$$\dot{x}(t) = rx(t)\left[1 - \frac{x(t-\tau) + c\dot{x}(t-\tau)}{K}\right] \tag{2.9}$$

where x is the population, r is the intrinsic growth rate and K the environmental carrying capacity. See Gopalsamy and Zhang [1988], Pielou [1977], Verriest and Pepe [2007] for more details.

For more details on neutral delay systems, please refer to [Bliman, 2002, Brayton, 1966, Fridman, 2001, Gopalsamy and Zhang, 1988, Han, 2002, 2005b, K.Hale and Lunel, 1991, Kolmanovskii and Myshkis, 1962, Niculescu, 2001, Picard et al., 1998, Verriest and Pepe, 2007] and references therein.

Amongst all these types of time-delay systems, the current thesis focuses on time-delay systems with discrete-delays, especially on the state but some applications to systems with delay on the input and distributed delays will be provided.

2.1.2 Constant Delays vs. Time-Varying Delays and Quenching Phenomenon

In the latter examples of time-delay systems represented in term of a functional differential equations, the delay is assumed to be constant. In some applications (networks, sampled-data control...) the delay is time-varying, making the system non-stationary. At first sight, it may appear as a technical detail but, actually, it leads to a complicated phenomena called 'Quenching' (see [Louisell, 1999, Papachristodoulou et al., 2007]). Indeed, there is a gap between constant and time-varying delays and it is possible to find systems which are stable for constant delay $h \in [h_1, h_2]$ but unstable for time-varying delay belonging to the same interval. In such a phenomenon, the bound on the delay derivative plays an important role and remains still partially unexplained (some clues are provided in [Kharitonov and Niculescu, 2003, Papachristodoulou et al., 2007]).

In some systems, the delay may be a known function of time or some parameters. Moreover, methods to estimate the delay in real time are currently developed; see for instance [Belkoura et al., 2007, 2008, Drakunov et al., 2006, Veysset et al., 2006]. In these cases, it is imaginable to use this information to study stability and design specific control laws.

It is also possible to define systems in which the delay is a function of the state. This makes the stability analysis of the system extremely harder and only very few (uncomplete) results have been provided on that topic. See for instance [Bartha, 2001, Feldstein et al., 2005, Louihi and Hbid, 2007, Luzianina et al., 2000, 2001, Verriest, 2002, Walther, 2003] and references therein.

2.2 Stability Analysis of Time-Delay Systems

The stability analysis of time-delay systems is a very studied problem and has led to lots of approaches which can be classified in two main framework: the frequency-domain and time-domain analysis [Gu et al., 2003, Niculescu, 2001]. While the first one deals with characteristic quasipolynomial of the system, the second one considers directly the state-space domain and matrices. Before entering in more details, some preliminary definitions are necessary.

Definition 2.2.1 *If a time-delay system is stable for any delay values belonging to* $\mathbb{R}_+$*, the system is said to be delay-independent stable.*

Example 2.2.2 *A delay-independent stable time-delay system with constant time-delay is given by*

$$\dot{x}(t) = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t-h) \tag{2.10}$$

It seems obvious that if a system is delay-independent stable, then it must be stable for $h = 0$ *and* $h \to +\infty$ *which means that* A *and* $A + A_h$ *in (2.2) must be Hurwitz stable (all the eigenvalues lie in the open left-half plane). On the second hand, for any value of* h *from* 0 *to* $+\infty$*), the system must be stable too. It is shown in Appendix G.3 that a supplementary sufficient condition is given by*

$$\bar{\rho}[(j\omega - A)^{-1}A_h] < 1, \qquad \forall \omega \in \mathbb{R}$$

where $\bar{\rho}(\cdot)$ *denotes the spectral radius (i.e.* $\max_i |\lambda_i(\cdot)|$*).*

By verifying these conditions we find

$$\begin{aligned} \lambda(A) &= \{-5, -5\} \\ \lambda(A + A_h) &= \left\{ \frac{-13 \pm \sqrt{3}}{2} \right\} \\ \bar{\rho}[(j\omega - A)^{-1}A_h] &\sim 0.4739 < 1 \end{aligned}$$

The system is confirmed to be delay-independent stable.

The term 'delay-independent stable' has been introduced for the first time in [Kamen et al., 1985] and has become commonly used in the time-delay community.

Definition 2.2.3 *If a time-delay system is stable for all delay values belonging to a subspace* $D \subsetneqq \mathbb{R}_+$ *then the system is said to be delay-dependent stable.*

Example 2.2.4 *A well-known system being delay-dependent stable [Gouaisbaut and Peaucelle, 2006b] is given by*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h) \tag{2.11}$$

and is stable for any constant delay belonging to $[0, 6.17]$*. To see this note that* $A + A_h$ *is Hurwitz and hence the system is stable for zero delay. On the other hand,* $A - A_h$ *is not Hurwitz (has eigenvalues* $\{-1, 0.1\}$*) and shows that for some values of the delay the system has positive eigenvalues. This is explained further in the Appendix G which deals with quasipolynomials.*

When the lower bound of the interval of delay is 0, the term 'delay-margin' is often referred to the upper bound of the interval. It is possible to find systems for which the lower bound of the interval is non zero.

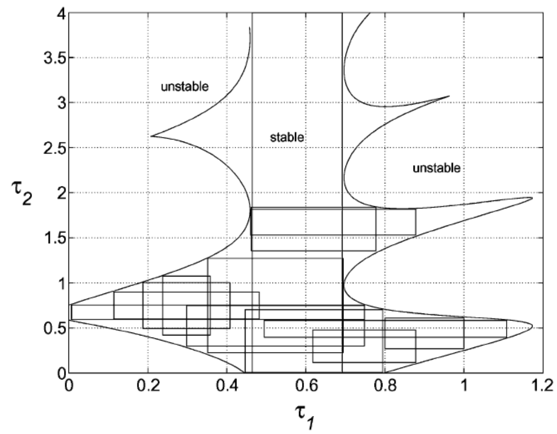

Figure 2.1: Stability regions of system (2.13) w.r.t. to delay values

Example 2.2.5

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-h) \tag{2.12}$$

This system is not stable for $h = 0$ since the matrix $A + A_h$ is not Hurwitz. Indeed, in [Gouaisbaut and Peaucelle, 2006a, Gu et al., 2003], it is shown that the system is stable for all constant delay in the interval $[0.10016826, 1.7178]$.

Other systems may exhibit (almost) periodicity in the intervals of stability: there exists a (finite or infinite) countable sequence of disjoint intervals for which the system is stable. Such a behavior most often occurs in systems with several delays.

Moreover, in the case of multiple delay systems the stability map (the set of delays for which the system is stable) can be very complicated as presented for instance in [Knospe and Roozbehani, 2006, Sipahi and Olgac, 2005, 2006]. The following example is borrowed from [Knospe and Roozbehani, 2006].

Example 2.2.6 *Let us consider the system with 2 delays*

$$\dot{x}(t) = \begin{bmatrix} -3.0881 & 2.6698 \\ -9.7383 & 2.8318 \end{bmatrix} x(t) + \begin{bmatrix} 0.5645 & 0.0178 \\ 1.2597 & 0.8020 \end{bmatrix} x(t-h_1) + \begin{bmatrix} 0.4176 & 0.0144 \\ 0.9432 & 0.5976 \end{bmatrix} x(t-h_2) \tag{2.13}$$

The stability map for this system is depicted on Figure 2.1. On this figure, it is possible to see that the there are notches which show that the stability set is not as regular as for system with single delay. The boxes are approximations of the stability set obtained using method of [Knospe and Roozbehani, 2006].

In the case of time-varying delays, the stability may depend or not on the rate of variation of the delay (the derivative of the delay) and in these cases, a similar vocabulary has been introduced.

Definition 2.2.7 *For a stable time-delay system with time-varying delays, if the stability depends on the rate of variation, then the system is said to be rate-dependent stable.*

Definition 2.2.8 *For a stable time-delay system with time-varying delays, if the stability does not depend on the rate of variation, then the system is said to be rate-independent stable.*

In most of the cases the bound on the rate of variation of the delay is closely related to the delay-margin, the greater the absolute value of the rate is, the lower the delay-margin will be. Papachristodoulou et al. [2007] have shown that system

$$\dot{x}(t) = -x(t - h(t)) \tag{2.14}$$

is unstable for a delay-rate bound greater than approximately 0.86 even though for a constant delay, the system is stable for $h < \pi/2$.

In [Kharitonov and Niculescu, 2003], analytical methods are provided to deal with uncertain delays around a fixed constant one. With such an approach it is possible to quantify and give bounds on the variation of the delay. For instance, the relevant example considered in Kharitonov and Niculescu [2003]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)) \tag{2.15}$$

is stable for a delay equal to 1. Using the method of [Kharitonov and Niculescu, 2003] where the time-varying delay is written as

$$h(t) = h_0 + \eta(t) \qquad \dot{h}(t) = \dot{\eta}(t) \tag{2.16}$$

it is shown that the stability is preserved for every $|\eta(t)| \leq \eta_0$ and $|\dot{\eta}(t)| \leq \dot{\eta}_0$ such that

$$\eta_0 < \frac{1}{640}\mu_0 \qquad \dot{\eta}_0 < 1 - 8\mu_0$$

with $\mu_0 \in (0, 1/40)$. From these inequalities we can see that the larger $\dot{\eta}_0$ is, the smaller η_0 must be to preserve stability. This illustrates the effect of a time-varying delay on the stability of time-delay systems.

2.2.1 Time-Domain Stability Analysis

Several frequency domain approaches have been provided and allow for more or less easy stability analysis of time-delay systems with constant delay. These methods cannot be applied for systems with time-varying delays or even for time-varying systems, uncertain systems with time-varying uncertainties and nonlinear systems (except locally). The idea now is to exploit state-space approaches to analyze stability which have a wider field of action. Many approaches have been developed these past years and amongst them, the extension of Lyapunov theory and Lyapunov functions play a central role.

This section is devoted to a presentation of many time-domain approaches. On the first hand, the extensions of Lyapunov theory through the celebrated Lyapunov-Krasovskii and Lyapunov-Razumikhin theorems are introduced. On the second hand, an historical review is developed in which the use of model transformations is introduced and justified. The concept of additional dynamics is then shown as a consequence of model transformations and as a limitation of some approaches. Still in the context of model transformations, the problem of bounding cross-terms is explained and solved in different manners exposed chronologically.

To conclude on the part on extensions of Lyapunov' theory, recent results does not involving model transformations are provided.

Finally more 'exotic' stability tests not directly based on extension of Lyapunov's theory but relying on well-posedness theory, Integral Quadratic Constraint theory or even small-gain theorems are introduced as an opening to new promising methods.

Remark 2.2.9 *All the definitions of stability of finite-dimensional systems given in Appendix B.4 can be generalized to time-delay systems by introducing the continuous norm* $||\cdot||_c$ *defined by*

$$||\phi||_c := \max_{a\leq\theta\leq b} ||\phi(\theta)||_2 \tag{2.17}$$

where $\phi \in \mathcal{C}([a,b],\mathbb{R}^n)$*.*

2.2.1.1 On the extension of Lyapunov Theory

Throughout this part, we will focus on the stability analysis of the general single delayed system

$$\begin{array}{rcl} \dot{x}(t) & = & f(x_t,t) \\ x_{t_0} & = & \phi \end{array} \tag{2.18}$$

where $x_t(\theta) = x(t+\theta)$ and $\phi \in \mathcal{C}([-h,\ 0],\mathbb{R})$ is the functional initial condition. We also assume that $x(t) = 0$ identically is a solution to (2.18), that will be referred to as the trivial solution.

As in the study of systems without delay, the Lyapunov method is an effective approach. For a system without delay, this consists in the construction of a Lyapunov function $V(t,x(t))$, which is some sense is a potential measure quantifying the deviation of the state $x(t)$ from the trivial solution 0. Since, for a delay-free system, $x(t)$ is needed to specify the system future evolution beyond t, and since in a time-delay system the 'state' at time required for the same purpose is the value of $x(\theta)$ in the interval $\theta \in [t-h,t]$ (i.e. x_t), it is natural to expect that for a time-delay system, the corresponding Lyapunov function be a functional $V(t,x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0. Such a functional is known as the Lyapunov-Krasovskii functional.

More specifically, let $V(t,\phi)$ be differentiable, and let $x_t(\tau,\phi)$ be a solution of (2.18) at time t with the initial condition $x_\tau = \phi$. We may calculate the derivative of $V(t,x_t)$ with respect to t and evaluate it at $t=\tau$. This gives rise to

$$\begin{array}{rcl} \dot{V}(\tau,\phi) & = & \left.\dfrac{d}{dt}V(t,x_t)\right|_{t=\tau,\ x_t=\phi} \\ & = & \limsup_{\Delta t\to 0} \dfrac{1}{\Delta t}[V(\tau+\Delta t, x_{t+\Delta t}(\tau,\phi)) - V(\tau,\phi)] \end{array} \tag{2.19}$$

Intuitively, a nonpositive $\dot{V}$ indicates that x_t does not grow with t, which in turn means that the system under consideration is stable in light of remark 2.2.9. The more precise statement of this observation is the following theorem.

Theorem 2.2.10 (Lyapunov-Krasovskii Stability Theorem) *Suppose* $f : \mathbb{R}\times\mathcal{C}_{[-h,\ 0]} \to \mathbb{R}^n$ *in (2.18) maps* $\mathbb{R} \times$ *(bounded sets of* $\mathcal{C}_{[-h,\ 0]}$*) into bounded sets of* $\mathbb{R}^n$*, and* $u,\ v,\ w : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ *are continuous nondecreasing functions,* $u(s)$ *and* $v(s)$ *are positive for* $s > 0$*, and*

$u(0) = v(0) = 0$. If there exists a continuous differentiable functional $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ such that

$$u(||\phi(0)|| \leq V(t, \phi) \leq v(||\phi||_c)$$

and

$$\dot{V}(t, \phi) \leq -w(||\phi(0)||)$$

then the trivial solution of (2.18) is uniformly stable. If $w(s) > 0$ for $s > 0$, then it is uniformly asymptotically stable. If, in addition, $\lim_{s\to+\infty} u(s) = +\infty$, then it is globally uniformly asymptotically stable.

In the special case of linear time-delay systems, it is possible to give a generic 'complete' Lyapunov-Krasovskii functional (see [Fridman, 2006a, Gu et al., 2003, Papachristodoulou et al., 2007] and references therein). The term 'complete' means that, if computed exactly, it provides necessary and sufficient conditions to the delay-dependent stability for such systems. Let us consider the following linear time-delay system:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t-h) \\ x(\theta) & = & \phi(\theta), \quad \theta \in [-h, 0] \end{array} \tag{2.20}$$

where $x \in x \subset \mathbb{R}^n$, $\phi \in \mathcal{C}_{[-h,\ 0]}$ and $h \in \mathbb{R}_+$ are respectively the system state, the functional initial condition and the constant time-delay. This leads to the following theorem

Theorem 2.2.11 *The latter system is delay-dependent asymptotically stable for a constant time-delay h if and only if there exist a constant matrix $P = P^T \in \mathbb{R}^{n\times n}$, a scalar $\varepsilon > 0$ and continuously differentiable matrix functions*

$$\begin{array}{rcl} Q(\xi) & : & [-h, 0] \to \mathbb{R}^{n\times n} \\ R(\xi, \eta) & = & R(\eta, \xi)^T, \quad \text{with } R(\xi, \eta) : [-h, 0]^2 \to \mathbb{R}^{n\times n} \\ S(\xi) & = & S(\xi)^T : [-h, 0] \to \mathbb{R}^{n\times n} \end{array}$$

such that

$$\begin{array}{rcl} V(x_t) & = & x(t)^T P x(t) + 2x(t)^T \displaystyle\int_{-r}^{0} Q(\xi) x(t+\xi) d\xi + \int_{-r}^{0} \left[\int_{-r}^{0} x(t+\xi)^T R(\xi, \eta) x(t+\eta) d\eta \right] d\xi \\ & & + \displaystyle\int_{-r}^{0} x(t+\xi)^T S(\xi) x(t+\xi) d\xi \geq \varepsilon ||x(t)||^2 \end{array} \tag{2.21}$$

is a Lyapunov-Krasovskii functional. Moreover its time derivative satisfies

$$\begin{array}{rcl} \dot{V}(x_t) & = & x(t)^T [PA + A^T P + Q(0) + Q^T(0) + S(0)] x(t) - x(t-h)^T S(-h) x(t-h) \\ & & - \displaystyle\int_{-h}^{0} x(t+\xi)^T \dot{S}(\xi) x(t+\xi) d\xi + 2x(t)^T [PA_h - Q(-h)] x(t-h) \\ & & - \displaystyle\int_{-h}^{0} d\xi \int_{-h}^{0} x(t+\xi)^T \left[\frac{\partial}{\partial \xi} R(\xi, \eta) + \frac{\partial}{\partial \eta} R(\xi, \eta) \right] x(t+\eta) d\eta \\ & & + 2x(t)^T \displaystyle\int_{-h}^{0} [A^T Q(\xi) - \dot{Q}(\xi) + R(0, \xi)] x(t+\xi) d\xi \\ & & + 2x(t)^T \displaystyle\int_{-h}^{0} [A_h^T Q(\xi) - R(-h, \xi)] x(t+\xi) d\xi \leq -\varepsilon ||x(t)||^2 \end{array} \tag{2.22}$$

In practice, it is numerically difficult to check the existence of such a quadratic functional. Indeed, it describes an infinite dimensional problem since decision variables are functions (i.e. Q, R, S). To overcome this problem a discretization scheme may be adopted [Fridman, 2006b, Gu et al., 2003, Han and Gu, 2001] or a Sum-of-Squares based relaxation [Papachristodoulou and Prajna, 2002, Papachristodoulou et al., 2005, 2007, Prajna et al., 2004]. Section 3.6.1 will be devoted to a particular discretized Lyapunov-Krasovskii functional.

Note that the Lyapunov-Krasovskii functional requires the state variable $x(t)$ in the interval $[-h, 0]$ and necessitates the manipulation of functionals, which consequently makes the application of the Lyapunov-Krasovskii theorem rather difficult. This difficulty may sometimes be circumvented using the Razumikhin theorem, an alternative result invoking only functions rather than functionals.

The key idea behind the Razumikhin theorem also focuses on a function $V(x)$ representative of the size of $x(t)$. For such a function,

$$\bar{V}(x_t) = \max_{\theta \in [-h,0]} V(x(t+\theta)) \tag{2.23}$$

serves to measure the size of x_t. If $V(x(t)) < \bar{V}(x_t)$, then $\dot{V}(x) > 0$ does not make $\bar{V}(x_t)$ grow. Indeed, for $\bar{V}(x_t)$ to not grow, it is only necessary that $\dot{V}(x(t))$ is not positive whenever $V(x(t)) = \bar{V}(x_t)$. The precise statement is as follows.

Theorem 2.2.12 (Lyapunov-Razumikhin Stability Theorem) *Suppose $f : \mathbb{R} \times \mathcal{C}_{[-h,\ 0]} \to \mathbb{R}^n$ in (2.18) takes $\mathbb{R} \times$ (bounded sets of $\mathcal{C}_{[-h,\ 0]}$) into bounded sets of $\mathbb{R}^n$, and u, v, w : $\bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$, v strictly increasing.*

If there exists a continuously differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$u(||x||) \leq V(t, x) \leq v(||x||), \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \tag{2.24}$$

and the derivative of V along the solution $x(t)$ of (2.18) satisfies

$$\dot{V}(t, x(t)) \leq -w(||x(t)||) \text{ whenever } V(t+\theta, x(t+\theta)) \leq V(t, x(t)) \tag{2.25}$$

for $\theta \in [-h, 0]$, then the system (2.18) is uniformly stable.

If, in addition, $w(s) > 0$ for $s > 0$, and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (2.25) is strengthened to

$$\dot{V}(t, x(t)) \leq -w(||x(t)||) \quad \text{if} \quad V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))) \tag{2.26}$$

for $\theta \in [-h, 0]$, then the system (2.18) is uniformly asymptotically stable.

If in addition $\lim_{s \to +\infty} u(s) = +\infty$, then the system (2.18) is globally uniformly asymptotically stable.

Lyapunov-Krasovskii and Lyapunov-Razumikhin are the most famous results concerning stability of time-delay systems in the time-domain. However, there exists several others results, see for instance [Barnea, 1969]. In Sections 2.2.1.4 and 2.2.1.9 different stability tests will be derived using both theorems.

2.2.1.2 About model transformations

Model-transformations have been introduced early in the stability analysis of time-delay systems. They allow to turn a time-delay system into a new system, which is referred to as a *comparison system*. Finally, the stability of the original system is determined through the stability analysis of the comparison model. They are generally used to remove annoying terms in the equations or to turn the expression of the system in a more convenient form. Comparison systems may be of different types, (uncertain) finite dimensional linear systems (see [Gu et al., 2003, Knospe and Roozbehani, 2006, 2003, Roozbehani and Knospe, 2005, Zhang et al., 1999, 2001]), time-delay systems (see [Fridman and Shaked, 2001, Gu et al., 2003]). In our papers [Briat et al., 2007a, 2008b], a time-delay system is turned into an uncertain finite dimensional LPV systems from which a new control strategy is developed; this will developed in Section 5.1.7.

Some model transformations are introduced here, although the list is non exhaustive due to the important work that has been done in that field, it will be focused on two initial first-order model transformations [Goubet-Batholoméus et al., 1997, Kolmanovskii and Richard, 1997, 1999, Kolmanovskii et al., 1998, Li and de Souza, 1996, Niculescu, 1999, Niculescu and Chen, 1999, Su, 1994, Su and Huang, 1992] and a recent one [Fridman, 2001, Fridman and Shaked, 2001] which will be detailed in the following. The motivation for which only three model transformations have been chosen to be presented, comes from the fact the two first ones are simple but may induce some conservatism through the creation of additional dynamics which may be unstable. It will be shown in Section 2.2.1.3 than the second one is less conservative than the first and the last one does not induce some conservatism despite of its apparent complexity.

First of all, let us consider the linear time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t-h) \\ x_0 & = & \phi \end{array} \tag{2.27}$$

where A, A_h are given $n \times n$ real matrices and ϕ is the functional initial condition.

The following model transformations have been chosen to be illustrated in this section.

Euler formula : The Euler formula is the oldest model transformation which has been introduced [Goubet-Batholoméus et al., 1997, Kolmanovskii and Richard, 1997, 1999, Li and de Souza, 1996, Su, 1994, Su and Huang, 1992] and is still in use for different purposes [Gu et al., 2003, He et al., 2004, Niculescu, 2001]:

$$x(t-h) = x(t) - \int_{t-h}^{t} \dot{x}(\theta) d\theta \tag{2.28}$$

It allows to turn the time-delay system with discrete delay (2.27) into the following system with distributed delay:

$$\dot{x}(t) = (A + A_h)x(t) - A_h \int_{t-h}^{t} [Ax(s) + A_h x(s-h)] ds \tag{2.29}$$

Parametrized Euler formula : This model transformation [Kolmanovskii et al., 1998, Niculescu, 1999, Niculescu and Chen, 1999] improves the result obtained from the Euler formula by introducing a free parameter:

$$Cx(t-h) = Cx(t) - C \int_{t-h}^{t} \dot{x}(\theta) d\theta \tag{2.30}$$

where $C \in \mathbb{R}^{n\times n}$ is a free matrix. It allows to turn the time-delay system with discrete delay into a system with distributed delay:

$$\dot{x}(t) = (A+C)x(t) + (A_h - C)x(t-h) - C\int_{t-h}^{t}[Ax(s) + A_h x(s-h)]ds \tag{2.31}$$

Note that for $C = 0$ the original system is recovered and for $C = A_h$ the system obtained from Euler formula.

Descriptor Model Transformation : This model transformation [Fridman, 2001, Fridman and Shaked, 2001] allows to turn a time-delay system into a singular system with distributed delay

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ A+A_h & -I \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{t-h}^{t}\begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix}\begin{bmatrix} x(s) \\ y(s) \end{bmatrix}ds \tag{2.32}$$

where $y(t) = \dot{x}(t)$. By substituting the expression of $y(t)$ defined by the second row in the first row, the original system (2.27) is retrieved.

Many other model transformations have been provided in the literature and the author should refer, for instance, to [Gouaisbaut and Peaucelle, 2006a,b, 2007, Goubet-Batholoméus et al., 1997, Gu et al., 2003, Kao and Rantzer, 2007, Knospe and Roozbehani, 2006, 2003, Kolmanovskii and Myshkis, 1962, Kolmanovskii and Richard, 1997, 1998, Kolmanovskii et al., 1998, Li and de Souza, 1996, Niculescu, 2001, 1997, 1999, Niculescu and Chen, 1999, Roozbehani and Knospe, 2005, Su, 1994, Su and Huang, 1992, Zhang et al., 1999]. Many of model transformations introduced in the latter references have not been introduced in same spirit as the model-transformations detailed above, but in view of turning the system into another form in order to analyze it in a different framework. This will be detailed in Sections 2.2.1.4 to 2.2.1.8.

2.2.1.3 Additional Dynamics

Stability tests obtained from comparison systems are, in most of the cases, outer approximations of the original system only. This means that if the comparison model is stable then the original system is stable too but the converse does not necessary hold. The following development is borrowed from [Gu and Niculescu, 2000, 1999, Gu et al., 2003] and some precisions on additional dynamics can also be found in [Kharitonov and Melchor-Aguila, 2003] and references therein.

For instance the simpler model transformation (i.e. the Euler formula) leads to the comparison system

$$\dot{z}(t) = (A + A_h)z(t) - A_h\int_{t-h}^{t}[Az(t) + A_h z(s-h)]ds \tag{2.33}$$

where the instantaneous state is set to z to emphasize the difference between the original and comparison model.

The characteristic polynomial of the latter comparison system is then given by

$$\Delta_c(s) := \det(s^2 I - (A+A_h)s + A_h A(1-e^{-sh}) + A_h^2 e^{-sh}(1-e^{-sh})) \tag{2.34}$$

while the one of the original system is

$$\Delta_o := \det(sI - A - A_h e^{-sh}) \tag{2.35}$$

Therefore, it seems evident that the behavior of both systems may be different. To see this, we will emphasize that the comparison system quasipolynomial exhibits supplementary zeros, and hence additional dynamics. The idea is to factorize the comparison polynomial by the original quasipolynomial as

$$\Delta_c(s) = \Delta_o(s)\Delta_a(s) \tag{2.36}$$

where

$$\Delta_a(s) := \det\left(I - \frac{1-e^{-sh}}{s}A_h\right) \tag{2.37}$$

This shows that the set of zeros of the quasipolynomial of the comparison system is composed by the set of zeros of the quasipolynomial of the original system and some additional zeros. The idea is then to determine the location of these additional zeros in the complex plane. It is clear that if the real part of these zeros is nonnegative, the comparison system is unstable, even if the original system is stable (zeros of $\Delta_o(s)$ have strictly negative real part). Some results on this stability analysis are presented below.

Note that

$$\Delta_a(s) = \prod_{i=1}^{n}\left(1 - \lambda_i\frac{1-e^{-sh}}{s}\right) \tag{2.38}$$

where λ_i is the i^{th} eigenvalue of matrix A_h and let $s = s_{ik}$, $k = 1, 2, 3, \ldots$ be all the solutions of the equation

$$1 - \lambda_i\frac{1-e^{-sh}}{s} = 0 \tag{2.39}$$

Then s_{ik}, $i = 1, \ldots, n$, $k = 1, 2, 3, \ldots$ are all the additional poles of the comparison system.

Proposition 2.2.13 *For any given A_h, all the additional poles satisfy*

$$\lim_{h\to 0^+} \Re(s_{ik}) = -\infty$$

As a result, all the additional poles have negative real part for sufficiently small h. As h increases, some of the additional poles may cross the imaginary axis. It turns out that the exact crossing value can be analytically calculated. This is stated in the theorem below.

Theorem 2.2.14 *Corresponding to an eigenvalue λ_i of A_h, $\Im(\lambda_i) \neq 0$, there is an additional pole s_{ik} on the imaginary axis if and only if the time delay satisfies*

$$h = h_{ik} = \frac{k\pi + \arg(\lambda_i)}{\Im(\lambda_i)} > 0, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Corresponding to a positive real eigenvalue λ_i of A_h, there is an additional pole on the imaginary axis if and only if

$$h = \frac{1}{\lambda_i}$$

No additional poles corresponding to a negative real eigenvalue λ_i of A_h will reach the imaginary axis for any finite delay.

Therefore, if all the eigenvalues of the matrix A_h are real and negative, then the original and comparison system are equivalent for this particular model transformation.

On the second hand, some model transformations do not create such a gap between the original and comparison system. This is the case for the parametrized model transformation introduced in Goubet-Batholoméus et al. [1997], Niculescu [1999], Niculescu and Chen [1999], and for the descriptor model transformation introduced in [Fridman, 2001, Fridman and Shaked, 2001] which constructs a singular system with distributed delay that is equivalent, from a stability point of view, to the original system. Indeed, for the parametrized model transformation, it can be shown that the additional poles are solutions of the equation

$$\det\left(I - C\frac{1-e^{-sh}}{s}\right) = 0$$

This shows that for a judicious choice of the free matrix C, no unstable dynamics are generated which emphasizes the interest of the parametrized model transformation.

Finally, let us consider now the descriptor model transformation, the comparison model is governed by

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{t-h}^{t}\begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix}\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds$$

The corresponding characteristic polynomial is

$$\begin{array}{rcl} \Delta_{cd}(s) & := & \det\left(\begin{bmatrix} sI & -I \\ -(A+A_h) & I + A_h\dfrac{1-e^{-sh}}{s} \end{bmatrix}\right) \\ & := & \det\left(sI - A - A_h e^{-sh}\right) \end{array}$$

and we get the quasipolynomial of the original system by application of the determinant formula (see Appendix A.1). This shows that the systems are equivalent and is the great advantage of this model transformation. This has a great benefit in performances analysis of time-delay systems. Nevertheless the system is changed in a singular form and the delay is now a distributed delay, which may introduce some difficulties in the stability analysis.

2.2.1.4 Stability Analysis: Lyapunov-Razumikhin Functions

The two main extensions of Lyapunov's theory for time-delay systems have been introduced. It has been presented that stability criteria may be done using a reformulation of the system by a procedure called 'model-transformation'. This procedure, according to its type, may induce some additional dynamics leading to a non-equivalence between the original and the transformed system. This section is devoted to simple stability tests using Lyapunov-Razumikhin theorem and both delay-dependent and delay-independent tests are provided.

Let us consider here a general linear time-delay system of the form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h(t-h) \\ x(t+\theta) & = & \phi(\theta), \quad \theta[-h, 0] \end{array} \tag{2.40}$$

Delay-independent stability test via Lyapunov-Razumikhin theorem

A simple test on delay-independent stability using quadratic Lyapunov-Razumikhin function is provided here

$$V(x(t)) = x(t)^T Px(t) \tag{2.41}$$

The time-derivative of V along the trajectories solutions of system (2.40) is given by

$$\dot{V}(x(t)) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PA_h \\ A_h^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

Applying the Lyapunov-Razumikhin theorem 2.2.12, $\dot{V}(x(t))$ must be negative whenever $V(x(t+\theta)) < pV(x(t))$ for some $p > 1$ and for all $\theta \in [-h, 0]$. Since the latter inequality holds for all $\theta \in [-h, 0]$ then we have $V(x(t-h)) < pV(x(t))$ and by application of the $\mathcal{S}$-procedure (see [Boyd et al., 1994] or appendix E.9), we get

$$\begin{bmatrix} A^TP + PA + \tau pP & PA_h \\ A_h^TP & -\tau P \end{bmatrix} \prec 0 \tag{2.42}$$

with $\tau > 0$. Finally let $p = 1 + \delta$, for a small $\delta > 0$, we get the following result

Theorem 2.2.15 *System (2.40) is asymptotically stable independent of delay if there exists $P = P^T \succ 0$ and a scalar $\tau > 0$ such that*

$$\begin{bmatrix} A^TP + PA + \tau P & PA_h \\ A_h^TP & -\tau P \end{bmatrix} \prec 0 \tag{2.43}$$

Note that the feasibility of matrix inequality (2.43) implies the feasibility of matrix inequality (2.42).

It is clear that the latter inequality provides a delay-independent stability test since the matrix inequality does not depend on the delay. Moreover, it is worth noting, that (2.43) is not a LMI due to bilinear term τP but fall into the framework of generalized eigenvalues problem (see [Boyd et al., 1994, Gu et al., 2003, Nesterov and Nemirovskii, 1994]). Nevertheless, the problem is quasi-convex since if τ is fixed, then (2.43) becomes a LMI. This means that a suitable value for τ can be found using an iterative line search.

Delay-dependent stability test via Lyapunov-Razumikhin theorem

We give an example of delay-dependent result obtained from the application of the Lyapunov-Razumikhin theorem 2.2.12. The proof can be found in [Gu et al., 2003] and is omitted since it requires preliminary results on stability of distributed delay which are not of interest. However, it is important to say that it is based on the Euler model transformation.

Theorem 2.2.16 *System (2.40) is delay-dependent asymptotically stable is there exists $P = P^T \succ 0$ and scalars $\alpha, \alpha_0, \alpha_1 > 0$ such that*

$$\begin{bmatrix} M & P(\alpha I - A_h)A & P(\alpha I - A_h)A_h \\ \star & -\alpha_0 P - \alpha h A_0^T PA_0 & -\alpha h A^T PA_h \\ \star & \star & -\alpha_1 P - \alpha h A_h^T PA_h \end{bmatrix} \prec 0$$

holds with $M = \dfrac{1}{r}\left[P(A + A_h) + (A + A_h)^T P\right] + (\alpha_0 + \alpha_1)P$

A discussion on the choice of scalars $\alpha, \alpha_i,\ i = 0, 1$ is provided in [Gu et al., 2003]. As previously, the computation of $P, \alpha, \alpha_i,\ i = 0, 1$ is not an easy task since the resulting condition is not a LMI. The problem is quasi-convex and an iterative procedure should be performed in order to find suitable values for $\alpha, \alpha_i,\ i = 0, 1$. However, this iterative procedure is more difficult than in the delay-independent case since the search has to be performed over a three-dimensional space (instead of a one-dimensional), which is more involved from a algorithmic and computational point of view.

2.2.1.5 Stability Analysis: Lyapunov-Krasovskii Functionals

Despite of the simplicity of Lyapunov-Razumikhin functions, they generally lead to nonlinear matrix inequalities and to conservative results due to the use of non-equivalent model transformations. The use of Lyapunov-Krasovskii functionals, even if historically were used with identical model-transformations, have led to more and more accurate LMI results by applying either more precise bounding techniques of cross-terms (as of [Park, 1999, Park et al., 1998]), or more exact model transformations (as of [Fridman, 2001, Fridman and Shaked, 2001]) or also other methods without any model transformations (see for instance [Briat et al., 2008a, Gouaisbaut and Peaucelle, 2006b, Han, 2005a, Xu and Lam, 2007, Xu et al., 2006]).

This section is devoted to provide, in a chronological path, different results on delay-independent and delay-dependent stability based on Lyapunov-Krasovskii theorem 2.2.10. First of all, a simple delay-dependent stability test will be provided and secondly a delay-dependent stability test will be developed. The delay-dependent stability test is based on the Euler model transformation and induces cross terms in the equations. These terms involving products of signals at time t and the integral of same signals over the $[t-h, t]$ are of great difficulty. Different bounds have been provided in the literature to avoid overcome these difficulties and are of interest since they had led to more and more accurate results. Finally, several other stability tests not based on model transformation and avoiding them are introduced.

A more complete review on Lyapunov-Krasovskii functionals is given in Appendix **??**.

Delay-Independent stability test via Lyapunov-Krasovskii theorem

Consider the Lyapunov-Krasovskii functional given by

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta \tag{2.44}$$

where $P, Q \in \mathbb{S}^n_{++}$ are constant decision matrices.

Computing the derivative of the Lyapunov-Krasovskii functional $V(x_t)$ along the trajectories solutions of system (2.27) yields

$$\begin{array}{rcl}\dot{V}(x_t) &=& \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) + x(t)^T Q x(t) - x(t-h)^T Q x(t-h)\\ &=& [Ax(t) + A_h x(t-h)]^T P x(t) + x(t)^T P [Ax(t) + A_h x(t-h)]\\ && +x(t)^T Q x(t) - x(t-h)^T Q x(t-h)\\ &=& \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \begin{bmatrix} A^T P + PA + Q & PA_h \\ A_h^T P & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}\end{array}$$

By enforcing the latter quadratic form to be negative definite we get

$$\begin{bmatrix} A^T P + PA + Q & PA_h \\ A_h^T P & -Q \end{bmatrix} \prec 0 \tag{2.45}$$

Moreover by continuity of the eigenvalues this is equivalent to say that

$$\begin{bmatrix} A^T P + PA + Q + \varepsilon I & PA_h \\ A_h^T P & -Q \end{bmatrix} \prec 0 \tag{2.46}$$

for some $\varepsilon > 0$. This implies that $\dot{V}(t) \leq -\varepsilon ||x(t)||^2$ and the Lyapunov-Krasovskii theorem is satisfied. We then obtain the following result:

Theorem 2.2.17 *System (2.40) is asymptotically stable for any delay if there exist matrices $P = P^T \succ 0$ and $Q = Q^T \succ 0$ such that*

$$\begin{bmatrix} A^T P + PA + Q & PA_h \\ \star & -Q \end{bmatrix} \prec 0 \tag{2.47}$$

holds.

It is worth noting that the structure recalls the one obtained from the application of the Lyapunov-Razumikhin theorem, but is LMI in the current case. Moreover, this test is less conservative than the delay-independent Lyapunov-Razumikhin test since matrix Q is free and independent of P in the Lyapunov-Krasovskii test while the matrix is τP is the Lyapunov-Razumikhin test and strongly correlated to P. As a conclusion the Lyapunov-Krasovskii based test includes the Lyapunov-Razumikhin test as a particular case $Q = \tau P$.

Delay-Dependent stability test via Lyapunov-Krasovskii theorem

Many studies have dealt with the problem of determination of the delay-margin for time-delay systems. The aim of this paragraph is to provided an evolutive point of view of methods used to determine the delay-margin of a time-delay through the use of the Lyapunov-Krasovskii theorem 2.2.10. In this objective, model transformations have played a central role (and sometimes still play an important role in certain approaches). Despite of their effect of inducing additional dynamics leading to a certain conservatism they have facilitated the derivation of delay-dependent stability conditions. However, additional dynamics are not the only difficulties that they induce, they also generate cross-terms in the the mathematical development of the stability condition. While additional dynamics are a hidden problem which is not viewed directly by in the mathematical proof of stability tests, cross-terms are mathematical difficulties that have needed to be overcome or avoided.

Let us consider the following Lyapunov-Krasovskii functional

$$\begin{array}{rcl} V(x_t, \dot{x}_t) & = & V_1(x_t) + V_2(x_t) + V_3(x_t, \dot{x}_t) \\ V_1(x_t) & = & x(t)^T P x(t) \\ V_2(x_t) & = & \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta \\ V_3(x_t, \dot{x}_t) & = & \displaystyle\int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T Z \dot{x}(\eta) d\eta d\theta \end{array} \tag{2.48}$$

with $P = P^T, Q = Q^T, Z = Z^T \succ 0$. According to Euler transformation, system (2.27) is turned into

$$\dot{x}(t) = (A + A_h)x(t) - \int_{t-h}^{t} x(\theta) d\theta \tag{2.49}$$

Computing the derivative of V along the trajectories solutions of the latter system yields

$$\begin{array}{rcl} \dot{V}_1(x_t) & = & \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) \\ & = & x(t)^T[(A+Ah)^T P + P(A+A_h)]x(t) - 2x(t)^T PA_h \int_{t-h}^{t} \dot{x}(\theta)d\theta \\ & = & x(t)^T[(A+Ah)^T P + P(A+A_h)]x(t) \\ & & -2x(t)^T PA_h \int_{t-h}^{t} [Ax(\theta) + A_h x(\theta-h)]d\theta \\ \dot{V}_2(x_t) & = & x(t)^T Qx(t) - x(t-h)^T Qx(t-h) \\ \dot{V}_3(x_t,\dot{x}_t) & = & h\dot{x}(t)^T Z\dot{x}(t) - \int_{t-h}^{t} \dot{x}(\theta)^T Z\dot{x}(\theta)d\theta \end{array} \tag{2.50}$$

It is possible to see that a cross-term appears in $\dot{V}_1$ and is a coupling between the state at time t and an integral of $Ax(\theta) + A_h x(\theta - h)$ over $[t-h, t]$. This annoying term must be bounded in order to decouple the integral for $x(t)$. A simple bound can be provided by noting that

$$\int_{t-h}^{t} \begin{bmatrix} x(t) \\ x(\theta) \\ x(\theta-h) \end{bmatrix}^T \begin{bmatrix} PA_h \\ A^T \\ A_h^T \end{bmatrix} Z \begin{bmatrix} PA_h \\ A^T \\ A_h^T \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(\theta) \\ x(\theta-h) \end{bmatrix} d\theta \geq 0 \tag{2.51}$$

for some $Z = Z^T \succ 0$ and hence

$$\begin{array}{rcl} -2x(t)^T PA_h \int_{t-h}^{t} \dot{x}(\theta)d\theta & = & -2x(t)^T PA_h \int_{t-h}^{t} Ax(\theta) + A_h x(\theta-h)d\theta \\ & \leq & \int_{t-h}^{t} x(t)^T PA_h Z^{-1} A_h^T Px(t)d\theta + \int_{t-h}^{t} \dot{x}(\theta)^T Z\dot{x}(\theta)d\theta \\ & \leq & hx(t)^T PA_h Z^{-1} A_h^T Px(t) + \int_{t-h}^{t} \dot{x}(\theta)^T Z\dot{x}(\theta)d\theta \end{array}$$

And thus we have

$$\begin{array}{rcl} \dot{V} & \leq & x(t)^T[(A+Ah)^T P + P(A+A_h) + Q]x(t) + hx(t)^T PA_h Z^{-1} A_h^T Px(t) \\ & & -x(t-h)^T Qx(t-h) + h\dot{x}(t)^T Z\dot{x}(t) \end{array}$$

Finally since

$$+h\dot{x}(t)^T Z\dot{x}(t) = h \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^T ZA & A^T ZA_h \\ \star & A_h^T ZA_h \end{bmatrix} \tag{2.52}$$

we get

$$\dot{V} \leq \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} \Psi & hA^T ZA_h \\ \star & -Q + hA_h^T ZA_h \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \tag{2.53}$$

where $\Psi = (A+Ah)^T P + P(A+A_h) + Q + hA^T ZA + hPA_h Z^{-1} A_h^T P$ and therefore system (2.27) is delay-dependent stable with delay margin h if there exists symmetric positive definite matrices P, Q, Z such that the LMI

$$\begin{bmatrix} (A+Ah)^T P + P(A+A_h) + Q + hA^T ZA & hA^T ZA_h & +hPA_h \\ \star & -Q + hA_h^T ZA_h & 0 \\ \star & \star & -hZ \end{bmatrix} \prec 0 \tag{2.54}$$

holds.

Through the use of the Lyapunov-Krasovskii theorem 2.2.10 and the Euler model transformation we have developed a delay-dependent stability test. This model transformation has introduced cross terms which have been bounded by a technique based on a completion of the squares. This bound allowed to compensate the integral term coming the differentiation of V_3 and then remove the annoying integral term

$$\int_{t-h}^{t} \dot{x}(\theta)^T Z \dot{x}(\theta) d\theta \tag{2.55}$$

of the expression of $\dot{V}$.

Obviously, the bound on cross-terms is very conservative since, while the left-hand side of the inequality may be negative, the right-hand side is always nonnegative. One of the great improvement of the Lyapunov-Krasovskii based methods was the introduction of better bounds on cross-terms. Some additional material is detailed in Appendix F.2 on bounding cross-terms.

Park's Bounding Method A seminal result on time-delay system (from my point of view) is provided here and has been introduced in [Park, 1999, Park et al., 1998]. The idea was based on a more accurate bounding of cross terms in the derivative of the Lyapunov-Krasovskii functional (see appendix F.2 or [Park, 1999, Park et al., 1998]).

Park [1999] introduced the following lemma:

Lemma 2.2.18 *Assume that $a(\alpha) \in \mathbb{R}^{n_x}$ and $b(\alpha) \in \mathbb{R}^{n_y}$ are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$ and any matrix $M \in \mathbb{R}^{n_y \times n_y}$, the following holds*

$$-2 \int_{\Omega} b(\alpha)^T a(\alpha) d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \Psi \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha \tag{2.56}$$

with

$$\Psi = \begin{bmatrix} X & XM \\ M^T X & (M^T X + I) X^{-1} (XM + I) \end{bmatrix}$$

The bound provided in the latter lemma is able to provide a better bound on the cross term by allowing negative values for the bound. With the following Lyapunov-Krasovskii functional

$$V(x_t, \dot{x}_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T A_h^T R A_h \dot{x}(\eta) d\eta d\theta \tag{2.57}$$

and the Park's bounding theorem together leads to the theorem

Theorem 2.2.19 *System (2.40) is asymptotically delay-dependent stable for all $h \in [0, \bar{h}]$ if there exist $P = P^T \succ 0$, $Q = Q^T \succ 0$, $R = R^T \succ 0$, $V = V^T \succ 0$ and W such that*

$$\begin{bmatrix} M_{11} & -W^T A_h & A^T A_h^T V & \bar{h}(W^T + P) \\ \star & -Q & A_h^T A_h^T V & 0 \\ \star & \star & -V & 0 \\ \star & \star & \star & -V \end{bmatrix} \prec 0 \tag{2.58}$$

holds with $M_{11} = (A + A_h)^T P + P(A + A_h) + W^T A_h + A_h^T W + V$.

Although this technique allows to consequently reduce the conservatism of the method by finding a more accurate bound on cross-terms, it is still limited by the use of the Euler model-transformation (which may introduce additional dynamics) and hence it would be more convenient to use Park's bounding method with a model transformation which does not generate additional dynamics.

Descriptor Model Transformation This model transformation has been introduced in [Fridman, 2001, Fridman and Shaked, 2001] and as shown in Section 2.2.1.3, it does not introduce any additional dynamics. It is briefly recalled here for system (2.40):

$$\mathcal{E}\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \mathcal{A}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \mathcal{A}_h \int_{t-h}^{t} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds \tag{2.59}$$

where $\mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix}$ and $\mathcal{A}_h = \begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix}$.

One of the earliest results in this framework considers the Lyapunov-Krasovskii functional

$$V(x_t, y_t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T E^T P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{-h}^{0}\int_{t+\theta} y(s)^T R y(s) ds d\theta \tag{2.60}$$

where $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$, $E^T P = P^T E$, $P_1 = P_1^T \succ 0$ and $R = R^T \succ 0$. It is proved in [Fridman and Shaked, 2001] that such a Lyapunov-Krasovskii functional leads to the following theorem

Theorem 2.2.20 *System (2.40) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exists matrices $P_1 = P_1^T \succ 0$, $R = R^T \succ 0$, P_2, P_3 such that the LMI*

$$\begin{bmatrix} (A+A_h)^T P_2 + P_2^T (A + A_h) & P_1 - P_2^T + (A+A_h)^T P_3 & \bar{h} P_2^T A_h \\ \star & -P_3 - P_3^T + hR & \bar{h} P_3^T A_h \\ \star & \star & -\bar{h} R \end{bmatrix} \prec 0 \tag{2.61}$$

holds.

This results is based on a bounding technique of cross terms involving a positive matrix as on page 97. However, results of [Fridman and Shaked, 2002b] involves Park's bounding technique and leads to less conservative stability conditions coupled with complete Lyapunov-Krasovskii functional [Fridman, 2006a]. Although this method is interesting and leads to results of quality, it still leads to cross terms which are difficult to bound and result in conservative conditions from an absolute point of view.

Method of Free Weighting Matrices The following approach has been introduced in [He et al., 2004] and consists in injecting additional constraints into the LMI in order to tackle relations between signals involved in the system. These constraints involve additional free variables adding extra-degree of freedom into the LMI and this motivates the denomination of *free weighting matrices approach.*

The Lyapunov-Krasovskii functional used in [He et al., 2004] is

$$V(x_t, \dot{x}_t) = x(t)^T P x(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta \tag{2.62}$$

and is very similar to (2.57).

It is important to note that the following equality holds for all signals $\dot{x}, x, x_h$ governed by the expression of system (2.27).

$$\begin{array}{lcl}
2\left[x(t)^T N_1 + x(t-h)^T N_2 + \dot{x}(t) N_3\right] \cdot \left[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}(s) ds\right] & = & 0 \\
2\left[x(t)^T T_1 + x(t-h)^T T_2 + \dot{x}(t) T_3\right] \cdot \left[\dot{x}(t) - Ax(t) - A_h x(t-h)\right] & = & 0 \\
\bar{h} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix}^T X \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix} - \int_{t-h}^{t} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix}^T X \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t) \end{bmatrix} d\theta & \geq & 0 \\
\text{for a free matrix } X = X^T \succeq 0 & &
\end{array} \tag{2.63}$$

Indeed, the first constraint defines the Euler integral formula, the second constraint defines the model of the system and the last one defines $\bar{h}$ as the maximal value of the time-delay h. The key idea in this method is to differentiate the Lyapunov-Krasovskii functional but do not substitute the values of signal $\dot{x}$ in it. The constraints are then injected and this results in a quadratic form in $\mathrm{col}(\dot{x}(t), x(t), x(t-h))$ involving a integral quadratic term with vector $\mathrm{col}(\dot{x}(t), x(t), x(t-h), \dot{x}(s))$. By an appropriate choice of the matrix X the integral term can be neglected and finally by a Schur complement the following result is obtained:

Theorem 2.2.21 *System (2.40) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exists matrices $P = P^T \succ 0$, $Q = Q^T \succ 0$, $R = R^T \succ 0$, $X = X^T \succeq 0$ $N_1, N_2, N_3, T_1, T_2, T_3$ such that the LMI*

$$\begin{bmatrix}
Q + N_1^H - (T_1 A)^H & N_2^T - N_1 - A^T T_2^T - T_1 A_h & P + N_3^T + T_1 - A^T T_3^T & \bar{h} N_1 \\
\star & -Q - N_2^H - (T_2 A_h)^H & -N_3^T + T_2 - A_h^T T_3^T & \bar{h} N_2 \\
\star & \star & \bar{h} R + T_3^H & \bar{h} N_3 \\
\star & \star & \star & -\bar{h} R
\end{bmatrix} \prec 0 \tag{2.64}$$

holds.

While the addition of free variables is an advantage in the stability analysis (especially for robust stability analysis [He et al., 2004]), it becomes a drawback in synthesis problems since these decision variables are coupled to the system matrices (hence to the controller of observers gain) preventing to find a linearizing change of variable. A usual method consists in assuming a common simplification $T_i = \varepsilon_i K$ where ε_i are chosen fixed scalars and K is a decision matrix.

Approach using Jensen's inequality We give here a result which is not based on a model transformation but uses the Jensen's inequality (see Appendix F.1) and allows to avoid the bounding of cross-terms and any use of model transformation. It has been provided in different papers for instance in [Gouaisbaut and Peaucelle, 2006b, Han, 2005a].

Let us consider the Lyapunov-Krasovskii functional (2.62) and computing its time-derivative along the trajectories solutions of system (2.27) gives

$$\begin{aligned}\dot{V} &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP+PA+Q & PA_h \\ A_h^TP & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} + h\dot{x}(t)^T R\dot{x}(t) \\ &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP+PA+Q+hA^TRA & PA_h+hA^TRA_h \\ A_h^TP+hA_h^TRA & -Q+hA_h^TRA_h \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ &\quad - \int_{t-h}^{t} \dot{x}(\theta)^T R\dot{x}(\theta)d\theta\end{aligned}$$

At first sight the integral term could be neglected (bounded above by 0) but this will result in a too conservative condition. A more tight solution is the use of the Jensen's inequality on this integral term. The Jensen's inequality allows to establish the following bound on the integral term

$$-\int_{t-h}^{t} \dot{x}(s)^T R\dot{x}(s)ds \le -\bar{h}^{-1}\left(\int_{t-h}^{t} \dot{x}(s)ds\right)^T R\left(\int_{t-h}^{t} \dot{x}(s)ds\right) \tag{2.65}$$

Finally this leads to the following result

Theorem 2.2.22 *System (2.40) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exists matrices $P = P^T \succ 0$, $Q = Q^T \succ 0$, $R = R^T \succ 0$ such that the LMI*

$$\begin{bmatrix} A^TP+PA+Q-\bar{h}^{-1}R+\bar{h}A^TRA & PA_h+\bar{h}^{-1}R+\bar{h}A^TRA_h \\ \star & -Q-\bar{h}^{-1}R+\bar{h}A_h^TRA_h \end{bmatrix} \prec 0 \tag{2.66}$$

holds.

As remark, it is important to note that this result is identical to the method of free weighting matrices presented in the previous paragraph on page on page 100. Indeed, LMI (2.64) can be written as

$$\Psi + U^TZV + V^TZ^TU \prec 0 \tag{2.67}$$

where

$$\Psi = \begin{bmatrix} Q & 0 & P & 0 \\ \star & -Q & 0 & 0 \\ \star & \star & \bar{h}R & 0 \\ \star & \star & \star & -\bar{h}R \end{bmatrix} \qquad Z = \begin{bmatrix} T_1 & N_1 \\ T_2 & N_2 \\ T_3 & N_3 \end{bmatrix} \qquad U = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} -A & -A_h & I & 0 \\ I & -I & 0 & \bar{h}I \end{bmatrix}$$

Here the matrix Z is an unconstrained matrix and hence the projection lemma applies (see appendix E.18). It states that there exist at least one solution Z to (2.67) if and only if the two following underlying LMIs hold

$$\begin{aligned}\mathrm{Ker}[U]^T\Psi\mathrm{Ker}[U] &\prec 0 \\ \mathrm{Ker}[V]^T\Psi\mathrm{Ker}[V] &\prec 0\end{aligned}$$

These basis of null-spaces can be expressed as

$$\mathrm{Ker}[U] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \qquad \mathrm{Ker}[V] = \begin{bmatrix} I & 0 \\ 0 & I \\ A & A_h \\ -\bar{h}^{-1} & \bar{h}^{-1}I \end{bmatrix} \tag{2.68}$$

The leads to

$$\mathrm{Ker}[U]^T \Psi \mathrm{Ker}[U] = -\bar{h}R \tag{2.69}$$

which is negative definite by definition of $R \succ 0$. This means that feasibility of (2.67) is equivalent to the feasibility of the second underlying LMI:

$$\mathrm{Ker}[V]^T \Psi \mathrm{Ker}[V] = \begin{bmatrix} A^T P + PA + Q - \bar{h}^{-1}R + \bar{h}A^T RA & PA_h + \bar{h}^{-1}R + \bar{h}A^T RA_h \\ \star & -Q - \bar{h}^{-1}R + \bar{h}A_h^T RA_h \end{bmatrix} \tag{2.70}$$

The latter LMI is equivalent to (2.66) (they are identical modulo a Schur's complement) which allows to conclude that (2.66) and (2.64) define the same stability criterium. The advantage of formulation (2.64) is the decoupling between Lyapunov matrices P, Q, R and data matrices A, A_h which allows to provide interesting robust stability result for polytopic type uncertainties (see for instance [Gouaisbaut and Peaucelle, 2006b, He et al., 2004]). For stability analysis, criterion (2.66) is more interesting since it has low computational complexity (due to the absence of 'slack' variables and therefore a lower number of decision matrices).

Actually, many results in time-delay systems are related to each others modulo congruence transformations, Schur's complement or through the use of other theorems. This is emphasized in [Xu and Lam, 2007]. Other Lyapunov-Krasovskii based approaches avoiding model transformation have been provided in many research papers, see for instance [Xu et al., 2006].

2.2.1.6 Stability Analysis: (Scaled) Small-Gain Theorem

We have presented different results based on Lyapunov-Krasovskii functionals. It is aimed here at showing that similar results can be retrieved through the use of (scaled) small-gain theorem. Indeed, it is possible to provide delay-independent and delay-dependent stability tests based on the use of the small-gain theorem as emphasized for instance in [Zhang et al., 2001]. The correspondence between small-gain results and Lyapunov-Krasovskii will be emphasized.

Let us consider here the following operators:

$$\begin{array}{rcl} \mathscr{D}_h & : & x(t) \to x(t-h) \\ \mathscr{S}_h & : & x(t) \to \displaystyle\int_{t-h}^{t} x(s)ds \end{array}$$

Delay-Independent Stability Test using Scaled Small-Gain Theorem

This paragraph is devoted to delay-independent stability test using scaled-small gain theorem. First of all, system (2.27) must be rewritten as an interconnection of two subsystems (i.e. a linear finite dimensional systems and the delay operator $\mathscr{D}_h$) according to the framework of small-gain theorem. Hence (2.27) is rewritten as

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h w(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{D}_h(z(t)) \end{array} \tag{2.71}$$

It has been shown that the operator $\mathscr{D}_h(\cdot)$ is asymptotically stable and therefore has finite $\mathcal{H}_\infty$-norm. Indeed, if the input of the operator has finite $\mathcal{L}_2$-norm then the output, which is the delayed input with a constant delay, will have finite energy too; this shows stability. In order to determine the value of the $\mathcal{H}_\infty$-norm of $\mathscr{D}_h(\cdot)$ it suffices to compute the ratio of the output energy over the input energy:

$$\begin{aligned}\int_0^{+\infty} w(\theta)^T w(\theta) d\theta &= \int_0^{+\infty} z(\theta-h)^T z(\theta-h) d\theta \\ &= \int_{-h}^{+\infty} z(\theta')^T z(\theta') d\theta'\end{aligned}$$

with the change of variable $\theta' = \theta - h$. Hence assuming zero initial conditions (i.e. $z(t) = 0$ for all $t < 0$) we get

$$\int_0^{+\infty} w(\theta)^T w(\theta) d\theta = \int_0^{+\infty} z(\theta')^T z(\theta') d\theta'$$

showing that the operator $\mathscr{D}_h(\cdot)$ has unitary $\mathcal{H}_\infty$-norm.

Using this result it is possible to apply the small-gain result issued from the Hamiltonian function

$$H(x_t) = S(x) - \int_0^t s(x(\tau), x(\tau-h)) d\tau \tag{2.72}$$

where $S(x) = x(t)^T P x(t)$ is the storage function and the supply-rate is given by

$$s(x(t), x(t-h)) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

In the dissipativity framework, if the derivative $\dot{H}$ of the Hamiltonian function H is negative definite then this means that the interconnected system (2.71) is asymptotically stable and hence (2.27) is delay-independent stable. Differentiating H along the trajectories solution of system (2.71) gives

$$\dot{H} := \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \begin{bmatrix} A^T P + PA + L & PA_h \\ \star & -L \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \prec 0 \tag{2.73}$$

Finally we obtain the following theorem:

Theorem 2.2.23 *System (2.40) is delay-independent asymptotically stable if there exist matrices $P = P^T \succ 0$ and $L = L^T \succ 0$ such that the LMI*

$$\begin{bmatrix} A^T P + PA + L & PA_h \\ \star & -L \end{bmatrix} \prec 0 \tag{2.74}$$

holds.

It is easy to recognize the LMI obtained by application of the Lyapunov-Krasovskii theorem with Lyapunov-Krasovskii functional

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T L x(\theta) d\theta \tag{2.75}$$

as detailed in paragraph on page 96. This suggests that the Hamiltonian function H coincides with the above Lyapunov-Krasovskii functional. This is proved in what follows.

First of all rewrite H as

$$\begin{aligned} H(x_t) &= x(t)^T P x(t) + \int_0^t (x(s)^T L x(s) - x(s-h)^T L x(s-h)) ds \\ &= x(t)^T P x(t) + \int_0^t \int_{s-h}^s Y(\tau) d\tau ds \end{aligned}$$

where $Y(t) = \dfrac{d}{dt}(x(t)^T L x(t))$.

Now let $\tau' = \tau - s + h$ and in this case we have

$$H(x_t) = x(t)^T P x(t) + \int_0^t \int_0^h Y(\tau' + s - h) d\tau' ds$$

Now exchanging the order of integration yields

$$\begin{aligned} H(x_t) &= x(t)^T P x(t) + \int_0^h \int_0^t Y(\tau' + s - h) ds d\tau' \\ &= x(t)^T P x(t) + \int_0^h \left(x(\tau' + t - h)^T L x(\tau' + t - h) - x(\tau' - h)^T L x(\tau' - h) \right) d\tau' \end{aligned}$$

Assuming zero initial conditions (i.e. $x(s) = 0$ for all $s \leq 0$ hence $Y(s) = 0$ for all $s \leq 0$) then we have

$$\int_0^h x(\tau' - h)^T L x(\tau' - h) d\tau' = 0$$

and hence $H(x_t)$ reduces to

$$H(x_t) = x(t)^T P x(t) + \int_0^h x(\tau' + t - h)^T L x(\tau' + t - h) d\tau' \tag{2.76}$$

Finally let $\theta = \tau' + t - h$ and thus we obtain

$$H(x_t) = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T L x(\theta) d\theta \tag{2.77}$$

and Lyapunov-Krasovskii functional (2.75) is retrieved.

In [Zhang et al., 2001] the relation between Lyapunov-Krasovskii and small-gain results for time-delay, in general, is also emphasized. In [Bliman, 2001], less delay-independent stability tests are provided, based on extension of Lyapunov-Krasovskii functions which can also be viewed as an extension of small-gain based results introduced in this paragraph.

Delay-Dependent Stability Test using Scaled Small-Gain Theorem

According to operator $\mathscr{S}_h(\cdot)$, system (2.27) is rewritten as

$$\begin{aligned} \dot{x}(t) &= (A + A_h)x(t) - A_h w(t) \\ z(t) &= (A + A_h)x(t) - A_h w(t) \\ w(t) &= \mathscr{S}_h(z(t)) \end{aligned} \tag{2.78}$$

This reformulation is identical to the Euler model transformation (see Section 2.2.1.2) and then adds additional dynamics (see Section 2.2.1.3). Hence systems (2.27) and (2.78) are not equivalent. The operator $\mathscr{S}_h$ is LTI and it has be shown that it is stable; therefore it has finite $\mathcal{H}_\infty$ norm. First, note that the corresponding transfer function is given by

$$\hat{\mathscr{S}_h}(s) = \frac{1-e^{-sh}}{s} \tag{2.79}$$

The $\mathcal{H}_\infty$ norm γ_∞ is defined as

$$\begin{aligned} \gamma_\infty &:= \sup_{s\in\mathbb{C}^+}\left|\frac{1-e^{-sh}}{s}\right| = \sup_{\omega\in\mathbb{R}}\left|\frac{1-e^{-j\omega h}}{j\omega}\right| \\ &= \sup_{\omega\in\mathbb{R}}\frac{\left|1-e^{-j\omega h}\right|}{\omega} = \lim_{\omega\to 0^+}\frac{\left|1-e^{-j\omega h}\right|}{\omega} \\ &= h \leq \bar{h} \end{aligned} \tag{2.80}$$

For any $h \in [0, \bar{h}]$, the worst-case $\mathcal{H}_\infty$ norm of the operator $\mathscr{S}_h$ is $\bar{h}$. This interesting fact allows to express delay-dependent result from scaled small-gain theorems. Define the storage function $S(x) = x^T P x$ and the supply-rate

$$s(\dot{x}(t), x(t), x(t-h)) = \begin{bmatrix} \dot{x}(s) \\ x(s)-x(s-h) \end{bmatrix}^T \begin{bmatrix} \bar{h}L & 0 \\ 0 & -L \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ x(s)-x(s-h) \end{bmatrix} \tag{2.81}$$

to construct the Hamiltonian function

$$H(\dot{x}, x_t) = S(x) - \int_0^t s(\dot{x}(\tau), x(\tau), x(\tau-h))d\tau \tag{2.82}$$

Finally differentiating H along the trajectories solution of system (2.78) leads to the following theorem.

Theorem 2.2.24 *System (2.40) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exist matrices $P = P^T \succ 0$ and $L = L^T \succ 0$ such that the LMI*

$$\begin{bmatrix} (A+A_h)^T P + P(A+A_h) & -PA_h & (A+A_h)^T L \\ \star & -\bar{h}L & -A_h^T L \\ \star & \star & -L \end{bmatrix} \prec 0 \tag{2.83}$$

holds.

A connection between Lyapunov-Krasovskii functionals and small-gain results has also been provided in [Zhang et al., 2001] in the delay-dependent framework.

2.2.1.7 Stability Analysis: Padé Approximants

Still in the family of approaches considering a time-delay system into an interconnection of two subsystems, namely a finite dimensional system and a delay operator, the method provided in [Zhang et al., 1999] is of great interest. This method actually holds only for constant delay but leads to very interesting delay-dependent stability results that deserve to be presented. The system that will be considered is given below

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t-h) \\ x(\theta) & = & \phi(\theta), \ \theta \in [-h, 0] \end{array} \tag{2.84}$$

It is rewritten as in (2.71):

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h w(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{D}_h(z(t)) \end{array} \tag{2.85}$$

Since $\mathcal{D}_h$ is a time-invariant linear operator, the corresponding transfer function is then

$$H_d(s) := \frac{W(s)}{Z(s)} = e^{-sh} - 1 \tag{2.86}$$

Due to the complexity of exponential term, Zhang et al. [1999] propose to approximate the delay operator by a parameter dependent filter coinciding with the Padé approximant of e^{-sh} (see Appendix F.4). The idea of using Padé approximants to deal with time-delay is not new and the reader should refer for instance to [Lam, 1990, Saff and Varga, 1975]) but the current solution is interesting since it involves LMIs.

Formally, Padé approximants aim at approximating continuous functions, over a certain domain, by a rational function and this is the reason why it is an interesting tool in systems and control theory. Indeed, as a transfer function should be (strictly) proper, power series for instance cannot be applied as an approximant but Padé approximants can.

From this approximation, the system can be rewritten into an interconnection of a finite dimensional LTI system and a parameter dependent filter (i.e. the Padé approximation).

Let us consider system (2.84) and define the matrices $\bar{A} = A + A_h$ and $A_h = HF$ where H, F are full-rank factors of A_h. Let $\Psi(s, h) = \det(sI - A - A_h e^{-sh})$ be the characteristic quasipolynomial of (2.84). It is well-known that system (2.84) is asymptotically stable for all $h \in [0, \bar{h}]$ if and only if

$$\Psi(j\omega, h) \neq 0, \ \forall \omega \geq 0, \ h \in [0, \bar{h}]$$

Assuming that $G = F(sI - \bar{A})^{-1}H$ and $\Phi(hs) = (e^{-hs} - 1)I$, it is possible to rewrite the system as an interconnection of these two subsystems and then the delay-dependent stability condition is equivalent to the following statement:

$$\det[I - G(j\omega)\Phi(j\omega h)] \neq 0, \ \forall \omega \geq 0, \ h \in [0, \bar{h}] \tag{2.87}$$

Since this statement is very difficult to be checked exactly, then the idea is to provide an inner and outer approximation of the set defining the set of delay-operators for each delay from 0 to $\bar{h}$

$$\Omega_A(\omega, \bar{h}) := \left\{ e^{-j\omega h} : \ h \in [0, \ \bar{h}] \right\} \tag{2.88}$$

Using the Padé approximation, the inner and outer sets are given by

$$\begin{array}{rcl} \Omega_B(\omega, \bar{h}) & := & \left\{ R_m(j\theta\alpha_m\omega) : \ \theta \in [0, \bar{h}] \right\} \\ \Omega_C(\omega, \bar{h}) & := & \left\{ R_m(j\theta\omega) : \ \theta \in [0, \bar{h}] \right\} \end{array} \tag{2.89}$$

where $R_m(s) = \dfrac{N_m(s)}{N_m(-s)}$ is the m^{th} order $(m \geq 3)$ Padé approximation of e^s and

$$\alpha_m := \frac{1}{2\pi}\min\{\omega > 0 : \ R_m(j\omega) = 1\}$$

The following lemma, proved in [Zhang et al., 1999], is useful for comprehensive purpose

Lemma 2.2.25 *For every integer $m \geq 3$, the following statements hold:*

1. *All poles of $R_m(s)$ are in the open left half complex plane.*
2. $\Omega_C(\omega, \bar{h}) \subseteq \Omega_A(\omega, \bar{h}) \subseteq \Omega_B(\omega, \bar{h}), \ \forall \omega \geq 0.$
3. $\lim_{m \to +\infty} \alpha_m = 1$

This result says that the Padé approximmation $R_m(s)$ is a stable operator but, overall that the greater the order is, the better the approximation of the set is. Indeed, if the condition

$$\det[I - G(j\omega)R_m(j\theta\omega)] \neq 0, \ \forall \omega \geq 0, \ \theta \in [0, \bar{h}] \tag{2.90}$$

is a necessary condition for stability since $\Omega_C(\omega, \bar{h})$ is included in $\Omega_A(\omega, \bar{h})$. On the other hand, since $\Omega_B(\omega, \bar{h})$ contains $\Omega_A(\omega, \bar{h})$, therefore

$$\det[I - G(j\omega)R_m(j\theta\alpha_m\omega)] \neq 0, \ \forall \omega \geq 0, \ \theta \in [0, \bar{h}] \tag{2.91}$$

is a sufficient condition only. But when $m \to +\infty$ then $\alpha_m \to 1$ and hence the sets $\Omega_B(\omega, \bar{h})$ and $\Omega_C(\omega, \bar{h})$ converge to each other, to finally coincide with $\Omega_A(\omega, \bar{h})$ showing that, at infinity, the stability of the interconnected system over $\Omega_A(\omega, \bar{h})$, $\Omega_B(\omega, \bar{h})$ and $\Omega_C(\omega, \bar{h})$ are equivalent.

Since we are interested in a delay-dependent stability sufficient condition, the set $\Omega_B(\omega, \bar{h})$ is considered. Let (A_P, B_P, C_P, D_P) be the minimal realization of $P(s) := (R_m(\alpha_m s) - 1)I$ and denote n_P be the order of A_P. Note that in $P(s)$ the set $\Omega_B(\omega, \bar{h})$ is considered due to the presence of α_m. Also introduce $A_s := \bar{A} + HD_PD$, $B_s := B_PF$ and $C_s := HC_P$. Using this formulation, Zhang et al. [1999] provide this very interesting result:

Theorem 2.2.26 *System (2.84) is delay-dependent asymptotically stable for all $h \in [0, \bar{h}]$ if there exist matrices $X_0 \in \mathbb{S}^n_{++}$, $X_{22} \in \mathbb{S}^{n_P}_{++}$ and $X_1 \in \mathbb{R}^{n \times n}$, $X_{12} \in \mathbb{R}^{n_P \times n_P}$ such that*

$$\Pi(0) \prec 0, \qquad \Pi(\bar{h}) \prec 0$$

and

$$\begin{bmatrix} X_0 + \bar{h}X_1 & \bar{h}X_{12} \\ \star & \bar{h}X_{22} \end{bmatrix} \succ 0$$

where

$$\Pi(\theta) := \begin{bmatrix} \Pi_{11}(\theta) & \Pi_{12}(\theta) \\ \star & \Pi_{22}(\theta) \end{bmatrix}$$

with

$$\begin{array}{rcl} \Pi_{11}(\theta) & := & (X_0 + \theta X_1)A_s + X_{12}B_s + A_s^T(X_0 + \theta X_1)^T + B_s^T X_{12}^T \\ \Pi_{12}(\theta) & := & (X_0 + \theta X_1)C_s + X_{12}A_P + \theta A_s^T X_{12} + B_s^T X_{22} \\ \Pi_{22}(\theta) & := & \theta X_{12}^T C_s + \theta C_s^T X_{12} + X_{22}A_P + A_P^T X_{22} \end{array}$$

While considering system (2.11), using the latter theorem with $m = 5$ the delay margin is estimated as $\bar{h} = 6.150$ while the actual delay margin is 6.172. The computed delay-margin is very close to the theoretical one. This result, in year 2000, leads to very good result compared to existing works and this result is still competitive with recent works. Many results based on 'complete' discretized Lyapunov-Krasovskii functionals lead to similar result but with a larger computational complexity.

It is worth noting that in this approach a model transformation is used (expressed through the operator $e^{-sh} - 1$) but does not introduce any conservatism (i.e. additional dynamics). The only constraint imposed by the method is the asymptotic stability of the system for zero delay (since the matrix $\bar{A}$ needs to be Hurwitz). This is not a problem since stability over an interval including 0 is sought. For a more general approach using similar results, the reader should refer to [Knospe and Roozbehani, 2006, 2003, Roozbehani and Knospe, 2005].

2.2.1.8 Stability Analysis: Integral Quadratic Constraints

The approach based on Integral Quadratic Constraints (IQC) [Rantzer and Megretski, 1997] has led to more and more interest since they provide an efficient way to study stability of a wide variety os systems, including time-delay systems [Fu et al., 1998, Jun and Safonov, 2001, 2002, Kao and Rantzer, 2007]. The key idea behind IQC analysis is the $\mathcal{L}_2$ stability of an interconnected system. Indeed, if for exogenous $\mathcal{L}_2$ inputs, the loop-signals have bounded energy this means that the interconnection of systems is stable. The reader should refer to Section 1.3.4.6 for some brief explanations on IQC method.

Part of the results of [Kao and Rantzer, 2007], in the constant-delay case, is presented here. Indeed, Kao and Rantzer [2007] has provided very efficient criteria for stability analysis of time-delay systems which leads to impressive results, sometimes very near of the theoretical ones. Let us consider the delay-operators

$$\begin{array}{lcl} x(t) - x(t-h) & := & \mathscr{S}_h(x(t)) \\ x(t-h) & := & \mathscr{D}_h(x(t)) \end{array} \tag{2.92}$$

Note that these delay operators are equivalent those proposed, for instance, in [Zhang et al., 1999]. But the operators above can be extended to the time-varying delay case while the use of Padé approximation restricts the approach to constant delay case. This suggests that the IQC approach provided by Kao and Rantzer [2007] can be viewed as a generalization of the approach of [Zhang et al., 1999] to the time-varying delay case, although different techniques are used to study stability. Another comparison can be made between results that can be obtained with scaled-small gain, IQC techniques [Jun and Safonov, 2001, 2002] and Lyapunov-Krasovskii functionals which lead to similar (even identical) stability tests.

Using these operators, time-delay system (2.84) can be rewritten as an interconnection of two subsystems:

$$\begin{array}{rcl} \dot{x}(t) & = & (A + A_h)x(t) - A_h w(t) \\ z(t) & = & x(t) \\ w(t) & = & \mathscr{S}_h(z(t)) \end{array} \tag{2.93}$$

In the IQC analysis, the operators involved in the interconnections are defined by their input/output behavior through IQC. The following propositions introduce one IQC for each operator:

Proposition 2.2.27 *The operator $\mathscr{D}_h$ satisfies the IQC defined by*

$$\int_{-\infty}^{+\infty} \begin{bmatrix} v(t) \\ \mathscr{D}_h(v(t)) \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} \begin{bmatrix} v(t) \\ \mathscr{D}_h(v(t)) \end{bmatrix} dt \geq 0 \tag{2.94}$$

for any $X_1 = X_1^T \succeq 0$.

Proposition 2.2.28 *Suppose $h \in [0, \bar{h}]$, then the operator $\mathscr{S}_h$ satisfies any IQC defined by*

$$\int_{-\infty}^{+\infty} \begin{bmatrix} v(t) \\ \mathscr{S}_h(v(t)) \end{bmatrix}^T \begin{bmatrix} |\psi(j\omega)|^2 Y & 0 \\ 0 & -Y \end{bmatrix} \begin{bmatrix} v(t) \\ \mathscr{S}_h(v(t)) \end{bmatrix} dt \geq 0 \tag{2.95}$$

for any $Y = Y^T \succeq 0$ and where $|\psi(j\omega)| \geq g(\omega) + \delta$, for all $\omega \in \mathbb{R}$. The function $g(\omega)$ is defined below

$$g(\omega) := \begin{cases} 2 & \text{if } |\omega| > \dfrac{\pi}{\bar{h}} \\ 2\left|\sin\left(\dfrac{\omega\bar{h}}{2}\right)\right| & \text{if } |\omega| \leq \dfrac{\pi}{\bar{h}} \end{cases} \tag{2.96}$$

A good example of $\psi(s)$ satisfying the above conditions is

$$\psi(s) = 2\frac{\bar{h}^2 s^2 + c\bar{h}s}{\bar{h}^2 s^2 + a\bar{h} + b} + \delta \tag{2.97}$$

where $a = \sqrt{6.5 + 2b}$, $b = \sqrt{50}$, $c = \sqrt{12.5}$ and δ is an arbitrary small positive number.

Using these two IQCs (in the constant delay case), the criterium obtained from the KYP Lemma (see Appendix E.3 and Section 1.3.4.6) leads to a computation of the theoretical the delay margin (or very near) for system (2.11). This result is very effective since the model transformation used to rewrite the time-delay system as an interconnection of a linear system and the delay operator $\mathscr{S}_h(x(t))$ does not introduce any additional dynamics. Hence the interconnected system is completely equivalent to the original system. Moreover, the characterization of the operator $\mathscr{S}_h$ in terms of IQCs is sufficiently tight to remove any conservatism. Finally, in this case, performance analysis would be exact by analyzing the interconnected system.

This makes, at this time and from my point of view, the best numerical tool to analyze stability of a time-delay system since, compared to approaches such as discretized functionals (see [Gu et al., 2003]) or Padé approximation (see [Zhang et al., 1999]), the computational complexity is very low and the method allows for an easy extension to time-varying delays.

Example 2.2.29 *As an example let us consider the system*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)) \tag{2.98}$$

where the delay satisfies $\dot{h} \leq \mu$. It is shown in [Kao and Rantzer, 2007] that the IQC approach presented above leads to the results of Table 2.2.29 using the IQCβ toolbox [Jonsson et al., 2004]. Clearly, the result obtained for $\mu = 0$ is very close to the theoretical one and is computed with only two decision variable introduced by the use of two IQCs. This demonstrates the possibilities of the approach in terms of computational complexity and efficiency.

μ	0	0.1	0.2	0.5	0.8	0.999
[Kim, 2001]	1	0.974	0.883	0.655	0.322	0.001
[Wu et al., 2004]	4.4772	3.604	3.033	2.008	1.364	1.001
[Fridman and Shaked, 2002a]	4.4772	3.604	3.033	2.008	1.364	1.001
[Kao and Rantzer, 2007]	6.117	4.4714	3.807	2.280	1.608	1.360

Table 2.1: Comparison of different stability margins of system (2.98) with respect to the upper bound μ on the derivative of the delay $h(t)$

2.2.1.9 Stability Analysis: Well-Posedness Approach

Finally, the section on stability analysis of time-delay systems is ended by the stability analysis through well-posedness analysis of interconnections; see Section 1.3.4.4, on page 58 or [Iwasaki and Hara, 1998] for more details on well-posedness of feedback systems.

The result provided here is borrowed from [Gouaisbaut and Peaucelle, 2006a] and is an application of results on well-posedness to the interconnection of an uncertain matrix and an implicit linear transformation (see [Peaucelle et al., 2007]) as we will see below:

Let us consider the interconnected system:

$$\begin{array}{rcl} w & = & \Delta(z+v) \\ Ez & = & H(w+u) \end{array} \tag{2.99}$$

where w, z are loop signals, u, v exogenous input signals and Δ the uncertain matrix. The corresponding set-up is depicted in Figure 2.2.

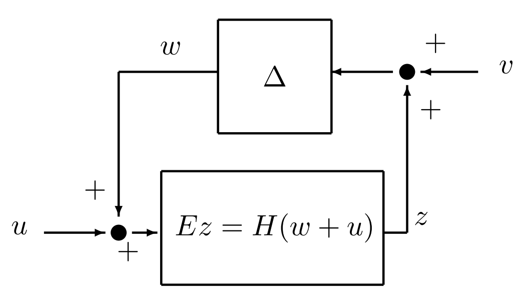

Figure 2.2: Interconnection of the uncertain matrix Δ and the implicit linear transformation $Ez = H(w+u)$

The following result on stability of (2.99) has been proved in [Peaucelle et al., 2007].

Theorem 2.2.30 *The closed-loop system (2.99) is well-posed if and only if there exists an Hermitian matrix $X = X^*$ such that*

$$\begin{bmatrix} EE^{\circledast} & -H \end{bmatrix}_{\perp}^{*} X \begin{bmatrix} EE^{\circledast} & -H \end{bmatrix}_{\perp} \succ 0 \tag{2.100}$$

$$\begin{bmatrix} 0 & I \\ \Delta E_{\perp} & \Delta E^{\circledast} \end{bmatrix}^{*} X \begin{bmatrix} 0 & I \\ \Delta E_{\perp} & \Delta E^{\circledast} \end{bmatrix} \preceq 0 \quad \text{for all } \Delta \in \mathbf{\Delta} \tag{2.101}$$

where $\mathbf{\Delta}$ *is the set of uncertainties,* E° *denotes a full-rank matrix whose columns span the same space as the columns of* E *and* $E^{\circledast} = E^{\circ *}$*. Moreover, if* E *and* H *are real, the equivalence still holds for* X *restricted to be real.*

We aim here at developing a simple delay-dependent stability result from the latter theorem. Define

$$E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ I_n & 0 & -I_n \end{bmatrix} \qquad H = \begin{bmatrix} A & A_h & 0 \\ I_n & 0 & 0 \\ -I_n & I_n & hI_n \end{bmatrix} \qquad \Delta(s) = \begin{bmatrix} s^{-1}I_n & 0 & 0 \\ 0 & e^{-sh}I_n & 0 \\ 0 & 0 & \dfrac{1-e^{-sh}}{sh}I_n \end{bmatrix} \tag{2.102}$$

By substituting these matrices into (2.99) it can be shown that system (2.84) is retrieved. We aim now at giving sufficient conditions of stability using Theorem 2.2.30.

Inequality (2.101) is always verified if X is chosen as

$$X = \begin{bmatrix} 0 & 0 & 0 & -P & 0 & 0 \\ \star & -Q & 0 & 0 & 0 & 0 \\ \star & \star & -R & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & Q & 0 \\ \star & \star & \star & \star & \star & R \end{bmatrix} \tag{2.103}$$

where $P = P^T \succ 0$, $Q = Q^T \succ 0$ and $R = R^T \succ 0$. In this case, (2.100) is equivalent to LMI (2.66) obtained with a Lyapunov-Krasovskii functional. This result has been extended, in a similar fashion as of [Bliman, 2002], to obtain a more accurate delay margin in [Gouaisbaut and Peaucelle, 2006a,b]) by considering higher order derivatives. However this has been provided for constant delays only and Ariba and Gouaisbaut [2007] have extended the results to the time-varying delay case.

The key idea in well-posedness based results is to use Taylor expansions to approximate the time-varying delay operator and the greater the order of the Taylor expansion is, the smaller the conservatism is. While the approach of Zhang et al. [1999] considers the frequency domain, the approach of Ariba and Gouaisbaut [2007], Gouaisbaut and Peaucelle [2006b] lies in the time-domain and hence allows for time-varying delays.

2.2.2 Robustness with respect to delay uncertainty

Stability with respect to delay uncertainty is an important problem which is still not really investigated. Some papers are devoted to or use results on robust stability analysis with respect to delay uncertainty [Kharitonov and Niculescu, 2003, Michiels et al., 2005, Sename and Briat, 2006, Verriest et al., 2002]. The idea (interest) behind of robust stability of systems with uncertain delay is double:

- Assuming that the stability of the system (2.104) is known for a nominal delay value h_0 the maximal deviation δ from this nominal value for which the system remains is stable is sought. Therefore the system will be shown to be stable for any delay belonging to $[h_0 - \delta^+, h_0 - \delta^-]$. In the case of a time-varying delay, the bound on the derivative of the variation η can also be considered.

$$\dot{x}(t) = Ax(t) + A_h(x - h_0 + \theta(t)), \quad \theta(t) \in [\delta^-, \delta^+], \ |\dot{\theta}| < \eta \tag{2.104}$$

- Assuming that a controlled time-delay system (with delay h) by a controller with memory but involving a different time-delay value, say h_c, takes the form (2.105). The implemented delay h_c can be decomposed into a sum of the real delay h and an uncertain value θ, representing the knowledge error on the delay value. In this case, the closed-loop system (2.105) involves two-delays which are interrelated by the latter equality. Here also, the delays can be chosen time-varying and a bounds on the derivatives η, ν can be taken into account:

$$\begin{array}{rcl} \dot{x} & = & Ax(t) + A_h^1 x(t - h(t)) + A_h^2 x(t - h(t) - \theta(t)) \\ & & h(t) \in [0, h_{max}], \ |\dot{h}| < \eta, \ \theta(t) \in [-\delta, \delta], \ |\dot{\theta}| < \upsilon \end{array} \tag{2.105}$$

In both case, some solutions exist and expressed in both frequency and time domains. Indeed, the frequency domain approaches are restricted to deal with constant time-delay while time-domain are not. In the following, we aim at providing 4 methods covering the all the possible scenarios.

2.2.2.1 Frequency domain: Matrix Pencil approach

The approach provided here has been introduced in the nice paper proposed by [Kharitonov and Niculescu, 2003]. The idea is to analyze the stability of perturbed delay system, assuming the stability of the nominal one. The interest of this approach is to provide *necessary and sufficient* conditions in terms of generalized eigenvalue distribution of some (finite dimensional) constant matrix pencil.

Let us consider system (2.104) with constant delays $h - \theta$ which is assumed to be stable for $\theta = 0$. Hence this means that the characteristic quasipolynomial

$$\det(sI_n - A - A_h e^{-sh}) = 0$$

has no solutions with $\Re(s) \geq 0$. Consider now

$$\det(sI_n - A - A_h e^{-s(h-\theta)}) = 0$$

and in this case we are interested to find all terms $\zeta := h - \theta$ such that

$$\det\left(j\omega I - A - A_h e^{-j\omega\zeta}\right) \neq 0, \qquad \forall \omega \in \mathbb{R} \tag{2.106}$$

Note that if (2.106) is guaranteed for all $\zeta \geq 0$ then the system is delay independent stable and else we have a delay-dependent stability result. The following theorem proved in [Kharitonov and Niculescu, 2003] is based on matrix pencils [Chen et al., 1995, Niculescu, 2001] and provides a necessary and sufficient condition to stability of uncertain system (2.106).

Theorem 2.2.31 *The linear time-delay system (2.104) with constant delay perturbation θ is robustly stable if and only if the nominal system (2.104) is stable (i.e. for $\theta = 0$) and the following inequality hold*

$$h - \inf\{\beta : \ (\beta, \alpha) \in \Pi_{h,+}\} < \theta < h - \sup\{\beta : \ (\beta, \alpha) \in \Pi_{h,-}\} \tag{2.107}$$

where

$$\Pi(z) = z \begin{bmatrix} I_p & 0 \\ 0 & \phi_{\otimes}(A_h, I_n) \end{bmatrix} + \begin{bmatrix} 0 & -I_p \\ \phi_{\otimes}(I_n, A_h^T) & \phi_{\oplus}(A, A^T) \end{bmatrix} \tag{2.108}$$

$$\begin{aligned} \Pi_{h,+} &= \Big\{(h_{k_i}, \alpha_k) : \ h_{k_i} = \frac{\alpha_k}{\omega_{k_i}} > h : \ e^{-j\alpha_k} \in \tilde{\sigma}(\Pi), \ j\omega_{k_i} \in \tilde{\sigma}\left(A + e^{-j\alpha_k} A_h\right) - \{0\}, \\ &\qquad 1 \leq k \leq 2p, \ 1 \leq i \leq n \Big\} \\ \Pi_{h,-} &= \Big\{(h_{k_i}, \alpha_k) : \ h_{k_i} = \frac{\alpha_k}{\omega_{k_i}} < h : \ e^{-j\alpha_k} \in \tilde{\sigma}(\Pi), \ j\omega_{k_i} \in \tilde{\sigma}\left(A + e^{-j\alpha_k} A_h\right) - \{0\}, \\ &\qquad 1 \leq k \leq 2p, \ 1 \leq i \leq n \Big\} \end{aligned} \tag{2.109}$$

where $\tilde{\sigma}(\cdot)$ denotes the set of (generalized) eigenvalues of corresponding matrix (pencil) and $\phi_{\otimes}, \phi_{\oplus}$ correspond to the following special matrix tensor product and sum, see Appendix A.5 or [Niculescu, 2001].

This result could be used to analyze stability for systems of the form where two delays are interrelated by an equality and this deserves future attention…

2.2.2.2 Frequency domain: Rouché's Theorem

The Rouché's Theorem, a celebrated result of complex analysis [Levinson and Redheffer, 1970] allows to compute a bound on the variation on the delay for systems of the form (2.105). It provides a sufficient condition only but a bound can be easily computed from the computation of norms of operators. It has been employed in [Sename and Briat, 2006, Verriest et al., 2002].

The Rouché's Theorem [Levinson and Redheffer, 1970] is recalled for reader ease and the proof is provided in Appendix F.7:

Theorem 2.2.32 *Given two functions f and g analytic (holomorphic) inside inside and on a contour γ. If $|g(z)| < |f(z)|$ for all z on γ, then f and $f+g$ have the same number of roots inside γ.*

Let us consider system (2.105) with constant delay. We tacitly assume that it is asymptotically stable system for $h = h_c$, i.e.

$$\dot{x}(t) = Ax(t) + (A_h^1 + A_h^2)x(t-h)$$

is asymptotically stable.

Since we have $h_c = h + \theta$ hence we can write

$$e^{-sh_c} = e^{-sh} + (e^{-s(h+\theta)} - e^{-sh}) = e^{-sh}(1 - \Delta(s)) \tag{2.110}$$

where $\Delta(s) = 1 - e^{-s\theta}$. The characteristic quasipolynomial of the closed-loop system is given by

$$\begin{aligned} \chi(s) &= \det(sI - A - A_h^1 e^{-sh} - A_h^2 e^{-sh_c}) \\ &= \det(sI - A - A_h^1 e^{-sh} - A_h^2 e^{-sh}(1 - \Delta(s))) \\ &= \det((sI - A - A_h^1 e^{-sh} - A_h^2 e^{-sh}) + A_h^{(2)} e^{-sh}\Delta(s)) \\ &= \det(\Psi(s)) \det(I + \Psi(s)^{-1} A_h^2 e^{-sh}\Delta(s)) \end{aligned} \tag{2.111}$$

where $\Psi(s) = sI - A - \left(A_h^1 + A_h^2\right) e^{-sh}$.

As the 'exact' design gives a stable system then $\det(\Psi(s))$ does not change sign when s sweeps the imaginary axis. Then the perturbed closed-loop remains stable if $\det(1 + \Psi(s)^{-1}A_h^2\Delta(s))$ does not change sign for all $s = j\omega$, $\omega \in \mathbb{R}$.

Invoking Rouché's theorem (see appendix F.7) it follows that a stability condition is

$$\left|\left|\Psi(s)^{-1}A_h^2 e^{-sh}\Delta(s)\right|\right|_\infty < 1 \tag{2.112}$$

First recall that $|\Delta(s)| \leq |\delta_h s| \leq \delta_h^+|s|$ for all $s = j\omega$, $\omega \in \mathbb{R}$ and where δ_h^+ is an upper bound on the absolute value of delay uncertainty. Finally we have

$$\left|\left|\Psi(s)^{-1}A_h^2 e^{-sh}\Delta(s)\right|\right|_\infty \leq \delta_h \left|\left|\Psi(s)^{-1}A_h^2 e^{-sh}s\right|\right|_\infty \tag{2.113}$$

and gives the following bound preserving stability

$$\theta_h < 1/||\Psi(s)^{-1}A_h^{(2)}e^{-sh}s||_\infty \tag{2.114}$$

Hence, for any $\theta \in [-\theta_h, \theta_h]$, the determinant has fixed sign, implying the absence of zero crossings, and henceforth the stability of the perturbed system (provided the nominal one is stable). This approach allows to give an analytic bound on the delay error value but when done in the stabilization framework, it is difficult to address a robust stabilization problem directly since the analysis has to be done a posteriori (on the closed-loop system). For this reason, the development of an iterative algorithm seems to be a difficult task.

2.2.2.3 Time-Domain: Small Gain Theorem

Time-domain methods have interesting properties, first they allow for time-varying delays and second it is possible to consider the uncertainty on the delay in the synthesis framework, guaranteeing a prescribed bound on the delay uncertainty. The first method to be investigated is an application of the small-gain theorem.

Note that it is possible to rewrite system (2.105) using Euler transformation as

$$\begin{array}{rcl}\dot{x}(t) & = & Ax(t) + (A_h^1 + A_h^2)x(t-h(t)) - \delta^+ A_h^2 w(t)\\ z(t) & = & \dot{x}(t)\\ w(t) & = & \dfrac{1}{\delta^+}\displaystyle\int_{t-h_c(t)}^{t-h(t)} z(s)ds\end{array} \tag{2.115}$$

With a similar reasoning as of [Gu et al., 2003], the $\mathcal{H}_\infty$ norm of the integral operator can be bounded by δ_h^+ and hence a simple application of the scaled small gain theorem allows to provide a robustness analysis by considering the Hamiltonian function

$$H(x_t) = S(x_t) - \int_0^t \begin{bmatrix} x(s-h(s)) - x(s-h_x(s)) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} L & 0 \\ 0 & -L \end{bmatrix} (\star)^T \tag{2.116}$$

where $S(x)$ is the storage function and the integral is the dissipativity condition related to scaled-small gain. Note that any Lyapunov-Krasovskii functional may play the role of the storage function $S(x)$. A similar results has been provided in [Gu et al., 2003] where a time-varying delay is approximated by a constant one and where the uncertainty represents the time-varying part.

2.2.2.4 Time-Domain: Lyapunov-Krasovskii functionals

Since the scaled-small gain theorem may lead to conservative results it would be more convenient to use a Lyapunov-Krasovskii approach to deal with such a problem of stabilization with incorrect delay value. Considering again system (2.105), the Lyapunov-Krasovskii functional

$$\begin{aligned} V(x_t) &= V_n(x_t) + V_u(x_t) \\ V_n(x_t) &= x(t)^T P x(t) + \int_{t-h}^{t} x(s)^T Q x(s) ds + \int_{-h}^{0}\int_{t+\beta}^{t} \dot{x}(s)^T R x(s) ds d\beta \\ V_u(x_t) &= \int_{t-h_c}^{t} x(s)^T S x(s) ds + \int_{\delta^-}^{\delta^+}\int_{t+\beta-h}^{t} \dot{x}(s) T \dot{x}(s) ds d\beta \end{aligned} \tag{2.117}$$

can lead to a robust stability analysis criterium for system (2.105). This will be detailed in Section 3.7. Note that Kharitonov and Niculescu [2003] have also provided a solution in terms of a complete Lyapunov-Krasovskii functional for such a goal.

2.3 Chapter Conclusion

This ends the section about time-delay systems and their stability analysis. A brief first section has been devoted to different types of time-delay systems representations: systems over a ring, infinite-dimensional systems over a abstract space and functional differential equations. The latter representation has been chosen so to be considered since many tools exist, such as Lyapunov-Krasovskii theorem, and can be extended to LPV case.

In Appendix G, simple methods for constant delays are provided for curiosity purpose only, in order to show that frequency domains exist and that they can provide necessary and sufficient conditions for stability. Since these methods cannot be extended to the time-varying delay case, time-domain approaches have been privileged in this chapter since they apply very well to LPV time-delay systems. Even if all examples of criteria have been developed for systems with constant delays, most of them can be, more or less easily, extended to time-varying delays (except Padé approximation which is actually extended, in a somewhat certain sense, either in [Gouaisbaut and Peaucelle, 2006a] or [Kao and Rantzer, 2007] as explained in the above section).

Among time-domain techniques, fundamental theorems extending Lyapunov's theory have been provided and illustrated through examples of stability tests. While Lyapunov-Razumikhin is a simple test involving to the use of function, the Lyapunov-Krasovskii employs functionals. However, while Lyapunov-Razumikhin tests are not LMIs, Lyapunov-Krasovskii tests are and provide more general results. This has been illustrated that the Razumikhin delay-independent stability test is a particular case of the Krasovskii one. The evolution of Lyapunov-Krasovskii criteria has been discussed by a successive introduction of model transformations, additional dynamics and the problem of cross-terms.

Scaled small gain can be used to develop stability criteria for time-delay systems and a connection between Lyapunov-Krasovskii results has been emphasized. Moreover, these results have also been derived in the IQC framework in [Jun and Safonov, 2001].

A technique based on a an approximation of the delay element by Padé approximants has been presented and shown as an interesting and efficient technique but limited to constant delay-case.

In order to relieve this lack, IQC techniques using the efficient input/output behavior point of view provide very tight solution to the stability analysis of time-delay systems using same operators as in [Zhang et al., 1999] but extended in the time-varying case.

Then, a recent result based on well-posedness has been introduced and is related to recent results based on Lyapunov-Krasovskii functionals.

Finally, results about robust stability of systems with respect to delay uncertainty have been provided as a anticipation of next use in this thesis. Both frequency and time domain techniques have been provided as point of comparison.

All the time-domain techniques have not been introduced in this section and, as an opening, the reader should refer to [Briat et al., 2007a, 2008b, Gouaisbaut and Peaucelle, 2007, Han and Gu, 2001, He et al., 2007, Jiang and Han, 2006, 2005, Kharitonov and Niculescu, 2003, Knospe and Roozbehani, 2006, 2003, Michiels et al., 2005, Roozbehani and Knospe, 2005] and references therein for other techniques. Among them it is important to distinguish range-stability analysis which addresses the problem of finding a compact set of delay value, possibly not including 0, for which the system is stable (similarly as for robustness analysis with delay uncertainty). Most of these results are based on Lyapunov-Krasovskii functionals or approximation of delay elements.

Chapter 3

Definitions and Preliminary Results

THIS CHAPTER aims at introducing some basic concepts and fundamental results used along the thesis. It contains contribution contained below.

Section 3.1 provides redundant notions such as delay and parameter spaces and the class of LPV time-delay systems under consideration throughout this thesis. These definitions are quite common and the relevance of such sets will be emphasized briefly as a justification. Finally an example of LPV time-delay is given in order to motivate the interest of our work on this kind of systems.

Section 3.2 will provide a new relaxation method for polynomially parameter dependent Linear Matrix Inequalities. Indeed, it is well known that parametrized LMIs consist in an infinite (uncountable) number of LMIs that have to be satisfied. When the dependence is linear a convex argument, as used in the polytopic approach (see Section 1.3.2) allows to conclude on the feasibility of the whole set of LMIs only by considering a particular finite set of LMIs (actually the LMIs evaluated at the vertices of the convex polyhedral set containing parameters values). On the other hand, when the dependence is polynomial it is not necessary and sufficient to consider the vertices of the set of values of the parameters. Indeed, such a relaxation can only be done under certain strong assumptions on the degree of polynomials and some matrices. There exists different approaches to solve very efficiently and accurately this type of problems (see Sections 1.3.3.2, 1.3.3.3 and 1.3.3.4). We will provide here a new one based on spectral factorization of parameter dependent matrices and the Finsler's lemma (see Appendix E.16). This approach will turn the polynomially parameter dependent LMI into a slightly more conservative LMI involving 'slack' variables. Such a LMI will have the interest of having a parameter linear dependence on which convex relaxation can be applied without any conservatism. Such an approach has been introduced in [Briat et al., 2008b].

Section 3.3 is devoted to the development of a new relaxation for concave nonlinearity of the form $-\alpha^T\beta^{-1}\alpha$ with $\beta = \beta^T \succ 0$ in negative definite LMIs. Several approach to deal with such non linearities have been provided in the literature: the hyperplane bound and an application of the cone complementary algorithm. While the first one is too conservative since it corresponds to a linearization of the nonlinearity around some fixed point, the second one cannot be applied on parameter-varying matrices. These two limitations motivated us to introduce a new method based on the introduction of a 'slack' variable with the drawback of keeping a nonlinear structure of the problem (the problem becomes BMI). However, even if the structure remains complex and cannot be efficiently solved by interior point algorithms as LMIs, it has a nicer form that the initial problem and can be efficiently solved with iterative LMIs procedures. A discussion is then provided in order to explain the algorithm, its initialization step and optimality gap compared to the initial problem.

Section 3.4 aims at providing a simple algebraic approach in order to compute bounds on the rate of variation of parameters in the polytopic framework. Indeed, in the literature, most of the approaches consider LPV polytopic systems with unbounded parameter variation rates which is rather conservative since it consider constant Lyapunov functions and hence conclude on quadratic stability. When a general parameter dependent system is turned into a polytopic formulation, the values and the dependence is hidden into the new parameters since a mix of all parameters is performed. From this consideration it is difficult to make a correspondence between the derivative of the initial parameters and the derivative of the polytopic parameters. This section provides then a simple methodology to compute these bounds.

Section 3.5 aims at providing a simple stability/performances test expressed through parameter dependent LMIs for LPV time-delay systems. This approach is based on the use of a simple Lyapunov-Krasovskii functional which has been introduced for instance in [Gouaisbaut and Peaucelle, 2006b, Han, 2005a]. This result has the benefit of being interesting from a computational point of view since it involves a few matrix variables and no model transformation is employed. However, it is difficult to use it for synthesis purposes and this motivates the development of an associated relaxation leading to another LMI which can be efficiently used to design controllers and observers.

Section 3.6 extends the 'simple' approach to a discretized version of a 'complete' Lyapunov-Krasovskii functional. The same relaxation scheme is then applied in order to get LMI adapted to design objectives.

Finally, Section 3.7 develops a new Lyapunov-Krasovskii functional for systems with two delays where the delays satisfy an algebraic constraint. Such configuration occurs whenever a time-delay system is controlled/observed by a controller/observer with memory implementing a delay which is different from the system delay. In this case it is important to take into account this specific problem in order to ensure robustness of the closed-loop stability/performances.

3.1 Definitions

This section is devoted to the introduction of the definitions which will be used along the thesis. First of all, delay spaces under consideration will be defined. Restrictions on these sets will be introduced and justified through simple examples. Then, parameters sets will be introduced and a particular class, the delayed parameters, will be introduced and their properties analyzed (continuity, differentiability, set of values...). Finally, the class of LPV systems which will be analyzed in the thesis will be introduced with an example of a milling process borrowed from [Zhang et al., 2002].

3.1.1 Delay Spaces

We will consider throughout this thesis several delay spaces. Each delay-space considers a particular stability result: delay-dependent/independent and rate dependent/independent. Due to the large diversity of these spaces, only some of them are described below:

$$\mathscr{H}_1 := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, [h_{min}, h_{max}]) : \ |\dot{h}| < \mu \right\} \tag{3.1}$$

which defines bounded delay with bounded derivative. It is assumed that when $\mu = 0$ then the delay is constant. We will denote further the set $\mathscr{H}_1^\circ$ the particular case when $h_{min} = 0$. Then,

$$\mathscr{H}_2 := \{ h : \mathbb{R}_+ \to [h_{min}, h_{max}] \} \tag{3.2}$$

defines the set of bounded delays with unbounded derivatives. We will denote further the set $\mathscr{H}_2^\circ$ the particular case when $h_{min} = 0$. Then,

$$\mathscr{H}_3 := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+) : \ |\dot{h}| < \mu \right\} \tag{3.3}$$

defines the set of unbounded delays with bounded derivatives. Finally,

$$\mathscr{H}_4 := \{ h : \mathbb{R}_+ \to \mathbb{R}_+ \} \tag{3.4}$$

corresponds to the set of unbounded delays with unbounded derivatives.

Among them, the most relevant and useful sets are $\mathscr{H}_1$ and $\mathscr{H}_2$. In many cases, $\mathscr{H}_2$ is useful when no information is available on the rate of variation of the delay and then no constraint can be considered. On the second hand, when dealing with delays with bounded derivatives the Lyapunov-Krasovskii functional approach can only be used whenever the delay derivative in less than 1 (or in some cases between -1 and 1), which is very constraining since it appears to be difficult to deal with between delay derivatives between 1 and $+\infty$. Model transformations can be used in order to deal with such cases; see for instance [Gu et al., 2003, Jiang and Han, 2005, Shustin and Fridman, 2007].

The argument that the delay derivative must be greater than 1 can be justified by considering input delay systems and is not of interest in the case of state-delayed system. However this will be explained for completeness. To see this, consider the problem of Figure 3.1 where an transmitter sends data to a receiver continuously (the data is a continuous flow). The data are driven through a medium of length ℓ with with a finite variable speed $v(t)$ depending on the time instant of emission (as in a network where the speed of propagation depends on the occupation of the servers). Hence, the time of transmission is given by $h(t) = \ell v(t)$. When a data is transmitted at times t and $t + \delta t$ then they will be received at times $t + h(t)$ and $t + \delta t + h(t + \delta t)$ respectively.

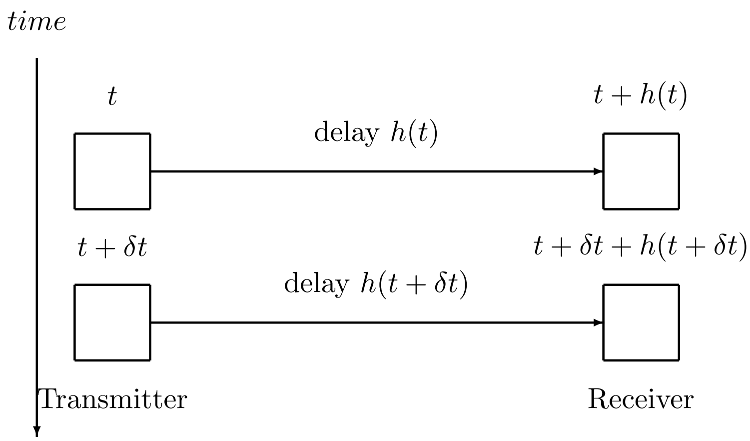

Figure 3.1: Illustration of continuous data transmission between two entities

The causality of the signals claim that if a data value is emitted at time t, it will reach the receiver before the data emitted at time $t + \delta t$ for every $\delta t > 0$. This is translated in the formal expression

$$t + h(t) < t + \delta t + h(t + \delta t) \tag{3.5}$$

then we have

$$-\delta t < h(t + \delta t) - h(t) \tag{3.6}$$

and thus

$$-1 < \frac{h(t + \delta t) - h(t)}{\delta t} \tag{3.7}$$

Since the inequality is true for every $\delta t > 0$ then we get

$$-1 < \dot{h}(t) \tag{3.8}$$

This condition ensures that once emitted the data will be received in a correct order. Note that it may not be the case when considering the control of system over a network using packet switching. In this case, since the data may not follow the same path, then it is not guaranteed that the data will be received in a correct order (this is the reason why the TCP protocol implements a packet counter allowing to reorganize the packets once received).

The second idea, which is important for state-delayed systems, is to look at the evolution of the function $f(t) = t - h(t)$ compared to t. It is clear that $f(t) \leq t$ which means that $h(t) \geq 0$ but it is also interesting to have $f(t)$ increasing. Indeed, having $f(t)$ increasing means that there exists an inverse function $f^{-1}(\cdot)$ and in some applications and computations this property is important. If for some time values t, $f(t)$ is locally decreasing, then this means that there exist $t_1 < t_2$ such that $t_2 - h(t_2) = t_1 - h(t_1)$. This would mean that the same data is considered at different times which may be incorrect.

Let $t_2 = t_1 + \delta t$ with $\delta t > 0$ and thus we have $t_1 + \delta t - h(t_1 + \delta t) = t_1 - h(t_1)$ which is equivalent to

$$\delta t - h(t_1 + \delta t) = -h(t_1)$$

and finally

$$1 = \frac{h(t_1 + \delta t) - h(t_1)}{\delta t}$$

If δt tends to 0, we get

$$1 = \dot{h}(t)$$

This shows that if the delay derivative reaches 1 for some time-instants, then the same data will be used at different times. If this has to be avoided, by continuity, it suffices to restrict $\dot{h}$ to satisfy the inequality

$$\dot{h}(t) < 1 \tag{3.9}$$

Such a function $t - h(t)$ satisfying this property is depicted on Figure 3.2.

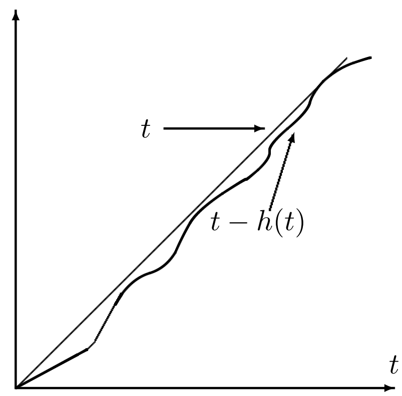

Figure 3.2: Illustration of the nondecreasingness of the function $t - h(t)$

In some specific applications, for instance control of systems with varying sampling-rate (sampled-data systems) [Fridman et al., 2004, Suplin et al., 2007] time delay systems having a derivative equals to 1 almost everywhere are used. Indeed, systems with zero-order hold on the input is turned into a time-delay system with time-varying delay on the input where the time-varying delay describes the zero-order hold with variable period. Since a zero-order hold maintains a specific value during a certain amount of time (the period) it seems obvious that $h(t) = 1$ all over the period (in this case all works as if the time was frozen over the period).

3.1.2 Parameter Spaces

This section is devoted to describing the considered sets of parameters. Only common sets will be briefly introduced and more details are given in Section 1.1. Along this thesis we will mainly focus on continuous parameters (smooth and nonsmooth). In some applications, delayed parameters are encountered and basic properties (continuity and differentiability) of such parameters will be discussed hereafter. First of all, let us introduce the following sets:

$$U_\rho := \times_{i=1}^{N_p}[\rho_i^-, \rho_i^+] \text{ compact of } \mathbb{R}^{N_p} \tag{3.10}$$

where $N_p > 0$ is the number of parameters.

$$U_\nu := \times_{i=1}^{N_p}\{\nu_i^-, \nu_i^+\} \tag{3.11}$$

The set U_ρ is the set of values taken by the parameters and is a bounded orthotope of $\mathbb{R}^{N_p}$. On the second hand, the set U_ν is a discrete set of $\mathbb{R}^{N_p}$ containing 2^{N_p} values. It contains the set of vertices of the orthotope where the parameter derivative values evolve. Hence this orthotope is defined as the convex hull of the points contained in U_ν and is denoted hull$[U_\nu]$.

Sometimes the delay may act on some parameters if the system involve such parameters or due to the use of a particular Lyapunov-Krasovskii functional as in [Zhang et al., 2002]. Thus it seems necessary to introduce this important case. Obviously, the set of values taken by delayed parameters is included into the set of non-delayed parameters. In an absolute point of view they coincide but it will be shown hereunder that it is not so simple and the delay will play an important role. The set of values that can take $\rho_h(t) = \rho(t-h(t))$ for each value of $\rho(t)$ is analyzed below.

Let us consider the case $N_p = 1$ and define the delayed parameter as $\rho(t-h(t))$. Moreover, without loss of generality let $\nu := \nu^+ = -\nu^-$ and $\bar{\rho} = \rho^+ = -\rho^-$. Hence, as the parameter has a bounded derivative, then it satisfies the so-called Lipschitz condition

$$|\rho(t_2) - \rho(t_1)| \leq \nu|t_2 - t_1| \tag{3.12}$$

for any $t_1 \neq t_2$, $t_1, t_2 \in \mathbb{R}_+$. Hence assuming that $t_2 > t_1$ then we have

$$-\nu(t_2 - t_1) \leq \rho(t_2) - \rho(t_1) \leq \nu(t_2 - t_1) \tag{3.13}$$

Let $t_2 = t$ and $t_1 = t - h(t)$ then we obtain

$$-\nu h(t) \leq \rho(t) - \rho(t - h(t)) \leq \nu h(t) \tag{3.14}$$

and hence we obtain

$$\rho(t) - \nu h(t) \leq \rho(t-h) \leq \rho(t) + \nu h(t) \tag{3.15}$$

Since in most cases the current value of the delay $h(t) \in [h_{min}, h_{max}]$ is generally unknown then it is more convenient to consider

$$\rho(t) - \nu h_{max} \leq \rho(t-h) \leq \rho(t) + \nu h_{max} \tag{3.16}$$

This shows that the set of values taken by the delayed parameters depend on the rate of variation of the parameters ν and the maximal delay value h_{max}. Hence for sufficiently small ν and h_{max} then the set of values of the delayed parameters is not absolutely identical to U_ν but can be reduced to a neighborhood of the value of $\rho(t)$ at time t. This neighborhood, in the one dimensional case, is an interval centered around $\rho(t)$ with radius νh.

The interest of the following results is to determine whether or not the domain of delayed parameters coincides with the parameters.

Proposition 3.1.1 *If $\nu h_{max} \geq 2\bar{\rho}$ then the set of value of $\rho(t-h)$ coincides with U_ρ for every $t \geq 0$.*

A direct analysis shows that if the parameters are discontinuous (i.e. unbounded derivatives) and/or the delay is unbounded (i.e. $h_{max} = +\infty$), then the set of delayed-parameters coincide with the set of non-delayed parameters.

Proposition 3.1.2 *If $\nu h_{max} < 2\bar{\rho}$ then the set of value of $\rho(t-h)$ is included in U_ρ for every $t \geq 0$ and is depicted in Figure 3.3.*

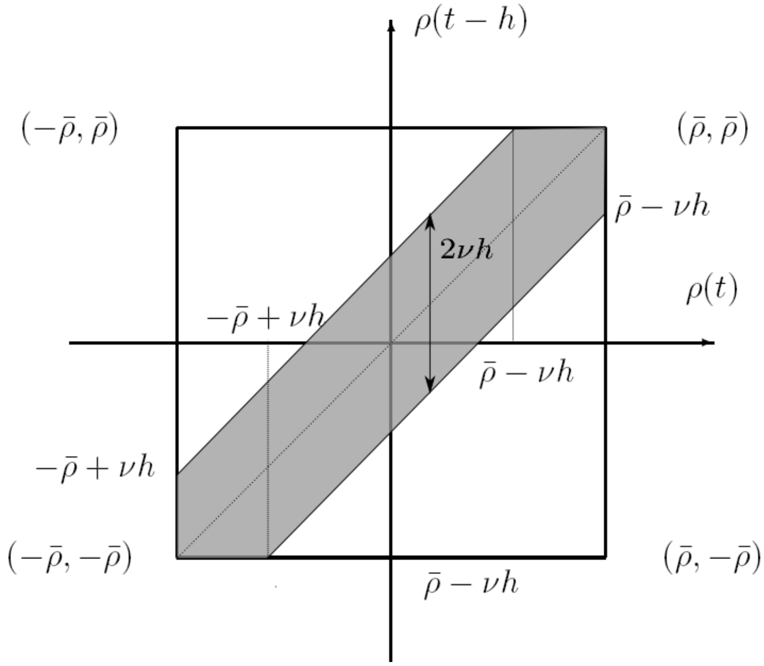

Figure 3.3: Set of the values of $\rho(t-h)$ (in grey) with respect to the set of value of $\rho(t)$ (the horizontal interval $[-\bar{\rho},\bar{\rho}]$

The set of generalized parameters (ρ,ρ_h) where ρ_h is the delayed parameter is a polyhedral with 6 vertices and 6 edges. Moreover the set of values of the delayed parameter ρ_h can be parametrized by ρ:

$$U_{\rho_h}(x) := \{y \in \mathbb{R} : \ |y-x| \leq \nu h\} \tag{3.17}$$

Hence the set of all values for ρ_h is given by

$$\bar{U}_{\rho_h} := \{y \in \mathbb{R} : \ |y-x| \leq \nu h, \ x \in U_\rho\} \tag{3.18}$$

and the whole set $\bar{U}_\rho$ of values of (ρ,ρ_h) is defined by

$$\bar{U}_\rho := \{(\rho_1,\rho_2) : \ \rho_1 \in U_\rho, \ \rho_2 \in U_{\rho_h}(\rho_1)\} \tag{3.19}$$

Let us consider now the derivative of the delayed parameters for the particular case of continuous parameters. In the case of constant delay, the set of delayed parameter derivative values coincides with the set $[-\nu,\nu] = \text{hull}[\{-\nu,\nu\}]$ since the delay is constant, i.e.

$$\frac{d}{dt}\rho(t-h) = \rho'(t-h) \in [-\nu,\nu]$$

However, in the case of varying delay two cases may appear according to the type of the rate of variation (bounded or unbounded) of the delay. Assume first that the rate is bounded and then we have

$$\frac{d}{dt}\rho(t-h(t)) = (1-\dot{h}(t))\rho'(t-h(t)) \tag{3.20}$$

and hence we have

$$-(1+\mu)\nu \leq \frac{d}{dt}\rho(t-h(t)) \leq (1+\mu)\nu \tag{3.21}$$

This shows that the set of values of the rate of variation of delayed parameters is larger than for the nondelayed ones. Finally, if the delay derivative is unbounded then the rate of variation of delayed parameters is unbounded too.

Different properties of delayed parameters have been determined and overall their set of values. We can now give a general formulation of the class of systems that will be considered in this thesis.

3.1.3 Class of LPV Time-Delay Systems

Throughout this thesis, the following class of LPV time-delay systems [Wu, 2001, Zhang and Grigoriadis, 2005] will be considered if not stated otherwise:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho,\rho_h)x(t) + A_h(\rho,\rho_h)x(t-h(t)) + E(\rho,\rho_h)w(t) \\ z(t) & = & C(\rho,\rho_h)x(t) + C_h(\rho,\rho_h)x(t-h(t)) + F(\rho,\rho_h)w(t) \\ x(\theta) & = & \phi(\theta), \quad \theta \in [-h_{max}, 0] \end{array} \tag{3.22}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $z \in \mathbb{R}^p$ are respectively the system state, the exogenous input and the controlled output. Such a class captures a wide class of LPV time-delay systems. Moreover, the delay is assumed to belong to $\mathscr{H}_1^\circ$ with $h_{min} = 0$ and the parameters $(\rho, \rho_h) \in \bar{U}_\rho$, $(\dot{\rho}, d\rho_h/dt) \in U_\nu \times (1+\mu)U_\nu$ where the sets are extended to $N_p > 1$. From these considerations it is clear that such systems merge all the particularities of LPV systems introduced in Chapters 1 and 2.

Such systems may occur in many nonlinear physical systems with delay approximated by LPV systems. For instance, in [Zhang et al., 2002] a milling process is modeled as a LPV time-delay systems as shown below:

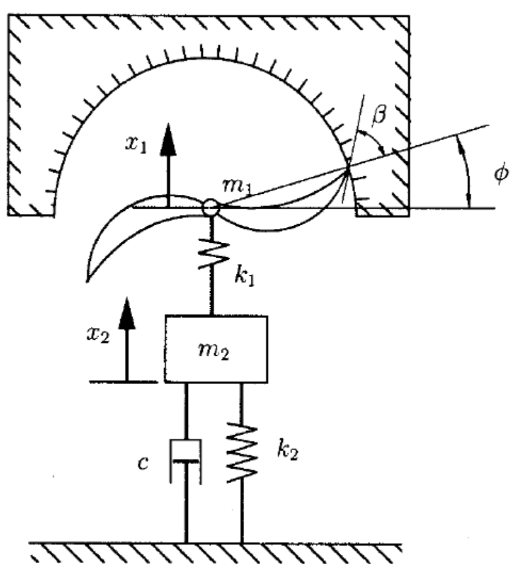

Figure 3.4: Simplified geometry of a milling process

The corresponding model is given by the expressions

$$\begin{aligned}
&m_1\ddot{x}_1 + k_1(x_1 - x_2) = f\\
&m_2\ddot{x}_2 + c\dot{x}_2 + k_2(x_2 - x_1) + k_2 x_2 = 0\\
&f = k\sin(\phi+\beta)z(t)\\
&z(t) = z_a + \sin(\phi)[x_1(t-h) - x_1(t)]
\end{aligned}$$

where k_1 and k_2 are the stiffness coefficients of the two springs, c is the damping coefficient, m_1 is the mass of the cutter, m_2 is the mass of the 'spindle'. The displacements of the blade and tool are x_1 and x_2 respectively. The angle β depends on the particular material and tool used, and is constant. The angle ϕ denotes the angular position of the blade and k is the cutting stiffness. z_a is the average chip thickness (here assumed, without loss of generality, $z_a = 0$) and $h = \pi/\omega$ is the delay between successive passes of the blades. This system can be modeled as a LPV system with delay of the form

$$\dot{x}(t) = (A + A_k k + A_\gamma \gamma + A_{k\gamma} k\gamma)x(t) + (A^h + A_k^h k + A_\gamma^h \gamma + A_{k\gamma}^h k\gamma)x(t-h) \tag{3.23}$$

where the parameters are the stiffness $k = k_1 = k_2$ and $\gamma = \cos(2\phi + \beta) \in [-1, 1]$. An interesting discussion about this process is provided in [Zhang et al., 2002].

3.2 Relaxation of Polynomially Parameter Dependent Matrix Inequalities

In this section, a new method of relaxation of polynomially parameter dependent LMIs is proposed in the following. It will be used to tackle parameter dependent LMIs (also called 'robust LMIs') that arise for instance in the (robust) stability analysis of uncertain and LPV systems.

Since several years, many results on relaxation of polynomially parameter dependent LMIs have been provided. Even if many of them were applied to polynomial of degree 2, arising for instance in gain-scheduled state-feedback controller for polytopic systems having a parameter dependent input matrix, most of them could be applied to polynomial of higher degree. For instance let us mention the following works on this topic [Apkarian and Tuan, 1998, Geromel and Colaneri, 2006, Oliveira et al., 2007, Oliveira and Peres, 2006, 2002, Scherer, 2008, Tuan and Apkarian, 1998, 2002].

The approach provided in this section is close to the Sum-of-Squares relaxation in the sense that the matrix of polynomials is represented in a spectral form (See Section 1.3.3.3). But at the difference of the classical SOS approach, this method does not involve any choice or decision of the designer (such as the degree of polynomials) except the choice of the basis in which the polynomial is expressed (by basis we mean the outer factor of the spectral form). We will also see that this method linearizes the dependence on the parameters, and thus turns a polynomially parameter dependent LMI into a linearly parameter dependent LMI with a slight conservatism. Finally the resulting conditions are directly written in terms of a linearly parameter dependent LMI involving a slack variable, which benefits of the simplicity of the affine dependence on the parameters. As an extension of the procedure, it will be possible to provide some ideas about a judicious choice of the basis which reduces (minimizes) the number of involved monomials.

Let us consider the parameter dependent LMI $\mathcal{M}(X,\rho)$ which writes as

$$\mathcal{M}(X,\rho) := \mathcal{M}_0(X) + \sum_{i=1}^{N} \mathcal{M}_i(X)u_i(\rho) \tag{3.24}$$

where $\mathcal{M}_i(X) \in \mathbb{S}^n$, X denotes the decision variables and $u_i(\rho)$ are monomials in $\rho = \mathrm{col}_i(\rho_i) \in U_\rho$.

The following result describes the transformation of the polynomially parameter dependent LMI into a linearly parameter dependent LMI:

Theorem 3.2.1 *Let us consider a polynomially parameter dependent matrix inequality of the form (3.24). It can be written into a spectral form*

$$\Theta_\perp(\rho)^T \mathcal{M}(X) \Theta_\perp(\rho) \prec 0 \tag{3.25}$$

where $\mathcal{M}$ is a parameter independent symmetric matrix constructed from $\mathcal{M}_0(X)$, $\mathcal{M}_i(X)$ and $\Theta_\perp(\theta)$ a rectangular matrix gathering monomials appearing in the parameter dependent LMIs (e.g. $\Theta_\perp(\rho) = \mathrm{col}(1, u_1(\rho), \ldots, u_n(\rho))$). Then (3.25) is feasible in X for all $\rho \in U_\rho$ if there exists X and a matrix $\mathcal{P}$ of appropriate dimensions such that

$$\mathcal{M}(X) + \mathcal{P}^T\Theta(\rho) + \Theta(\rho)^T\mathcal{P} \prec 0 \tag{3.26}$$

holds for all $\rho \in U_\rho$ and $\Theta(\rho)\Theta_\perp(\rho) = 0$. Moreover, with an appropriate choice of $\Theta_\perp(\rho)$ then $\Theta(\rho) = \sum_{i=1}^{N} \Theta_i \rho_i$ is affine in θ_i.

Proof: *The proof is a simple application of the Finsler's lemma to parameter dependent LMIs. Consider first, the parametrized LMI in its spectral form (3.25) and then invoking the Finsler's lemma (see Appendix E.16), we can claim that this is equivalent to the existence of $\mathcal{P}(\rho)$ such that*

$$\mathcal{M}(X) + \mathcal{P}(\rho)^T\Theta(\rho) + \Theta(\rho)^T\mathcal{P}(\rho) \prec 0 \tag{3.27}$$

holds.

However, since the aim of the procedure is the linearization of the parameter dependence then by restricting $\mathcal{P}$ to be parameter independent we get

$$\mathcal{M}(X) + \mathcal{P}(\rho)^T\Theta(\rho) + \Theta(\rho)^T\mathcal{P}(\rho) \prec 0 \Rightarrow \Theta_\perp(\rho)^T\mathcal{M}(X)\Theta_\perp(\rho) \prec 0 \tag{3.28}$$

It is aimed now that, for every polynomial, it is possible to construct $\Theta_\perp(\rho)$ for which $\Theta(\rho)$ is affine in ρ. To show this, note that the trivial basis for univariate polynomials $\Theta_\perp(\rho) = \mathrm{col}(1, \rho, \rho^2, \ldots, \rho^n)$ admits

$$\Theta(\rho) = \begin{bmatrix} -\rho & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & -\rho & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & -\rho & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -\rho & 1 & \ldots & 0 \\ \vdots & & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix} \tag{3.29}$$

Hence since the trivial basis is the kernel of an affine parameter dependent matrix then it is possible to find an affine $\Theta(\rho)$ for every univariate polynomial. This generalizes directly to the multivariate case. $\square$

It is worth noting that using the trivial functions $u_i(\rho) = \rho^i$ in $\Theta_\perp(\rho)$ (for the univariate case) may be not the best choice. Indeed, the most intuitive choice is to choose $\Theta_\perp(\rho) = col(1, \rho_1, \ldots, \rho_N, \rho_1^2, \ldots)$ which will give an affine $\Theta(\rho)$ but with increased complexity since the dimension of $\mathcal{M}$ is larger than in the case of taking the $u_i(\rho)$'s. Hence, it is important to point out the properties of a nontrivial basis $\Theta_\perp(\rho)$ (reduced dimension) for which $\Theta(\rho)$ is affine. Actually, if the polynomials (may not be exclusively monomials) $v_i(\rho)$ components of $\Theta_\perp(\rho)$ are chosen to satisfy

$$v_i(\rho) = \sum_j^n p_{ij}(\rho) v_j(\rho) \tag{3.30}$$

where the $p_{ij}(\rho)$'s are affine polynomials in ρ and N is the size of the basis, then there exists an affine $\Theta(\rho)$.

The latter equality can be rewritten into the compact form

$$v(\rho) = P(\rho)v(\rho) \tag{3.31}$$

where $v(\rho) = \text{col}_i(v_i(\rho))$ and $P(\rho) = \begin{bmatrix} p_{11}(\rho) & \ldots & p_{1N}(\rho) \\ \vdots & \ddots & \vdots \\ p_{N1}(\rho) & \ldots & p_{NN}(\rho) \end{bmatrix}$ or equivalently

$$(I - P(\rho))v(\rho) = 0 \tag{3.32}$$

It is worth mentioning that the computational complexity of the procedure depends on the number N of functions $u_i(\rho)$. Hence the problem results, for a given $\mathcal{M}(X, \rho)$, in finding the minimal N such that

$$\begin{array}{l} \det(I - P(\rho)) = 0 \\ \mathcal{M}(X, \rho) := \sum_{i=1}^N \mathcal{M}_i(X) v_i(\rho) \end{array} \tag{3.33}$$

for some $\mathcal{M}_i(X)$ and for all $\rho \in U_\rho$ with $P(\rho)$ affine in ρ. Indeed, if this condition is satisfied this means that there exists a $\Theta_\perp(\rho)$ which is a basis of the null space of an affine matrix $\Theta(\rho)$. This optimization problem is non trivial since it involves polynomials and a dimension of a basis is the cost to minimize. This gives rise to interesting optimization problem that will not be treated here but belongs to further works and investigations.

Coming back to theorem (3.2.1), it is possible to derive an important result for LMI involving quadratic polynomial dependence, useful in polytopic systems.

Corollary 3.2.2 *The following parameter dependent matrix inequality is feasible*

$$\mathcal{M}(\lambda) = \mathcal{M}_0 + \sum_{i=1}^N \lambda_i \mathcal{M}_i + \sum_{i,j=1}^N \lambda_i \lambda_j \mathcal{M}_{ij} \prec 0 \tag{3.34}$$

provided that $\sum_{i=1}^N \lambda_i = 1,\ \lambda_i \geq 0$ *if there exists* $\mathcal{Z}$ *such that*

$$\tilde{\mathcal{K}} + \mathcal{Z}^T \Pi(\lambda) + \Pi(\lambda)^T \mathcal{Z} < 0 \tag{3.35}$$

is feasible for all $\lambda \in U_\lambda^{N-1}$ *where*

$$\Pi(\lambda) = \begin{bmatrix} -\lambda_1 I & I & 0 & \ldots & 0 \\ -\lambda_2 I & 0 & I & \ldots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -\lambda_{N-1} I & 0 & 0 & \ldots & I \end{bmatrix}$$

$$\tilde{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_0 & \mathcal{K}_1/2 & \dots & \mathcal{K}_{N-1}/2 \\ \star & \mathcal{K}_{11} & \dots & \mathcal{K}_{1(N-1)}/2 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathcal{K}_{(N-1)(N-1)} \end{bmatrix}$$

with $\mathcal{K}_0 = \mathcal{M}_0 + \mathcal{M}_N + \mathcal{M}_{NN}$, $\mathcal{K}_i = \mathcal{M}_i - \mathcal{M}_N + 2\Sigma_{iN} - 2\mathcal{M}_{NN}$, $\mathcal{K}_{ij} = \mathcal{M}_{ij} - 2\Sigma_{Ni} + \mathcal{M}_{NN}$, $\Omega_{ij} = (\mathcal{K}_{ij} + \mathcal{K}_{ji})/2$ *and* $\Sigma_{ij} = (\mathcal{M}_{ij} + \mathcal{M}_{ji})/2$.

Proof: First note that the set of parameters λ_i, $i = 1, \dots, N$ *can be reduced to a set of* $N-1$ *parameters by using the constraint* $1 - \sum_{i=1}^{N} \lambda_i = 0$ *and hence we have*

$$\lambda_N = 1 - \sum_{i=1}^{N-1} \lambda_i$$

Hence (3.34) can be rewritten as

$$\begin{aligned} (3.34) \quad = \quad & \mathcal{M}_0 + \sum_{i=1}^{N-1} \mathcal{M}_i + (1 - \sum_{i=1}^{N-1} \lambda_i)\mathcal{M}_N + \sum_{i,j=1}^{N-1} \mathcal{M}_{ij}\lambda_i\lambda_j\mathcal{M}_{ij} \\ & + \sum_{i=1}^{N-1} \lambda_i \left(1 - \sum_{i=1}^{N-1} \lambda_i\right)(\mathcal{M}_{iN} + \mathcal{M}_{Ni}) \\ & + \left(1 - \sum_{i=1}^{N-1} \lambda_i\right)\left(1 - \sum_{i=1}^{N-1} \lambda_i\right)\mathcal{M}_{NN} \end{aligned}$$

Gathering the affine and quadratic together terms we get

$$\mathcal{K}_0 + \textstyle\sum_{i=1}^{N-1} \lambda_i \mathcal{K}_i + \sum_{i,j=1}^{N-1} \lambda_i\lambda_j \mathcal{K}_{ij} \tag{3.36}$$

where $\mathcal{K}_0 = \mathcal{M}_0 + \mathcal{M}_N + \mathcal{M}_{NN}$, $\mathcal{K}_i = \mathcal{M}_i - \mathcal{M}_N + 2\Sigma_{iN} - 2\mathcal{M}_{NN}$, $\mathcal{K}_{ij} = \mathcal{M}_{ij} - 2\Sigma_{Ni} + \mathcal{M}_{NN}$ *and* $\Sigma_{ij} = \dfrac{1}{2}(\mathcal{M}_{ij} + \mathcal{M}_{ji})$.

The rest of the proof is an application of theorem 3.2.1. First, choose $\Pi_\perp(\lambda) = col(I, \lambda_1 I, \dots, \lambda_{N-1} I)$ *and*

$$\tilde{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_0 & \mathcal{K}_1 & \dots & \mathcal{K}_{N-1} \\ \star & \mathcal{M}_{11} & \dots & \Omega_{1(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \star & \star & \dots & \mathcal{M}_{(N-1)(N-1)} \end{bmatrix} \tag{3.37}$$

such that

$$\Pi_\perp \tilde{\mathcal{M}} \Pi_\perp^T < 0$$

with $\Omega_{ij} = (\mathcal{K}_{ij} + \mathcal{K}_{ji})/2$.

Now compute Π *such that* $\Pi\Pi_\perp = 0$ *and we get*

$$\Pi = \begin{bmatrix} -\lambda_1 I & I & 0 & \dots & 0 \\ -\lambda_2 I & 0 & I & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -\lambda_{N-1} I & 0 & 0 & \dots & I \end{bmatrix} \tag{3.38}$$

Finally applying theorem 3.2.1 we get

$$\tilde{\mathcal{K}} + \mathcal{Z}^T \Pi(\lambda) + \Pi(\lambda)^T \mathcal{Z} < 0$$

which is exactly the result of corollary 3.2.2. □

The following example shows the interest of the approach:

Example 3.2.3 *Let us consider the univariate polynomial*

$$p(x) = -x^4 + 4x^3 + 43x^2 - 58x - 240 \tag{3.39}$$

whose graph is depicted on Figure 3.5

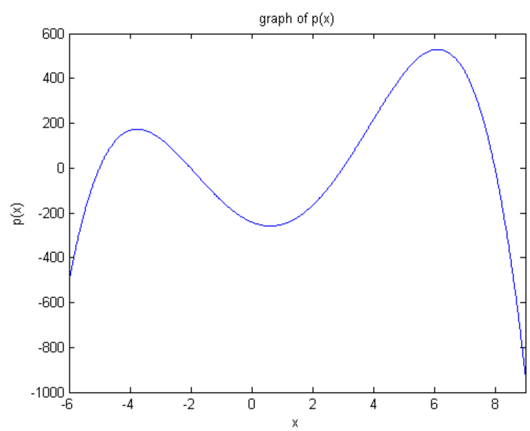

Figure 3.5: Graph of the polynomial $p(x)$ over $x \in [-6, 9]$

The goal is to find the supremum of $p(x)$ over the interval $[-6, 9]$, hence we are looking for the minimal value of γ such that

$$p(x) \leq \gamma \ \forall \ x \in [-6, 9]$$

which is equivalent to the following optimization problem

$$\begin{array}{l} \min \gamma \ s.t. \\ p(x) - \gamma \leq 0 \\ x \in [-6, 9] \end{array}$$

First of all, $p(x) - \gamma$ is rewritten in the spectral form (the repartition of the terms along anti-diagonals is arbitrary):

$$p(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} -\gamma - 240 & -29 & 20 \\ \star & 3 & 2 \\ \star & \star & -1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \leq 0 \tag{3.40}$$

Applying Theorem 3.2.1, we get the following LMI

$$\begin{bmatrix} -\gamma - 240 & -29 & 20 \\ \star & 3 & 2 \\ \star & \star & -1 \end{bmatrix} + N^T\Theta(x) + \Theta(x)^T N \preceq 0 \tag{3.41}$$

where N is a free matrix variable belonging to $\mathbb{R}^{2\times 3}$ and $\Theta(x)$ is defined such that $\Theta(x)\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = 0$. Then we have

$$\Theta(x) = \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \end{bmatrix}$$

Finally this leads to the parameter dependent LMI with linear dependence in x:

$$\mathcal{P}(\gamma, x) = \begin{bmatrix} -\gamma - 240 & -29 & 20 \\ \star & 3 & 2 \\ \star & \star & -1 \end{bmatrix} + N^T \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \end{bmatrix} + \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \end{bmatrix}^T N \preceq 0$$

Hence with a polytopic argument, the optimization problem becomes

$$\begin{array}{l} \min \gamma \ s.t. \\ \mathcal{P}(\gamma, -6) \preceq 0 \\ \mathcal{P}(\gamma, 9) \preceq 0 \end{array}$$

Solving this SDP we get $\gamma_{opt} = 529.6340928975$ found with

$$N = \begin{bmatrix} -10.0347 & 5.4752 & 0.2318 \\ -3.6163 & -0.0356 & -0.03792 \end{bmatrix}$$

The theoretical result is given by $s := \sup_{x\in[-6,9]} p(x) = 529.63265619463$ and the computation error is

$$\varepsilon := \gamma - s = 0.001436702914$$

We can see that the computed maximum is very close to the theoretical one. This shows that this relaxation leads to results with low conservatism.

3.3 Relaxation of Concave Nonlinearity

Concave nonlinearities are the most difficult nonlinearities to handle in the LMI framework. They may appear in many problems especially when congruence transformations are performed and occur for instance in the problems studied in [Briat et al., 2008d, Chen and Zheng, 2006] and maybe many others. First of all, known solutions will be presented and explained and finally the new 'exact' relaxation will be provided.

Indeed, it is well known that, even if the following problem in ε, α and β is nonlinear

$$\varepsilon + \alpha^T\beta^{-1}\alpha \prec 0, \quad \varepsilon = \varepsilon^T \prec 0, \quad \beta = \beta^T \succ 0 \tag{3.42}$$

the problem is convex since the nonlinearity $\alpha^T\beta^{-1}\alpha$ is convex. A Schur complement on this matrix inequality yields the matrix

$$\begin{bmatrix} \varepsilon & \alpha^T \\ \alpha & -\beta \end{bmatrix} \prec 0 \tag{3.43}$$

which is affine (and then convex) in the decision variable. But the question is what happens when the sign '+' is turned into a sign '-' ? The convex nonlinearity becomes concave and the Schur complement does not apply anymore. The following section aims at providing solutions on the relaxation of such nonlinearity.

Let us consider now the following nonlinear matrix inequality

$$\varepsilon - \alpha^T \beta^{-1} \alpha \prec 0, \quad \varepsilon = \varepsilon^T, \quad \beta = \beta^T \succ 0 \tag{3.44}$$

Note that the negative definiteness of ε is not assumed anymore in this case, hence the nonlinear term is needed for negative definiteness of the sum. Indeed, if $\varepsilon \prec 0$ there exists a trivial (conservative) bound on the nonlinear term which is 0 (since the nonlinear term is positive semidefinite). Moreover, the matrix α are not necessarily square and these facts show the wide generality of the proposed approach.

The following result has been often used in the literature to bound the nonlinear term.

Lemma 3.3.1 *The following relation holds*

$$-\alpha^T \beta^{-1} \alpha \preceq -\alpha - \alpha^T + \beta \tag{3.45}$$

Proof: *Since $\beta \succ 0$, then define the inequality*

$$(I - \beta^{-1}\alpha)^T \beta (I - \beta^{-1}\alpha) \succeq 0$$

and thus we have

$$\begin{array}{rll} & \beta - \alpha^T - \alpha + \alpha^T \beta^{-1} \alpha & \succeq 0 \\ \Rightarrow & -\alpha^T \beta^{-1} \alpha & \preceq \beta - \alpha^T - \alpha \end{array}$$

This concludes the proof. □

A direct extension of the latter results yields

Lemma 3.3.2 *The following relation holds for some $\omega > 0$*

$$-\alpha^T \beta^{-1} \alpha \preceq -\omega(\alpha + \alpha^T) + \omega^2 \beta \tag{3.46}$$

Using this lemma, the nonlinearity is bounded by an hyperplane as seen on figure 3.6 where the scalar case is considered with $\omega = 1$.

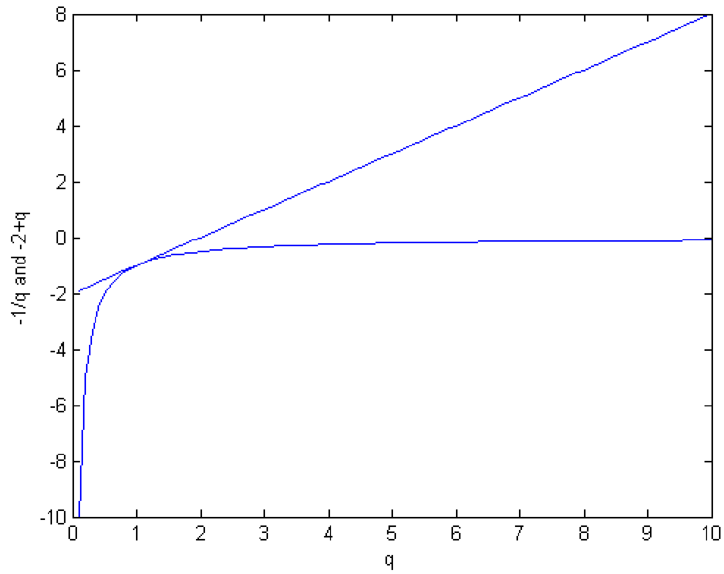

Figure 3.6: Evolution of the concave nonlinearity and the linear bound in the scalar case with fixed $\alpha = p = 1$, $\beta = q$ and $\omega = 1$

Actually these results are a linearization of the nonlinearities around some particular point. Hence, such a bound will be conservative when the computed matrices are far from the linearization point. In the second result, ω has the role of a tuning parameter which 'moves' the linearization point in order to decrease the conservatism of the bound.

In [Chen and Zheng, 2006] it has been proposed to use the Cone Complementary Algorithm [Ghaoui et al., 1997] as a relaxation result. Initially, this algorithm was developed to deal with static output feedback design or more generally to 'LMI' problems involving simultaneously matrices and their inverse. At the light of [Chen and Zheng, 2006], it turns out that it can also be applied efficiently to relax concave nonlinearity of the form $-\alpha^T\beta^{-1}\alpha$.

To adapt this algorithm to a relaxation scheme, let $\upsilon \leq \alpha^T\bar{\beta}\alpha$ and we have $\bar{\upsilon} \geq \bar{\alpha}\beta\bar{\alpha}^T$ where $\bar{\alpha}\alpha = I$, $\bar{\beta}\beta = I$ and $\bar{\upsilon}\upsilon = I$. Then we get the following problem of finding $\mathcal{X} = (\varepsilon, \alpha, \beta, \upsilon, \bar{\alpha}, \bar{\beta}, \bar{\upsilon})$ such that

$$\begin{array}{lcl} \varepsilon - \upsilon & \prec & 0 \\ \begin{bmatrix} \bar{\upsilon} & \bar{\alpha}^T \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} & \succeq & 0 \\ \bar{\alpha}\alpha & = & I \\ \bar{\beta}\beta & = & I \\ \bar{\upsilon}\upsilon & = & I \end{array} \tag{3.47}$$

which is a nonconvex problem due to nonlinear equalities. It is clear that the latter problem is identical to the initial one. Finally, using the Cone Complementary Algorithm it is possible to approximate the problem (3.44) by the following iterative procedure:

Algorithm 3.3.3 ***Adapted Cone Complementary Algorithm:***

1. *Initialize* $i = 0$, $\varepsilon = 1$ *and* $\mathcal{X}_0 := (\varepsilon_0, \alpha_0, \beta_0, \upsilon_0, \bar{\alpha}_0, \bar{\beta}_0, \bar{\upsilon}_0)$ *solution of*

$$\varepsilon - \upsilon \prec 0 \qquad \begin{bmatrix} \bar{\upsilon} & \bar{\alpha}^T \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} \succeq 0 \tag{3.48}$$

2. *Find* $\mathcal{X}_{i+1}$ *solution of*

$$\gamma_{i+1} := min_{\mathcal{X}} \operatorname{trace}(\upsilon_i \bar{\upsilon} + \bar{\upsilon}_i \upsilon + \alpha_i \bar{\alpha} + \bar{\alpha}_i \alpha + \beta_i \bar{\beta} + \bar{\beta}_i \beta)$$

such that

$$\varepsilon - \upsilon \prec 0 \qquad \begin{bmatrix} \bar{\upsilon} & \bar{\alpha}^T \\ \bar{\alpha} & \bar{\beta} \end{bmatrix} \succeq 0 \tag{3.49}$$

3. *if* $\gamma_{i+1} = 6n$ *then STOP: Solution Found*
   else if $\varepsilon > \varepsilon_{max}$ *then STOP: Infeasible problem*
   else $i = i + 1$, *goto Step 2.*

Although this algorithm does not converge systematically to a global optimum of the optimization problem, it gives quite good results in practice. However, this efficient approach suffers from two drawbacks:

1. It can be applied with square matrices only since the procedure needs the inversion of the matrix α.

2. It can only deal with constant matrices since they are needed to be inverted and the inverse of parameter dependent matrices cannot be expressed in a linear fashion with respect to the unknown matrices. As an examples, the inverse of the matrix $P(\rho) = P_0 + \rho P_1$ is defined by

$$P(\rho)^{-1} = P_0^{-1} - P_0^{-1}(P_0^{-1} - P_1^{-1}\rho^{-1})^{-1}P_0^{-1}$$

and cannot be expressed linearly for instance $P(\rho)^{-1} = S_0 + S_1 u(\rho)$ where $u(\rho)$ is a particular function.

The parameter dependent matrix case should be treated with the lemma 3.3.1 and 3.3.2 with a possibly parameter varying $\omega(\rho)$. However, due to the high conservatism of this bound, we have been brought to develop the following result to overcome these problems. Such a result has been published in [Briat et al., 2008d] and allows for a 'good' relaxation of the nonlinearity by bilinearities.

Theorem 3.3.4 *Consider a symmetric positive definite matrix function* $\beta(\cdot)$, *a matrix (non necessarily square) function* $\alpha(\cdot)$ *and a symmetric matrix function* $\varepsilon(\cdot)$ *then the following propositions are equivalent:*

a) $\varepsilon(\cdot) - \alpha^T(\cdot)\beta^{-1}(\cdot)\alpha(\cdot) \prec 0$

b) There exists a matrix function of appropriate dimensions $\eta(\cdot)$ *such that*

$$\begin{bmatrix} \varepsilon(\cdot) + \alpha(\cdot)^T\eta(\cdot) + \eta(\cdot)^T\alpha(\cdot) & \star \\ \beta(\cdot)\eta(\cdot) & -\beta(\cdot) \end{bmatrix} \prec 0 \tag{3.50}$$

Proof: *b)⇒ a)*

First we suppose that there exists $\eta(\cdot)$ such that (3.50) holds. Hence using Schur complement there exists a $\eta(\cdot)$ such that

$$\varepsilon(\cdot) + [\eta^T(\cdot)\alpha(\cdot)]^H + \eta^T(\cdot)\beta(\cdot)\eta(\cdot) \prec 0$$

Using the completion by the squares, this is equivalent to

$$\varepsilon(\cdot) + \zeta^T(\cdot)\beta^{-1}(\cdot)\zeta(\cdot) - \alpha^T(\cdot)\beta^{-1}(\cdot)\alpha(\cdot) \prec 0$$

with $\zeta(\cdot) = \alpha(\cdot) + \beta(\cdot)\eta(\cdot)$. Finally we obtain

$$\varepsilon(\cdot) - \alpha^T(\cdot)\beta^{-1}(\cdot)\alpha(\cdot) \prec -\zeta^T(\cdot)\beta^{-1}(\cdot)\zeta(\cdot) \tag{3.51}$$

Since $\beta(\cdot) > 0$ then the right-hand side of equation (3.51) is negative semidefinite for all $\eta(\cdot)$. Then we can conclude that if there exist a $\eta(\cdot)$ such that (3.51) is satisfied then a) is true. Moreover when $\zeta(\cdot)$ vanishes identically then no conservatism is induced and the bound equals the nonlinear term. That means that when $\eta(\cdot) = -\beta^{-1}(\cdot)\alpha(\cdot)$ the relaxation is exact.

a)⇒b)

First consider the matrix

$$\Theta(\cdot) = \begin{bmatrix} \varepsilon(\cdot) & \alpha(\cdot)^T \\ \alpha(\cdot) & \beta(\cdot) \end{bmatrix} \tag{3.52}$$

with $\beta(\cdot) \succ 0$ and $\varepsilon(\cdot) - \alpha(\cdot)^T\beta(\cdot)^{-1}\alpha(\cdot) \prec 0$. Let $\dim(\varepsilon(\cdot)) = n$ and $\dim(\beta(\cdot)) = l$ and note that $\Theta(\cdot)$ may be rewritten as

$$\Theta(\cdot) = \begin{bmatrix} \delta(\cdot)^{1/2} & \alpha(\cdot)^T\beta(\cdot)^{-1/2} \\ 0 & \beta(\cdot)^{1/2} \end{bmatrix} \begin{bmatrix} -I_n & 0 \\ 0 & I_l \end{bmatrix} \begin{bmatrix} \delta(\cdot)^{1/2} & 0 \\ \beta(\cdot)^{-1/2}\alpha(\cdot) & \beta(\cdot)^{1/2} \end{bmatrix} \tag{3.53}$$

where $\delta^{1/2}(\cdot)$ and $\beta^{1/2}(\cdot)$ define the symmetric positive definite square root of matrices $\delta(\cdot)$ and $\beta(\cdot)$ with $\delta(\cdot) = -\varepsilon(\cdot) + \alpha(\cdot)^T\beta^{-1}(\cdot)\alpha(\cdot)$. From this equality it is clear the matrix Θ has n negative eigenvalues and l positive eigenvalues since $\Theta(\cdot)$ is congruent to $\mathrm{diag}(-I_n, I_l)$. Then there exists a subspace with maximal rank of the form

$$\Lambda(\cdot) = \begin{bmatrix} \theta(\cdot) \\ \eta(\cdot) \end{bmatrix} \qquad \text{with } \mathrm{rank}\ \Lambda(\cdot) = n \tag{3.54}$$

with $\theta(\cdot) \in \mathbb{R}^{n\times n}$ and $\eta(\cdot) \in \mathbb{R}^{l\times n}$ such that $\Lambda(\cdot)^T\Theta(\cdot)\Lambda(\cdot) < 0$. Expand the latter inequality leads to (dropping the dependence $(\cdot)$):

$$\theta^T\varepsilon\theta + \theta^T\alpha^T\eta + \eta^T\alpha\theta + \eta^T\beta\eta \prec 0 \tag{3.55}$$

Rearranging the terms using the fact that $\beta(\cdot) \succ 0$ is symmetric leads to

$$\theta^T(\varepsilon - \alpha^T\beta^{-1}\alpha)\theta + (\alpha\theta + \beta\eta)^T\beta^{-1}(\alpha\theta + \beta\eta) \prec 0 \tag{3.56}$$

Since $\varepsilon - \alpha^T\beta^{-1}\alpha \prec 0$ and since $\beta(\cdot) \succ 0$ then it implies that $\theta^T(\varepsilon - \alpha^T\beta^{-1}\alpha)\theta < 0$. Hence θ is of full rank (nonsingular in the square case). Now let $\mathcal{K}$ be the set such that

$$\mathcal{K} := \{\kappa :\ \theta^T(\varepsilon - \alpha^T\beta^{-1}\alpha)\theta + \kappa^T\beta^{-1}\kappa \prec 0\} \tag{3.57}$$

It is clear that the set $\mathcal{K}$ is nonempty since it includes $\kappa = 0$. It is not reduced to a singleton since it exists a neighborhood $\mathcal{N}$ centered around $\kappa = 0$ for which (3.57) is satisfied for all $\kappa \in \mathcal{N}$. Now we will show that for all nonsingular θ there exist values for κ (and hence values for η) for which (3.57) holds.

First note that $\beta > 0$ is nonsingular, then the equation

$$\alpha\theta + \beta\eta = \kappa \tag{3.58}$$

for given θ and κ has the solution $\eta = \beta^{-1}(\kappa - \alpha\theta)$. Hence this means that for given $\varepsilon, \beta, \alpha, \theta$ such that $\varepsilon - \alpha^T\beta^{-1}\alpha \prec 0$, $\beta \succ 0$, $rank(\theta) = n$, there exist η such that (3.56) is satisfied. The existence of such a η is thus shown.

Now fix $\theta = I$ for simplicity and consider (3.56) we obtain

$$\varepsilon + \alpha^T\eta + \eta^T\alpha + \eta^T\beta\eta \prec 0 \tag{3.59}$$

Apply the Schur's Lemma to obtain

$$\begin{bmatrix} \varepsilon + \alpha^T\eta + \eta^T\alpha & \eta^T\beta \\ \beta\eta & -\beta \end{bmatrix} \prec 0 \tag{3.60}$$

This concludes the proof. □

This theorem has the benefit of allowing to deal with parameter varying matrices and non square α which is a great improvement compared to previous methods. Moreover, it involves only feasibility problems and this can be directly extended to an optimization problems. This is not the case for the cone complementary algorithm which involves already an optimization problem (i.e. the trace on some matrices is aimed to be minimized). Hence, if minimal $\mathcal{L}_2$-performances are sought then we will be in presence of a multi-objective optimization problem (the costs are the trace and the norm) which is not trivial. The tradeoff between the costs should be done with care, in order to not too penalize the trace cost which is the most important one.

Since (3.50) is bilinear (BMI) then no efficient algorithm is available to solve it in reasonable time. Nevertheless, the BMI structure is more convenient than the initial nonlinear expression involving the inverse of a decision matrix and this fact suggests that an iterative procedure should work to find a solution to the problem. Indeed, noting that by fixing the value of η the problem is convex in ε, α and β and vice-versa, it seems interesting to develop such an algorithm matching this particular form of BMI. Due to this property an algorithm in two steps can be used to find a solution iteratively such as the D-K iteration algorithm used in μ-synthesis [Apkarian et al., 1993, Balas et al., 1998].

Since every iterative procedures needs to find an initial feasible point in order to converge to a local/global minimum, the remaining problem is to find this initial feasible point. In the proof of theorem 3.3.4, it is shown that the relaxation is exact if and only if $\eta = -\beta^{-1}\alpha$ and hence finding an initial η_0 is equivalent to finding an initial α_0 and β_0. If all the matrices are square then lemmas 3.3.1 and 3.3.2 can be used to find an initial feasible point. If α is rectangular, then a nondeterministic approach can be used to find a good (random) value for η_0.

Finally, if parameter dependent matrices $\varepsilon(\rho)$ and $\alpha(\rho)$ are considered, then according to the exact relation $\eta(\rho) = -\beta^{-1}\alpha(\rho)$, the matrix $\eta(\rho)$ has the same parameter dependence as $\alpha(\rho)$. For simplicity of initialization, it is possible to define a constant η_0 which does not depend on the parameters but when the optimization procedure in η is launched a second time then η should be defined as parameter dependent.

Algorithm 3.3.5 *1. Let $i = 0$, fix $\eta_i(\rho)$*

2. Solve for $(\varepsilon_i(\rho), \alpha_i(\rho), \beta_i(\rho))$ solutions of

$$\begin{bmatrix} \varepsilon(\rho) + \eta_i(\rho)\alpha(\rho) + \alpha(\rho)^T\eta_i(\rho) & \eta_i(\rho)^T\beta(\rho) \\ \star & -\beta(\rho) \end{bmatrix} \prec 0 \tag{3.61}$$

3. Let $i = i + 1$, solve for η_i solution of

$$\begin{bmatrix} \varepsilon_{i-1}(\rho) + \eta_i(\rho)\alpha_{i-1}(\rho) + \alpha_{i-1}(\rho)^T\eta_i(\rho) & \eta_i(\rho)^T\beta_{i-1}(\rho) \\ \star & -\beta_{i-1}(\rho) \end{bmatrix} \prec 0 \tag{3.62}$$

If stop criterion is satisfied then STOP else go to Step 2.

It will be shown in Section 5.1.3 that such an algorithm leads to good results in a small number of iterations (between 1 and 4).

3.4 Polytopic Systems and Bounded-Parameter Variation Rates

In many of the papers, only parameter dependent polytopic system with arbitrary fast-varying parameter variation rate (unbounded rate of variation) are considered (see for instance [Oliveira et al., 2007]). However, some of them consider robust stability instead of quadratic stability [de Souza and Trofino, 2005]. In this section, a way to consider easily bounded parameter variation rate in the polytopic domain is introduced. The main difference between stability conditions expressed for arbitrary fast varying system and bounded rate parameters, is the presence, or not, of parameter derivatives into these conditions. The main difficulty is that derivatives of the polytopic variables have a non-straightforward relation with parameter derivatives. As an example, let us consider the following polytopic LPV system with $N = 2^s$ polytopic variables where s is a positive integer:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\lambda)x(t) + E(\lambda)w(t) \\ z & = & C(\lambda)x(t) + F(\lambda)w(t) \end{array} \tag{3.63}$$

The robust bounded-real lemma (see Section 1.3.2) is then given by the LMI condition

$$\begin{bmatrix} P(\lambda)A(\lambda) + A(\lambda)^TP(\lambda) + \mathcal{P}[\dot{\lambda}(t) \otimes I] & P(\lambda)E(\lambda) & C(\lambda)^T \\ \star & -\gamma I & F(\lambda)^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{3.64}$$

where $\mathcal{P} = \begin{bmatrix} P_1 & P_2 & \dots & P_N \end{bmatrix}$.

Now rewrite the matrix $A(\lambda)$ as the following:

$$A(\lambda) = \sum_{i=1}^{N} \lambda_i(t)V_iA_i \tag{3.65}$$

where the time-varying parameters are given by $\rho(t) = \sum_{i=1}^{N} \lambda_i(t)V_i$ where the V_i are the vertices of the polytope in which $\rho(t)$ evolve and $\lambda_i(t)$ the time-varying polytopic coordinates evolving over the unit simplex Γ:

$$\Gamma := \left\{ \lambda(t) \in [0,\ 1]^N : \sum_{i=1}^{N} \lambda_i(t) = 1,\ t \geq 0 \right\} \tag{3.66}$$

The extremal values of $\rho(t)$ are the V_i, $i = 1, \ldots, N$ but, on the other hand, provided that bounds on the rate of variation are known, then it is possible to define a polytope containing the parameter derivatives, i.e. $\dot{\rho}(t) \in \text{hull}[D]$. Indeed, differentiating the parameters $\rho(t)$ we get

$$\dot{\rho}(t) = \sum_{i=1}^{N} \dot{\lambda}_i(t) V_i \in \text{hull}[D_i] \tag{3.67}$$

and from this expression, the relation between the values $\dot{\lambda}_i(t)$ and the D_i is unclear. What are the extremal values for $\dot{\lambda}_i(t)$? A way to find them is to define

$$\dot{\rho}(t) := \sum_{i=1}^{N} \lambda_i'(t) D_i \tag{3.68}$$

where D_i, $i = 1 : \ldots, N$ are the vertices of the polytope containing all possible values of the parameter derivatives and $\lambda_i'(t)$ the time-varying polytopic coordinates evolving over the unit simplex Γ. In this case, we have the following equality

$$\sum_{i=1}^{N} \dot{\lambda}_i(t) V_i = \sum_{i=1}^{N} \lambda_i'(t) D_i \tag{3.69}$$

which is equivalently written in a compact matrix form

$$V\dot{\lambda}(t) = D\lambda'(t) \tag{3.70}$$

with $V = \begin{bmatrix} V_1 & V_2 & \ldots & V_N \end{bmatrix}$, $D = \begin{bmatrix} D_1 & D_2 & \ldots & D_N \end{bmatrix}$, $\dot{\lambda}(t) = \text{col}(\dot{\lambda}_i(t))$ and $\Lambda' = \text{col}(\lambda_i'(t))$. Note that we have the following equality constraints

$$\begin{array}{rcl} \sum_{i=1}^{N} \lambda_i'(t) & = & 1 \\ \sum_{i=1}^{N} \dot{\lambda}_i(t) & = & 0 \end{array} \tag{3.71}$$

Combined to (3.70), we get

$$\left[\begin{array}{cccc} & V & & \\ \hline 1 & 1 & \ldots & 1 \end{array}\right] \dot{\lambda}(t) = \left[\begin{array}{cccc} & D & & \\ \hline 1 & 1 & \ldots & 1 \end{array}\right] \lambda'(t) - \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] \tag{3.72}$$

which is rewritten compactly as

$$\bar{V}\dot{\lambda}(t) = \bar{D}\lambda'(t) - C \tag{3.73}$$

with $\bar{V} = \left[\begin{array}{cccc} & V & & \\ \hline 1 & 1 & \ldots & 1 \end{array}\right]$, $\bar{D} = \left[\begin{array}{cccc} & D & & \\ \hline 1 & 1 & \ldots & 1 \end{array}\right]$ and $C = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$. Such an equation has solutions in $\dot{\lambda}(t)$ if and only if one of the following statements holds (see Appendix A.8 or Skelton et al. [1997]):

1. $(I - \bar{V}\bar{V}^+)(\bar{D}\lambda'(t) - C) = 0$
2. $\bar{D}\lambda'(t) - C = \bar{V}\bar{V}^+(\bar{D}\lambda'(t) - C)$

In this case the set of solutions is given by

$$\dot{\lambda}(t) = \bar{V}^+(\bar{D}\lambda'(t) - C) + (I - \bar{V}^+\bar{V})Z \tag{3.74}$$

where Z is an arbitrary matrix with appropriate dimensions. It is clear that $V \in \mathbb{R}^{\log_2(N)\times N}$ and then $\text{rank}[V] = \dim(\rho) = \log_2(N)$. Finally, due to the structure of V, we have $\text{rank}[\bar{V}] = \log_2(N)+1$, hence $\bar{V}$ is a full row rank matrix and admits a right pseudoinverse $\bar{V}^+$ such that $\bar{V}\bar{V}^+ = I$. This shows that the first statement above holds for every $\bar{D}\Lambda'(t) - C$ and hence all the solutions of the problem write:

$$\dot{\lambda}(t) = \bar{V}^+(\bar{D}\lambda'(t) - C) + (I - \bar{V}^+\bar{V})Z \tag{3.75}$$

for a free matrix Z of appropriate dimensions. It is worth noting that the solution is affine in $\lambda'(t)$ (which seems logical since the equation is linear) and that Z can be removed from the solution since only a solution is needed. Z can be tuned in order to modulate the values of the vector $-\bar{V}^+C$ but it is not of great interest and then the term Z can be set to 0.

It is worth noting that terms $\dot{\lambda}_i(t)$ do not belong to the unitary simplex anymore and may take negative values since the constraint $\sum_{i=1}^N \dot{\lambda}_i(t) = 0$ must be satisfied at every time $t \geq 0$. Finally, substituting $\dot{\lambda}(t) = M\lambda'(t) + N$ with $M = \bar{V}^+(\bar{D}$ and $N = -\bar{V}^+\bar{D}C$ into the LMI (3.64) we get a new condition in terms of the λ_i and λ_i':

$$\begin{bmatrix} P(\lambda)A(\lambda) + A(\lambda)^T P(\lambda) + \mathcal{P}[(M\lambda'(t) + N) \otimes I] & P(\lambda)E(\lambda) & C(\lambda)^T \\ \star & -\gamma I & F(\lambda)^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0$$

The following example shows the effectiveness of the approach.

Example 3.4.1 *Consider a two parameter problem with* $(\rho_1, \rho_2) \in [-1, 1] \times [-2, 3]$ *and* $(\dot{\rho}_1, \dot{\rho}_2) \in [-2, 3] \times [-5, 6]$. *We have the following matrices*

$$V = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & -2 & 3 & 3 \\ -5 & 6 & -5 & 6 \end{bmatrix} \tag{3.76}$$

Thus we can choose

$$\bar{V}^+ = \frac{1}{10}\begin{bmatrix} -2.5 & -1 & 3 \\ -2.5 & 1 & 2 \\ 2.5 & -1 & 3 \\ 2.5 & 1 & 2 \end{bmatrix} \quad \text{and then } \bar{V}^+\bar{D} = \begin{bmatrix} 1.3 & 0.2 & 0.05 & -1.05 \\ 0.2 & 1.3 & -1.05 & 0.05 \\ 0.3 & -0.8 & 1.55 & 0.45 \\ -0.8 & 0.3 & 0.45 & 1.55 \end{bmatrix} \quad \bar{V}^+C = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.2 \end{bmatrix}$$

Finally using (3.74) we get

$$\dot{\lambda}(t) = \begin{bmatrix} 1.3 & 0.2 & 0.05 & -1.05 \\ 0.2 & 1.3 & -1.05 & 0.05 \\ 0.3 & -0.8 & 1.55 & 0.45 \\ -0.8 & 0.3 & 0.45 & 1.55 \end{bmatrix} \lambda'(t) - \begin{bmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.2 \end{bmatrix} \tag{3.77}$$

This ends the section on computing the bounds on polytopic parameter derivatives in terms of another polytopic variables.

3.5 $\mathcal{H}_\infty$ Performances Test via Simple Lyapunov-Krasovskii functional and Related Relaxations

In this section, simple Lyapunov-Krasovskii functionals are considered as in [Gouaisbaut and Peaucelle, 2006b, Han, 2005a]. Fundamental results are recalled and generalized in the LPV case and in the framework of time-varying delays. The type of Lyapunov-Krasovskii functionals proposed in these papers allows to avoid any model transformations or any bounding of cross terms. The only conservatism of the method comes from the initial choice of the Lyapunov-Krasovskii functionals (which is not complete) and the use of the Jensen's inequality (see [Gu et al., 2003] or Appendix F.1) used to bound an integral term in the derivative of the Lyapunov-Krasovskii functional. The main advantage of these functionals is based on their simplicity and the small number of Lyapunov-Krasovskii variables involved, thus minimizing products between data matrices and decision variables, making them potentially interesting criteria for stabilization problem.

As we shall see later, in the case of a simple Lyapunov-Krasovskii functional, two matrix couplings occur and thus a relaxation scheme must be performed in order to get tractable LMI condition for the stabilization problem. In the framework of a discretized Lyapunov-Krasovskii functional, many couplings would appear corresponding to the order of discretization that has been considered.

We will consider in this section the following LPV time-delay system:

$$\begin{array}{rcl} \dot{x}(t) &=& A(\rho)x(t) + A_h(\rho)x(t-h(t)) + E(\rho)w(t) \\ z(t) &=& C(\rho)x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t) \end{array} \tag{3.78}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ are respectively the system state, the exogenous inputs and the controlled outputs. The delay $h(t)$ is assumed to be long to the set $\mathscr{H}_1^\circ$, the parameters ρ satisfy $\rho \in U_\rho$ and $\dot{\rho} \in \text{hull}[U_\nu]$.

3.5.1 Simple Lyapunov-Krasovskii functional

The main result of this subsection is based on the use of the following Lyapunov-Krasovskii functional [Gouaisbaut and Peaucelle, 2006b, Han, 2005a]:

$$\begin{array}{rcl} V(t) &=& V_1(t) + V_2(t) + V_3(t) \\ V_1(t) &=& x(t)^T P(\rho) x(t)^T \\ V_2(t) &=& \displaystyle\int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta \\ V_3(t) &=& \displaystyle\int_{-h_{max}}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T h_{max} R \dot{x}(\eta) d\eta d\theta \end{array} \tag{3.79}$$

from which the following results is derived:

Lemma 3.5.1 *System (3.78) is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, $Q, R \in \mathbb{S}_{++}^n$ and $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} \Psi_{11}(\rho,\nu) & P(\rho)A_h(\rho) + R & P(\rho)E(\rho) & C(\rho)^T & h_{max}A(\rho)^T R \\ \star & -(1-\mu)Q - R & 0 & C_h(\rho)^T & h_{max}A_h(\rho)^T R \\ \star & \star & -\gamma I_m & F(\rho)^T & h_{max}E(\rho)^T R \\ \star & \star & \star & -\gamma I_p & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{3.80}$$

with

$$\Psi_{11}(\rho,\nu = A(\rho)^T P(\rho) + PA(\rho) + \partial_\rho P(\rho)\nu + Q - R \tag{3.81}$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$.

Proof: *Computing the time-derivative of (3.79) along the trajectories solutions of system (3.22) leads to*

$$\begin{array}{rcl} \dot{V}_1(t) & = & \dot{x}(t)^T P(\rho)x(t) + x(t)P(\rho)\dot{x}(t) + x(t)^T \partial_\rho P(\rho)\dot{\rho}x(t) \\ \dot{V}_2(t) & = & x(t)^T Qx(t) - (1-\dot{h})x(t-h(t))^T Qx(t-h(t)) \\ \dot{V}_3(t) & = & h_{max}^2 \dot{x}(t)^T R\dot{x}(t) - \displaystyle\int_{t-h_{max}}^{t} \dot{x}(\theta)^T h_{max} R\dot{x}(\theta)d\theta \end{array} \tag{3.82}$$

Note that as $|\dot{h}| < 1$ *then we have* $-(1-\dot{h}) \leq -(1-\mu)$. *Note also that as* $h(t) \leq h_{max}$ *then*

$$-\int_{t-h_{max}}^{T} \dot{x}(\theta)^T h_{max} R\dot{x}(\theta)d\theta \leq -\int_{t-h(t)}^{t} \dot{x}(\theta)^T h_{max} R\dot{x}(\theta)d\theta$$

.

Finally using the Jensen's inequality (see Appendix F.1) it is possible to bound the integral term in $\dot{V}_3(t)$ *as follows:*

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & h_{max}^2 \dot{x}(t)^T R\dot{x}(t) - \displaystyle\int_{t-h(t)}^{T} \dot{x}(\theta)^t h_{max} R\dot{x}(\theta)d\theta \\ & \leq & h_{max}^2 \dot{x}(t)^T R\dot{x}(t) - \dfrac{h_{max}}{h(t)} \left(\displaystyle\int_{t-h(t)}^{t} \dot{x}(\theta)d\theta\right)^T R \left(\displaystyle\int_{t-h(t)}^{t} \dot{x}(\theta)d\theta\right) \\ & = & h_{max}^2 \dot{x}(t)^T R\dot{x}(t) - \dfrac{h_{max}}{h(t)} \left(x(t) - x(t-h(t))\right)^T R\left(x(t) - x(t-h(t))\right) \end{array} \tag{3.83}$$

It should be proved now that the previous expression is well-posed in $h(t)$ *when* $h(t)$ *is zero. First denote* t_i *to be the time-instant such that* $h(t_i) = 0$, *we aim to prove that when* $t \to t_i$ *then the quantity*

$$\frac{1}{h(t)} \left(x(t) - x(t-h(t))\right)^T R\left(x(t) - x(t-h(t))\right) \tag{3.84}$$

is bounded. Rewrite it in the form

$$h(t)\left(\frac{x(t) - x(t-h(t))}{h(t)}\right)^T R\left(\frac{x(t) - x(t-h(t))}{h(t)}\right) \tag{3.85}$$

Then when $t \to t_i$ *we have* $\dfrac{x(t) - x(t-h(t))}{h(t)} \to \dot{x}(t_i)$ *since* $x(t)$ *is differentiable. Moreover, as* $x(t)$ *is finite for all* $t \in \mathbb{R}^+$ *this proves that (3.84) remains bounded when* $t \to t_i$. *Finally bounding* $-\dfrac{h_{max}}{h(t)}$ *by* -1 *we get*

$$\dot{V}_3(t) = h_{max}^2 \dot{x}(t)^T R\dot{x}(t) - \left(x(t) - x(t-h(t))\right)^T R\left(x(t) - x(t-h(t))\right) \tag{3.86}$$

Gathering all the derivative terms $\dot{V}_i$ *we get the following quadratic inequality:*

$$\dot{V}(t) \leq X(t)^T \Psi(\rho,\dot{\rho})X(t) < 0 \tag{3.87}$$

with

$$\Psi(\rho,\dot\rho) = \begin{bmatrix} \Psi_{11}(\rho,\dot\rho) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) \\ \star & -(1-\mu)Q-R & 0 \\ \star & \star & 0 \end{bmatrix} + h_{max}^2 \mathcal{T}(\rho)^T R \mathcal{T}(\rho)$$

$$\begin{array}{rcl} X(t) & = & \mathrm{col}(x(t), x(t-h(t)), w(t)) \\ \mathcal{T} & = & \begin{bmatrix} A(\rho) & A_h(\rho) & E(\rho) \end{bmatrix} \\ \Psi_{11}(\rho,\dot\rho) & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R \end{array}$$

To introduce $\mathcal{L}_2$ performances test into the LMI condition, an Hamiltonian function H is constructed and defined as

$$H(t) = V(t) - \int_0^t \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta \tag{3.88}$$

If the hamiltonian function satisfies $\dot H < 0$ for all non zero $X(t)$ then have

$$\lim_{t\to+\infty} H(t) = \lim_{t\to+\infty} V(t) - V(0) - \int_0^t \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta < 0 \tag{3.89}$$

Assuming zero initial conditions (i.e. $V(0) = 0$) and that the system is asymptotically stable ($\lim_{t\to+\infty} V(t) = 0$) then we get

$$\lim_{t\to+\infty} H = -\int_0^t \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta < 0 \tag{3.90}$$

Finally we have

$$\int_0^t \gamma w(\theta)^T w(\theta) - \gamma^{-1} z(\theta)^T z(\theta) d\theta > 0 \tag{3.91}$$

which is equivalent to

$$\gamma ||w||_{\mathcal{L}_2}^2 - \gamma^{-1} ||z||_{\mathcal{L}_2}^2 > 0 \tag{3.92}$$

and thus

$$\frac{||z||_{\mathcal{L}_2}}{||w||_{\mathcal{L}_2}} < \gamma^2 \tag{3.93}$$

Expanding $z(t)$ into the expression of $\dot H$ leads to

$$\dot H \leq \dot V - \gamma w(t)^T w(t) + \gamma^{-1} X(t)^T \begin{bmatrix} C(\rho)^T \\ C_h(\rho)^T \\ F(\rho)^T \\ \end{bmatrix} \begin{bmatrix} C(\rho) & C_h(\rho) & F(\rho) \end{bmatrix} X(t) \tag{3.94}$$

Finally performing a Schur complement onto term

$$-\begin{bmatrix} C(\rho)^T & h_{max}A(\rho)^T R \\ C_h(\rho)^T & h_{max}A_h(\rho)^T R \\ F(\rho)^T & h_{max}E(\rho)^T R \end{bmatrix} \begin{bmatrix} -\gamma^{-1} I & 0 \\ 0 & -R^{-1} \end{bmatrix} \begin{bmatrix} C(\rho) & C_h(\rho) & F(\rho) \\ h_{max}RA(\rho) & h_{max}RA_h(\rho) & h_{max}RE(\rho) \end{bmatrix}$$

leads to LMI (3.80). Finally, noting that $\dot\rho \in \mathrm{hull}[U\nu]$ enters linearly in the LMI, it suffices to check the LMI only at the vertices which are the elements of $U\nu$. This concludes the proof.
□

3.5.2 Associated Relaxation

It is clear from the expression of LMI (3.80) that this criterium is not very suited for stabilization purposes due to the product terms $PA, RA\dots$. Indeed, by introducing the closed-loop system state-space into LMI conditions, due to coupling terms, the linearization is an impossible task without considering (strong) assumptions. In many problems, the common simplification would be to consider 'proportional' matrices in the sense that

$$R = \varepsilon P$$

where $\varepsilon > 0$ is a chosen fixed scalar. It is clear that such a simplification is very conservative since the initial space of decision

$$\mathbb{S}^n_{++} \times \mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$$

is reduced to

$$\mathbb{S}^n_{++} \times \mathbb{S}^n_{++}$$

The idea that we propose here is, rather to simplify the stabilization conditions after introducing the closed-loop system expression, we turn the initial LMI condition into a form which better fits the stabilization problem [Tuan et al., 2001]. Roughly speaking, a LMI is efficient for a stabilization problem if there is only one coupling between a decision matrix and system variables. This decision matrix is not a Lyapunov variable but is a 'slack' variable introduced by applying the Finsler's Lemma (see Appendix E.16). Using this lemma we obtain the following relaxation to LMI (3.80):

Lemma 3.5.2 *System (3.78) is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^n_{++}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$, a matrix function $X : U_\rho \to \mathbb{R}^{n\times n}$ and $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} -X(\rho)^H & P(\rho)+X(\rho)^TA(\rho) & X(\rho)^TA_h(\rho) & X(\rho)^TE(\rho) & 0 & X(\rho)^T & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{3.95}$$

with

$$\Psi_{22}(\rho,\nu) = \partial_\rho P(\rho)\nu - P(\rho) + Q - R \tag{3.96}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$.

Proof: *The proof is inspired from Tuan et al. [2001]. Rewrite (3.95) as*

$$\mathcal{M}(\rho,\nu) + \left[\mathcal{P}(\rho)^TX(\rho,\rho_h)\mathcal{Q}\right]^H \prec 0$$

with

$$\begin{array}{rcl}\mathcal{M}(\rho,\nu) & = & \begin{bmatrix} 0 & P(\rho) & 0 & 0 & 0 & 0 & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q(\rho_h)-R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \\ \mathcal{P}(\rho) & = & \begin{bmatrix} -I & A(\rho) & A_h(\rho) & E(\rho) & 0 & I & 0 \end{bmatrix} \\ \mathcal{Q} & = & \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

Noting that explicit basis of the null-space of $\mathcal{P}$ and $\mathcal{Q}$ are given by

$$Ker(\mathcal{P}(\rho)) = \begin{bmatrix} A(\rho) & A_h(\rho) & E(\rho) & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \qquad Ker(\mathcal{Q}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{3.97}$$

and applying the projection lemma (see Appendix E.18) we get the two underlying LMIs

$$\begin{bmatrix} \Psi_{11}(\rho) & P(\rho)A_h(\rho)+R & P(\rho)E(\rho) & C(\rho)^T & P(\rho) & h_{max}A(\rho)^TR \\ \star & -(1-\mu)Q-R & 0 & C_h(\rho)^T & 0 & h_{max}A_h(\rho)^TR \\ \star & \star & -\gamma I & F(\rho)^T & 0 & h_{max}E(\rho)^TR \\ \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & -P(\rho) & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{3.98}$$

$$\begin{bmatrix} -P(\rho)+Q-R+\partial_\rho P(\rho)\nu & R & C(\rho)^T & 0 & 0 & 0 \\ \star & -(1-\mu)Q-R & C_h(\rho)^T & 0 & 0 & 0 \\ \star & \star & -\gamma I & F(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{3.99}$$

LMI (3.98) is equivalent to (3.95) modulo a Schur complement (see Appendix E.15). Hence this shows that feasibility of (3.95) implies feasibility of (3.98) and (3.99). This concludes the proof. □

Although (3.95) implies (3.80), it also implies LMI (3.99) which is not always satisfied. Thus conservatism is induced while imposing supplementary constraints: among others the left-upper block gives $-P(\rho)+Q(\rho)-R+\partial_\rho P(\rho)\nu < 0$ and the 2×2 right-bottom block gives $-P(\rho)+h_{max}^2R < 0$ (invoking the Schur's complement) which restrict the initial set for decision variables. Nevertheless, several tests have shown that it gives relatively good results in terms of $\mathcal{H}_\infty$ level bound and delay-margin computation, this will be illustrated in Section 5.1.1.

3.5.3 Reduced Simple Lyapunov-Krasovskii functional

Another result based on a simple Lyapunov-Krasovskii functional is provided. This results aims at reducing the computational complexity of the stability test obtained from Lyapunov-Krasovskii functional (3.79) when the matrices A_h and C_h take the following form:

Assumption 3.5.3

$$A_h(\rho) = \begin{bmatrix} A_h^{11}(\rho) & 0 \\ A_h^{21}(\rho) & 0 \end{bmatrix} \qquad C_h(\rho) = \begin{bmatrix} C_h'(\rho) & 0 \end{bmatrix} \tag{3.100}$$

it is possible to reduce the complexity of the Lyapunov-Krasovskii functional. Indeed, as illustrated above, the second part of the state is not affected by the delay and thus this state information can be removed from the part of the Lyapunov-Krasovskii functional dealing with the stability analysis of the delayed part. It is interesting to note that such a representation occurs in the filtering problem of time-delay systems using a memoryless filter [Zhang and Han, 2008] or by controlling a time-delay system using a memoryless dynamic controller. Indeed, in each of this case, the second part of the state is either the filter or controller state, which are not affected by the delay (provided that no delay acts on the control input of the system).

Note that it is possible to write $A_h(\rho) = A_h'(\rho)Z$ and $C_h(\rho) = C_h'(\rho)Z$ where

$$Z = \begin{bmatrix} I & 0 \end{bmatrix} \qquad A_h'(\rho) = \begin{bmatrix} A_h^{11} \\ A_h^{21} \end{bmatrix} \tag{3.101}$$

Hence there is no increase of conservatism by considering the Lyapunov-Krasovskii functional

$$\begin{array}{rcl} V(t) & = & V_1(t) + V_2(t) + V_3(t) \\ V_1(t) & = & x(t)^T P(\rho) x(t) \\ V_2(t) & = & \int_{t-h(t)}^{t} x(\theta)^T Z^T Q(\rho(\theta)) Z x(t\theta) d\theta \\ V_3(t) & = & \int_{-h_{max}}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T Z^T R Z \dot{x}(\eta) d\eta d\theta \end{array} \tag{3.102}$$

which gives rise to the following result:

Lemma 3.5.4 *System (3.78) with assumption 3.5.3 is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ if there exist matrix a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, constant matrices $Q, R \in \mathbb{S}_{++}^n$ and a scalar $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} \Psi_{11}'(\rho,\nu) & P(\rho)A_h'(\rho) + R & P(\rho)E(\rho) & C(\rho)^T & h_{max}A(\rho)^T Z^T R \\ \star & -(1-\mu)Q(\rho_h) - R & 0 & C_h'(\rho)^T & h_{max}A_h'(\rho)^T Z^T R \\ \star & \star & -\gamma I_m & F(\rho)^T & h_{max}E(\rho)^T Z^T R \\ \star & \star & \star & -\gamma I_p & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{3.103}$$

with

$$\Psi_{11}'(\rho,\nu = A(\rho)^T P(\rho) + PA(\rho) + \partial_\rho P(\rho)\nu + Z^T(Q(\rho) - R)Z \tag{3.104}$$

holds for all $(\rho, \rho_h, \nu) \in U_\rho \times U_{\rho_h} \times U_\nu$.

Proof: *Similarly as in the proof of Lemma 3.5.1, the time-derivative of the Lyapunov-Krasovskii functional (3.102) can be expressed and bounded as follows*

$$\begin{array}{rcl} \dot{V}_1(t) & = & \dot{x}(t)^T P(\rho)x(t) + \dot{x}(t)P(\rho)x(t)^T + x(t)^T \partial_\rho P(\rho)\dot{\rho}x(t) \\ \dot{V}_2(t) & \leq & x(t)^T Z^T Q(\rho) Z x(t) - (1-\mu)x(t-h(t))^T Z^T Q(\rho_h) Z x(t-h(t)) \\ \dot{V}_3(t) & \leq & h_{max}^2 \dot{x}(t)^T Z^T R Z \dot{x}(t) - (x(t)-x(t-h(t)))^T Z^T R Z (x(t)-x(t-h(t))) \end{array} \quad (3.105)$$

Gathering all the derivative terms $\dot{V}_i$ we get the following quadratic inequality:

$$\dot{V}(t) \leq X(t)^T \Psi'(\rho,\nu) X(t) < 0 \quad (3.106)$$

$$\begin{array}{rcl} \Psi'(\rho,\nu) & = & \begin{bmatrix} \Psi'_{11}(\rho,\dot{\rho}) & P(\rho)A'_h(\rho) + Z^T R & P(\rho)E(\rho) \\ \star & -(1-\mu)Q - R & 0 \\ \star & \star & 0 \end{bmatrix} + h_{max}^2 \mathcal{T}(\rho)^T R \mathcal{T}(\rho) \\ X(t) & = & \mathrm{col}(x(t), Zx(t-h(t)), w(t)) \\ \mathcal{T}(\rho) & = & \begin{bmatrix} A(\rho) & A'_h(\rho) & E(\rho) \end{bmatrix} \\ \Psi_{11}(\rho,\dot{\rho}) & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + Z^T Q Z - Z^T R Z \end{array}$$

Adding the constraint

$$\int_0^t \gamma w(\eta)^T w(\eta) - \gamma^{-1} z(\eta)^T z(\eta) d\eta > 0 \quad (3.107)$$

to the Lyapunov function in view of constructing the Hamiltonian function we get

$$\dot{H} \leq \dot{V} - \gamma w(t)^T w(t) + \gamma^{-1} X(t)^T \begin{bmatrix} C(\rho)^T \\ C'_h(\rho)^T \\ F(\rho)^T \end{bmatrix} \begin{bmatrix} C(\rho) & C'_h(\rho) & F(\rho) \end{bmatrix} X(t) \quad (3.108)$$

Finally performing a Schur complement onto term

$$-\begin{bmatrix} C(\rho)^T & h_{max}A(\rho)^T Z^T R \\ C'_h(\rho)^T & h_{max}A'_h(\rho)^T Z^T R \\ F(\rho)^T & h_{max}E(\rho)^T Z^T R \end{bmatrix} \begin{bmatrix} -\gamma^{-1}I & 0 \\ 0 & -R^{-1} \end{bmatrix} \begin{bmatrix} C(\rho) & C'_h(\rho) & F(\rho) \\ h_{max}RZA(\rho) & h_{max}RZA'_h(\rho) & h_{max}RZE(\rho) \end{bmatrix}$$

leads to LMI (3.103). Finally, noting that $\dot{\rho} \in \mathrm{hull}[U\nu]$ enters linearly in the LMI, it suffices to check the LMI only at the vertices which are the elements of $U\nu$. This concludes the proof. □

3.5.4 Associated Relaxation

Similarly as for lemma 3.5.1, it is convenient to construct a relaxation lemma which will be of interest further in the thesis.

Lemma 3.5.5 *System(3.78) with assumption 3.5.3 is asymptotically stable for all $h \in \mathscr{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function*

$P : U\rho \rightarrow \mathbb{S}^n_{++}$, a matrix function $X : U_\rho \rightarrow \mathbb{R}^{n\times n}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$ and $\gamma > 0$ such that the LMI

$$\begin{bmatrix} -X(\rho)^H & P(\rho)+X(\rho)^T A(\rho) & X(\rho)^T A_h(\rho) & X(\rho)^T E(\rho) & 0 & X(\rho)^T & h_{max} Z^T R \\ \star & \Psi'_{22}(\rho,\nu) & R & 0 & C(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q - R & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max} Z^T R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{3.109}$$

with

$$\Psi'_{22}(\rho,\nu) = \partial_\rho P(\rho)\nu - P(\rho) + Z^T(Q-R)Z \tag{3.110}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$.

Proof: *The proof is similar to the proof of lemma 3.5.2.* □

This ends the section on results based on simple Lyapunov-Krasovskii functionals. The interest of such functionals, despite of the conservatism, is to to not rely on model transformations and bounding of cross-terms. The next section generalizes these functionals to a more 'complete' form in order to obtain less and less conservative results.

3.6 Discretized Lyapunov-Krasovskii Functional for systems with time varying delay and Associated Relaxation

The current subsection aims at improving previous results based on simple Lyapunov-Krasovskii functionals of the form (3.79) and (3.102). It is clear that, compared to complete Lyapunov-Krasovskii functionals defined in [Gu et al., 2003, Han, 2005b], the conservatism comes from the fact that the matrices Q and R are constant with respect to the integration parameter. Moreover, another advantage of the discretization approach is to divide the delay into smaller fragments in order to reduce the conservatism induced by the Jensen's inequality. To see this, let us consider the following example:

Example 3.6.1 *In this example we will consider the function $\dot{x}(\theta) = \theta$ and we will analyze the gap between the following integral*

$$\begin{aligned} \mathcal{I}_1 &:= -\int_{t-h}^{t} \dot{x}(\theta)^2 d\theta \\ \mathcal{I}_2 &:= -\frac{1}{h}\left(\int_{t-h}^{t} \dot{x}(\theta) d\theta\right)^2 \end{aligned} \tag{3.111}$$

were $h > 0$ and $t \in \mathbb{R}_+$. Then we can write

$$\begin{array}{rcl}\mathcal{I}_1 & = & -\int_{t-h}^{t} \theta^2 d\theta \\ & = & \frac{1}{3}((t-h)^3 - t^3) \\ & = & \frac{1}{3}(3th^2 - 3t^2h - h^3) \\ \mathcal{I}_2 & = & -\frac{1}{h}\left(\int_{t-h}^{t} \dot{x}(\theta)d\theta\right)^2 \\ & = & -\frac{1}{4h}(t^2 - (t-h)^2)^2 \\ & = & \frac{1}{4}(4th^2 - 4t^2h - h^3)\end{array} \tag{3.112}$$

The Jensen's inequality claims that $\mathcal{I}_2 \geq \mathcal{I}_1$ and hence the conservatism gap is given by the positive difference between $\mathcal{I}_2$ and $\mathcal{I}_1$, namely $\delta\mathcal{I}_{21}$:

$$\begin{array}{rcl}\delta\mathcal{I}_{21} & := & \mathcal{I}_2 - \mathcal{I}_1 \\ & = & \frac{1}{4}(4th^2 - 4t^2h - h^3) - \frac{1}{3}(3th^2 - 3t^2h - h^3) \\ & = & \frac{1}{12}h^3\end{array} \tag{3.113}$$

This shows that the gap between the initial integral term and its corresponding bounds varies proportionally to the cube of the delay value. Hence, this suggests that by considering smaller delay values it might be possible to reduce the conservatism of the approach. First of all, decompose $\mathcal{I}_1$ into

$$\mathcal{I}_1 = \int_{t-h}^{t-h/2} \dot{x}(\theta)^2 d\theta + \int_{t-h/2}^{t} \dot{x}(\theta)^2 d\theta \tag{3.114}$$

Let us consider the sum of the Jensen's bound of each integral term

$$\mathcal{I}_3 := -\frac{2}{h}\left[\left(\int_{t-h}^{t-h/2} \dot{x}(\theta)d\theta\right)^2 + \left(\int_{t-h/2}^{t} \dot{x}(\theta)d\theta\right)^2\right] \tag{3.115}$$

Using the explicit expression of $\dot{x}(\theta)$ we get

$$\begin{array}{rcl}\mathcal{I}_3 & = & -\frac{2}{4h}\left[\left((t-h/2)^2 - (t-h)^2\right)^2 + \left(t^2 - (t-h/2)^2\right)^2\right] \\ & = & -t^2h + th^2 - \frac{5}{16}h^3\end{array} \tag{3.116}$$

The corresponding gap $\delta\mathcal{I}_{31} := \mathcal{I}_3 - \mathcal{I}_1$ is then given by

$$\delta\mathcal{I}_{31} = \frac{1}{48}h^3 \tag{3.117}$$

By fragmenting the delay up to order 3 we get

$$\begin{array}{rcl}\mathcal{I}_4 & := & -\frac{3}{4h}\left[\left((t-2h/3)^2 - (t-h)^2\right)^2 + \left((t-h/3)^2 - (t-2h/3)^2\right) + \left(t^2 - (t-h/3)^2\right)^2\right] \\ & = & -t^2h + th^2 - \frac{35}{108}h^3\end{array} \tag{3.118}$$

and the resulting gap $\delta\mathcal{I}_{41} := \mathcal{I}_4 - \mathcal{I}_1$ *is given by*

$$\delta\mathcal{I}_{41} = \frac{1}{108}h^3 \tag{3.119}$$

This example shows that by increasing the order of the fragmentation it is possible to reduce the conservatism brought by the use of the Jensen's inequality. It is interesting to note that since the gap evolves as a polynomial of degree 3 and for each fragmentation the degree will remain to 3 (this is an intrinsic property related to the fact that $\dot{x}(\theta)$ is of degree 1). Fragmenting the delay will decrease the coefficient only meaning that for greater order of fragmentation the conservatism will be reduced. This has been also noticed in [Gouaisbaut and Peaucelle, 2006a,b, Han, 2008]. As a conjectural result, it can be shown that

$$\begin{array}{rcl} \delta\mathcal{I}_{N1} & := & \mathcal{I}_N - \mathcal{I}_1 \\ & = & \dfrac{1}{12N^2}h^3 \end{array} \tag{3.120}$$

where N is the fragmentation order and $\mathcal{I}_N$ is given by the expression

$$\mathcal{I}_N := -\frac{N}{4h}\sum_{i=0}^{N-1}\left[\left(t-\frac{N-i-1}{N}h\right)^2-\left(t-\frac{N-i}{N}h\right)^2\right]^2 \tag{3.121}$$

3.6.1 Discretized Lyapunov-Krasovskii functional

According to latter remarks, we introduce the following Lyapunov-Krasovskii functional [Han, 2008]:

$$\begin{array}{rcl} V(x_t,\dot{x}_t) & = & V_1(x(t)) + V_2(x_t) + V_3(\dot{x}_t) \\ V_1(x(t)) & = & x(t)^T P x(t) \\ V_2(x_t) & = & \displaystyle\sum_{i=0}^{N-1}\int_{t-(i+1)h_N(t)}^{t-ih_N(t)} x(\theta)^T Q_i x(\theta)d\theta \\ V_3(\dot{x}_t) & = & \displaystyle\sum_{i=0}^{N-1}\int_{-(i+1)\bar{h}}^{-i\bar{h}}\int_{t+\theta}^{t} \dot{x}(\eta)^T \bar{h} R_i \dot{x}(\eta)d\eta d\theta \end{array} \tag{3.122}$$

with $h_N(t) \triangleq \dfrac{h(t)}{N}$ and $\bar{h} \triangleq \dfrac{h_{max}}{N}$. This Lyapunov-Krasovskii functional gives the following result:

Lemma 3.6.2 *System (3.78) is asymptotically stable for all* $h \in \mathscr{H}_1^\circ$ *and satisfies* $||z||_{\mathcal{L}_2} \le \gamma||w||_{\mathcal{L}_2}$ *if there exist a continuously differentiable matrix* $P : U_\rho \to \mathbb{S}^n_{++}$, *constant matrices* $Q_i, R_i \in \mathbb{S}^n_{++}$, $i \in \{0,\dots,N-1\}$ *and a scalar* $\gamma > 0$ *such that the LMI*

$$\begin{bmatrix} \mathcal{M}_{11} & \Gamma_2(\rho)^T & \bar{h}\Gamma_1(\rho)^T R_0 & \dots & \bar{h}\Gamma_1(\rho)^T R_{N-1} \\ \star & -\gamma I & 0 & \dots & 0 \\ \star & \star & -\bar{h}R_0 & & \\ \star & \star & & \ddots & \\ \star & \star & & & -\bar{h}R_{N-1} \end{bmatrix} \prec 0 \tag{3.123}$$

holds for all $(\rho,\nu)\in U_\rho\times U_\nu$ *where*

$$\mathcal{M}_{11}=\left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \dots & P(\rho)A_h(\rho) & P(\rho)E(\rho)\\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0\\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0\\ & & & \ddots & \ddots & \vdots & \vdots\\ & & & & \ddots & R_{N-1} & 0\\ & & & & & N^{(2)} & 0\\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right] \tag{3.124}$$

$$\begin{array}{rcl} M_{11} & = & A(\rho)^TP(\rho)+P(\rho)A(\rho)+\partial_\rho P(\rho)+Q_0-R_0\\ N_i^{(1)} & = & -(1-i\mu_N)Q_{i-1}+(1+i\mu_N)Q_i-R_{i-1}-R_i\\ N^{(2)} & = & -(1-\mu)Q_{N-1}-R_{N-1}\\ \Gamma_1(\rho) & = & \left[\begin{array}{cccccc|c} A(\rho) & 0 & 0 & 0 & \dots & A_h(\rho) & E(\rho)\end{array}\right]\\ \Gamma_2(\rho) & = & \left[\begin{array}{ccccc|c} C(\rho) & 0 & 0 & \dots & C_h(\rho) & F(\rho)\end{array}\right]\\ \mu_N & = & \mu/N \end{array}$$

Proof:

Computing the derivative of (3.122) along the trajectories solutions of system (3.22) and with similar arguments as for the proof of lemma 3.5.1 we get:

$$\begin{array}{rcl}\dot V(t) & \leq & Y(t)^T\left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \dots & P(\rho)A_h(\rho) & P(\rho)E(\rho)\\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0\\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0\\ & & & \ddots & \ddots & \vdots & \vdots\\ & & & & \ddots & R_{N-1} & 0\\ & & & & & N^{(2)} & 0\\ \hline \star & \star & \star & \dots & 0 & 0 & 0 \end{array}\right]Y(t)\\ & & +\bar h\sum_{i=0}^{N-1}Y(t)^T\Gamma_1^T(\rho)\Gamma_1(\rho)Y(t)\end{array} \tag{3.125}$$

with

$$\begin{array}{rcl} M_{11} & = & A(\rho)^TP(\rho)+P(\rho)A(\rho)+\partial_\rho P(\rho)+Q_0(\rho_0)-R_0\\ N_i^{(1)} & = & -(1-i\mu_N)Q_{i-1}+(1+i\mu_N)Q_i-R_{i-1}-R_i\\ N^{(2)} & = & -(1-\mu)Q_{N-1}-R_{N-1}\\ \Gamma_1(\rho) & = & \left[\begin{array}{cccccc|c} A(\rho) & 0 & 0 & 0 & \dots & A_h(\rho) & E(\rho)\end{array}\right]\\ Y(t) & = & \mathrm{col}(x(t),x_1(t),x_2(t),\dots,x_N(t),w(t))\\ x_i(t) & = & x(t-ih_n(t)) \end{array}$$

The time-derivative of the Hamiltonian function is negative definite if and only if

$$\dot{H}(t) \leq Y(t)^T \left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \dots & P(\rho)A_h(\rho) & P(\rho)E(\rho) \\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right] Y(t) \quad (3.126)$$
$$+\bar{h}\sum_{i=0}^{N-1} Y(t)^T\Gamma_1^T(\rho)\Gamma_1(\rho)Y(t) + \gamma^{-1}Y(t)^T\Gamma_2^T(\rho)\Gamma_2(\rho)Y(t)$$

with

$$\Gamma_2(\rho) = \left[\begin{array}{ccccc|c} C(\rho) & 0 & 0 & \dots & C_h(\rho) & F(\rho) \end{array}\right] \quad (3.127)$$

Then in virtue of the Schur complement with respect to terms

$$+\bar{h}\sum_{i=0}^{N-1} Y(t)^T\Gamma_1^T(\rho)\Gamma_1(\rho)Y(t) + \gamma^{-1}Y(t)^T\Gamma_2^T(\rho)\Gamma_2(\rho)Y(t)$$

we get

$$\begin{bmatrix} \mathcal{M}_{11} & \Gamma_2(\rho)^T & \bar{h}\Gamma_1(\rho)^T R_0 & \dots & \bar{h}\Gamma_1(\rho)^T R_{N-1} \\ \star & -\gamma I & 0 & \dots & 0 \\ \star & \star & -\bar{h}R_0 & & \\ \star & \star & & \ddots & \\ \star & \star & & & -\bar{h}R_{N-1} \end{bmatrix} \prec 0 \quad (3.128)$$

with

$$\mathcal{M}_{11} = \left[\begin{array}{cccccc|c} M_{11} & R_0 & 0 & 0 & \dots & P(\rho)A_h(\rho) & P(\rho)E(\rho) \\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right] \quad (3.129)$$

This concludes the proof. □

This result allows to obtain less conservative results than by using lemma 3.5.1 since on one hand, extra degree of freedom are added by fragmenting the delay which is equivalent to choose piecewise constant continuous functions $Q(\theta)$ and $R(\theta)$. On the second hand, the the fragmentation of the delay reduced the conservatism of the Jensen's inequality.

It is also important to notice that similar results are obtained in [Gouaisbaut and Peaucelle, 2006b, Peaucelle et al., 2007]. However, these results are based on translation of the state by fragmented time-invariant delays which makes the problems more difficult when time-varying delays are considered. The approach provided here is not based on any translation of the state and hence the problem of time-varying delays does not hold. The derived results are actually based only on the application of the Lyapunov-Krasovskii's theorem using the functional (3.122), as done in [Han, 2008].

N	1	2	3	4
h_{max}	4.4721	5.7175	5.9678	6.0569
nb vars	9	15	21	27
h_{max} [Gouaisbaut and Peaucelle, 2006b]	4.4721	5.71	5.91	6.03
nb vars [Gouaisbaut and Peaucelle, 2006b]	9	50	147	324

Table 3.1: Delay margin of system (3.130) using lemma 3.6.2 compared to results of [Gouaisbaut and Peaucelle, 2006b]

Example 3.6.3 *Let us consider the time-delay system [Gouaisbaut and Peaucelle, 2006b]*

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t-h) \tag{3.130}$$

where the delay is constant. The analytical maximal delay value for which the system is asymptotically stable is $h_{analytical} = 6.17$. Table 3.1 provides results using lemma 3.6.2. For $N = 1$, lemma 3.6.2 coincides with lemma 3.5.1.

On the other hand, by increasing N, the bound on the delay margin is less and less conservative which shows the interest of the approach. Compared to results of [Gouaisbaut and Peaucelle, 2006b], the results are roughly identical for each fragmentation order. On the other hand, since in [Gouaisbaut and Peaucelle, 2006b] for each fragmentation number, the state of the system is augmented in order to contain every delayed states (for each fragment), then the number of decision matrices grows very quickly. Indeed, the number of decision variables with lemma 3.6.2 is given by

$$\frac{1}{2}(2N+1)n(n+1) \tag{3.131}$$

and a size of LMI constraint (3.123)

$$n(2N+1) \times n(2N+1) \tag{3.132}$$

where n is the dimension of the system and N the order of fragmentation. For instance, the number of variables is 27 for a system dimension $n = 2$ and an order of discretization $N = 4$; as shown in the previous example. On the other hand, the approach in [Gouaisbaut and Peaucelle, 2006b] results in a number of decision variables

$$\frac{1}{2}Nn(1+2N)(NR+1) \tag{3.133}$$

and a size of the principal LMI of

$$2Nn \times 2Nn \tag{3.134}$$

For instance, the number of variables is 27 for a system dimension $n = 2$ and an order of discretization $N = 4$; as shown in the previous example.

When solving LMI problem with interior point algorithms, the complexity (and thus the time of computation) of the algorithm highly depend on the size of LMIs. Hence, the size of LMIs is an important criterium to compare different methods. Actually, the LMI (3.123) can be reduced to a lower size by a Schur complement (see Appendix E.15) which results in a LMI of size $n(N+1) \times n(N+1)$ which is more competitive compared to method of [Gouaisbaut and Peaucelle, 2006a].

3.6.2 Associated Relaxation

As for lemma 3.5.1, due to the multiple products between Lyapunov matrices ($P(\rho)$, R_i) and data matrices A, A_h and E, the linearization procedure is a difficult task in the control synthesis problem. A relaxed version is provided in order to decouple these terms by introducing a slack variable.

Lemma 3.6.4 *System (3.78) is asymptotically stable for all $h \in \mathcal{H}_1^\circ$ and satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : U\rho \to \mathbb{S}_{++}^n$, a matrix function $X : U_\rho \to \mathbb{R}^{n\times n}$, constant matrices $Q_i, R_i \in \mathbb{S}_{++}^n$, $i \in \{0,\dots,N-1\}$ and a scalar $\gamma > 0$ such that the LMI*

$$\left[\begin{array}{cccc|ccc} -X(\rho)^H & U_{12}(\rho) & 0 & X(\rho)^T & \bar{h}_1 R_0 & \dots & \bar{h}_1 R_{N-1} \\ \star & U_{22}(\rho,\nu) & U_{23}(\rho) & 0 & 0 & \dots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \dots & 0 \\ \star & \star & \star & -P(\rho) & -\bar{h}_1 R_0 & \dots & -\bar{h}_1 R_{N-1} \\ \hline \star & \star & \star & \star & & -\operatorname{diag}_i R_i & \end{array}\right] \prec 0 \quad (3.135)$$

holds for all $(\rho, \rho_h, \nu) \in U_\rho \times U_{\rho_h} \times U_\nu$ and where

$$U_{22} = \left[\begin{array}{cccccc|c} U'_{11} & R_0 & 0 & 0 & \dots & 0 & 0 \\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right] \quad (3.136)$$

$$\begin{array}{rcl} U'_{11} & = & \partial_\rho P(\rho)\dot{\rho} - P(\rho) + Q_0 - R_0 \\ N_i^{(1)} & = & -(1 - i\mu_N)Q_{i-1} + (1 + i\mu_N)Q_i - R_{i-1} - R_i \\ N^{(2)} & = & -(1-\mu)Q_{N-1} - R_{N-1} \\ U_{12}(\rho) & = & \left[\begin{array}{cccccccc} P(\rho) + X(\rho)^T A(\rho) & 0 & 0 & X(\rho)^T A_h(\rho) & \dots & 0 & X(\rho)^T E(\rho) \end{array}\right] \\ U_{23}(\rho) & = & \left[\begin{array}{ccccc|c} C(\rho) & 0 & \dots & 0 & C_h(\rho) & F(\rho) \end{array}\right]^T \end{array}$$

Proof: *The proof is similar to the proof of Lemma 3.5.2.* □

3.7 Simple Lyapunov-Krasovskii functional for systems with delay uncertainty

We consider here LPV time-delay systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h^1(\rho)x(t-h(t)) + A_h^2(\rho)x(t-h_c(t)) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h^1(\rho)x(t-h(t)) + C_h^2(\rho)x(t-h_c(t)) + F(\rho)w(t) \end{array} \quad (3.137)$$

where the delays $h(t)$ and $h_c(t)$ are assumed to satisfy the relation $h_c(t) = h(t) + \theta(t)$ where $\theta(t) \in [-\delta, \delta]$, $\delta > 0$. The problem addressed in this section is the production of a delay-dependent test for a time-delay system involving two-delays which are related by an algebraic equation. Actually, this problem arises when stabilizing, observing or filtering a time-delay systems by a process (controller, observer or filter) involving a delay which is different from the system. In this problem the objectives can be:

1. Given h_{max}, find the maximal uncertainty bound δ for which the system remains stable

2. Given δ find the delay value h_{max} for which the system remains stable

When dealing with performances criterium such as $\mathcal{H}_\infty$ level γ. Other combinations are seeked:

1. Given h_{max} and γ, find the maximal uncertainty bound δ such that the LMI conditions remain feasible

2. Given δ and γ, find the delay value h_{max} such that the LMI conditions remain feasible

3. Given h_{max} and δ, find the minimal $\mathcal{L}_2$ performances index γ for which the LMI conditions remain feasible.

The following sections address the problem of finding a Lyapunov-Krasovskii functional capturing both the stability/performances of system (3.137) and the algebraic equality $h_c(t) = h(t) + \theta(t)$. The last equality makes of this problem a new open problem which is has not been addressed to our best knowledge.

3.7.1 Lyapunov-Krasovskii functional

The main idea in this problem is to capture both the maximal delay value for h but also capture the fact that the relation $h_c(t) = h(t) + \theta(t)$ exists with $\theta(t) \in [-\delta, \delta]$.

If a Lyapunov-Krasovskii functional of the form

$$\begin{array}{rcl} V(x_t, \dot{x}_t) & = & V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(\dot{x}_t) + V_5(\dot{x}_t) \\ V_1(x_t) & = & x(t)^T P x(t) \\ V_2(x_t) & = & \displaystyle\int_{t-h(t)}^{t} x(\theta)^T Q_1 x(\theta) d\theta \\ V_3(x_t) & = & \displaystyle\int_{t-h_c(t)}^{t} x(\theta)^T Q_2 x(\theta) d\theta \\ V_4(\dot{x}_t) & = & \displaystyle\int_{h_{max}}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T R_1 \dot{x}(\eta) d\eta d\theta \\ V_5(\dot{x}_t) & = & \displaystyle\int_{h_{max}+\delta}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T R_2 \dot{x}(\eta) d\eta d\theta \end{array} \tag{3.138}$$

were considered, it is clear that only the condition $h_{c_{max}} = h_{max} + \delta$ would be taken into account, but the 'global' constraint $h(t) = h_c(t) + \theta(t)$ would not. In such a case, the delays would be considered as independent and only their maximal amplitude (i.e. h_{max} and $h_{max} + \delta$) would be mutually dependent. This shows that a new specific Lyapunov-Krasovskii functional should be considered instead:

$$
\begin{array}{rcl}
V(x_t) & = & V_n(x_t) + V_u(x_t) \\
\text{where} & & \\
V_n(x_t) & = & x(t)^T P(\rho)x(t) + \int_{t-h(t)}^{t} x(s)^T Q_1 x(s)ds + \int_{-h_{max}}^{0}\int_{t+\beta}^{t} \dot{x}(s)^T h_{max} R_1 x(s) ds d\beta \\
V_u(x_t) & = & \int_{t-h_c(t)}^{t} x(s)^T Q_2 x(s)ds + \int_{-\delta}^{\delta}\int_{t+\beta-h(t)}^{t} \dot{x}(s) R_2 \dot{x}(s) ds d\beta
\end{array}
\tag{3.139}
$$

The main difference here is the term last term of $V_u(x_t)$ which introduces terms in $t - h(t) + \delta$ and $t - h(t) - \delta$ which can be bounded by terms involving $h_c(t)$. This will be better explained in the proof of the following theorem obtained from this specific functional (3.139):

Theorem 3.7.1 *System (3.137) is delay-dependent stable with $h(t) \in [0, h_{max}]$, $h_c(t) = h_c(t) + \theta(t)$, $\theta(t) \in [-\delta, \delta]$, $|\dot{h}(t)| < \mu$ and $|\dot{h}_c(t)| < \mu_c$ such that $||z||_{\mathcal{L}_2} \le \gamma ||w||_{\mathcal{L}_2}$ if there exists a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^n_{++}$ and symmetric matrices $Q_1, Q_2, R_1, R_2 \succ 0$ and a scalar $\gamma > 0$ such that*

$$
\begin{bmatrix}
\Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h(\rho) + R_1 & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 & C(\rho)^T \\
\star & -(1-\mu)(Q_1+Q_2) - R_1 & 0 & h_{max}A_h(\rho)^T R_1 & A_h(\rho)^T R_2 & C_h(\rho)^T \\
\star & \star & -\gamma I & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 & F(\rho)^T \\
\star & \star & \star & -R_1 & 0 & 0 \\
\star & \star & \star & \star & -(2\delta)^{-1}R_2 & 0 \\
\star & \star & \star & \star & \star & -\gamma I
\end{bmatrix} \prec 0
\tag{3.140}
$$

$$
\begin{bmatrix}
\Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho) + R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 & C(\rho)^T \\
\star & \Psi_{22} & (1-\mu)R_2/\delta & 0 & h_{max}A_h^1(\rho)^T R_1 & A_h^1(\rho)^T R_2 & C_h^1(\rho)^T \\
\star & \star & \Psi_{33} & 0 & h_{max}A_h^2(\rho)^T R_1 & A_h^2(\rho)^T R_2 & C_h^2(\rho)^T \\
\star & \star & \star & -\gamma I & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 & F(\rho)^T \\
\star & \star & \star & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -R_1 & 0 & 0 \\
\star & \star & \star & \star & \star & -\frac{R_2}{2\delta} & 0 \\
\star & \star & \star & \star & \star & \star & -\gamma I
\end{bmatrix} \prec 0
\tag{3.141}
$$

hold for all $\rho \in U_\rho$ and $\nu = \mathrm{col}(\nu_i) \in U_\nu$ where

$$
\begin{array}{rcl}
\Psi_{11}(\rho,\nu) & = & A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q_1 + Q_2 + \sum_{i=1}^{N} \frac{\partial P}{\partial \rho_i}\nu_i - R_1 \\
\Psi_{22} & = & -(1-\mu)(Q_1 + R_2/\delta) - R_1 \\
\Psi_{33} & = & -(1-\mu_c)Q_2 - (1-\mu)R_2/\delta \\
A_h & = & A_h^1 + A_h^2 \\
C_h & = & C_h^1 + C_h^2
\end{array}
$$

Proof: *Differentiating (3.139) along the trajectories solutions of the system (3.137) yields:*

$$\begin{aligned}
\dot{V}_n &\leq Y(t)^T \left(\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) - Q_2 & P(\rho)A_h^1(\rho) + R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) \\ \star & -(1-\dot{h})Q_1 - R_1 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \right. \\
&\quad \left. + h_{max}^2 \begin{bmatrix} A(\rho)^T \\ A_h^1(\rho)^T \\ A_h^2(\rho)^T \\ E(\rho)^T \end{bmatrix} R_1 \begin{bmatrix} A(\rho) & A_h^1(\rho) & A_h^2(\rho) & E(\rho) \end{bmatrix} \right) Y(t) \\
\dot{V}_u &= x(t)^T Q_2 x(t) - (1-\dot{h}_c)x(t-h_c(t))^T Q_2 x(t-h_c(t)) + 2\delta \dot{x}(t)^T R_2 \dot{x}(t) \\
&\quad -(1-\dot{h}(t)) \int_{t-\delta-h(t)}^{t+\delta-h(t)} \dot{x}(s)^T R_2 \dot{x}(s) ds
\end{aligned} \tag{3.142}$$

where $\Psi_{11}(\rho,\dot{\rho}) = A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q_1 + Q_2 + \frac{\partial P}{\partial \rho}\dot{\rho} - R_1$ *and* $Y(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_c(t)) \\ w(t) \end{bmatrix}$.

Moreover, note that we have the inequality

$$\begin{aligned}
-\int_{t-\delta-h(t)}^{t+\delta-h(t)} \dot{x}(s)^T R_2 \dot{x}(s) ds &\leq -\operatorname{sgn}(h(t) - h_c(t)) \int_{t-h(t)}^{t-h_c(t)} \dot{x}(s)^T R_2 \dot{x}(s) ds \\
&\leq -\frac{1}{|h(t) - h_c(t)|} \left(\int_{t-h(t)}^{t-h_c(t)} \dot{x}(s) ds \right)^T R_2 \left(\int_{t-h(t)}^{t-h_c(t)} \dot{x}(s) ds \right) \\
&\leq -\frac{1}{\delta} \left(\int_{t-h(t)}^{t-h_c(t)} \dot{x}(s) ds \right)^T R_2 \left(\int_{t-h(t)}^{t-h_c(t)} \dot{x}(s) ds \right)
\end{aligned} \tag{3.143}$$

This shows that two cases must be treated separately:

1. *either when* $h_c(t_i) = h(t_i)$ *for some* $t_i \geq 0$ *and in this case* $x(t_i - h(t_i)) = x(t_i - h_c(t_i))$, *or*

2. *when* $h_c(t) \neq h(t)$ *for all* $t \geq 0$ *and* $t \neq t_i$.

Case. 1: *When* $h_c(t_i) = h(t_i)$ *the derivative of the Lyapunov-Krasovskii functional reduces to*

$$\begin{aligned}
\dot{V} &\leq X(t_i)^T \left(\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h(\rho) + R_1 & P(\rho)E(\rho) \\ \star & -(1-\dot{h}(t_i))(Q_1+Q_2) & 0 \\ \star & \star & 0 \end{bmatrix} \right. \\
&\quad \left. + \left(\begin{bmatrix} h_{max}A(\rho)^T & h_{max}A(\rho)^T \\ h_{max}A_h(\rho)^T & h_{max}A_h(\rho)^T \\ h_{max}E(\rho)^T & h_{max}E(\rho)^T \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & 2\delta R_2 \end{bmatrix} \begin{bmatrix} h_{max}A(\rho) & h_{max}A_h(\rho) & h_{max}E(\rho) \\ h_{max}A(\rho) & h_{max}A_h(\rho) & h_{max}E(\rho) \end{bmatrix} \right) X(t_i)
\end{aligned} \tag{3.144}$$

where $X(t) = \operatorname{col}(x(t), x(t-h(t)), w(t))$ *and* $A_h(\rho) = A_h^1(\rho) + A_h^2(\rho)$.

And finally, a Schur complement yields LMI

$$X(t_i)^T \begin{bmatrix} \Psi_{11}(\rho,\dot\rho) & P(\rho)A_h(\rho)+R_1 & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 \\ \star & -(1-\dot h(t_i))(Q_1+Q_2)-R_1 & 0 & h_{max}A_h(\rho)^T R_1 & A_h(\rho)^T R_2 \\ \star & \star & 0 & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 \\ \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & -(2\delta)^{-1}R_2 \end{bmatrix} X(t_i) \prec 0 \tag{3.145}$$

Adding the input/output constraint

$$-\gamma w(t_i)^T w(t_i) + \gamma^{-1} z(t_i)^T z(t_i) = -\gamma w(t_i)^T w(t_i) + \gamma^{-1} X(t_i)^T \begin{bmatrix} C(\rho)^T \\ C_h(\rho)^T \\ F(\rho)^T \end{bmatrix} \begin{bmatrix} C(\rho)^T \\ C_h(\rho)^T \\ F(\rho)^T \end{bmatrix}^T X(t_i) \tag{3.146}$$

with $C_h(\rho) = C_h^1(\rho) + C_h^2(\rho)$.

A Schur complement leads to the final LMI for the case 1:

$$\begin{bmatrix} \Psi_{11}(\rho,\dot\rho) & P(\rho)A_h(\rho)+R_1 & P(\rho)E(\rho) & h_{max}A(\rho)^T R_1 & A(\rho)^T R_2 & C(\rho)^T \\ \star & -(1-\dot h(t_i))(Q_1+Q_2)-R_1 & 0 & h_{max}A_h(\rho)^T R_1 & A_h(\rho)^T R_2 & C_h(\rho)^T \\ \star & \star & -\gamma I & h_{max}E(\rho)^T R_1 & E(\rho)^T R_2 & F(\rho)^T \\ \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & -(2\delta)^{-1}R_2 & 0 \\ \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{3.147}$$

Case. 2: *When* $t \geq 0$ *and* $t \neq t_i$ *then we have*

$$\begin{aligned} \dot V \leq\; & Y(t)^T \left(\begin{bmatrix} \Psi_{11}(\rho,\dot\rho) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) \\ \star & -(1-\dot h(t))Q_1 - R_1 & 0 & 0 \\ \star & \star & -(1-\dot h_c(t))Q_2 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} A(\rho)^T & A(\rho)^T \\ A_h^1(\rho)^T & A_h^1(\rho)^T \\ A_h^2(\rho)^T & A_h^2(\rho)^T \\ E(\rho)^T & E(\rho)^T \end{bmatrix} \begin{bmatrix} h_{max}R_1 & 0 \\ 0 & 2\delta R_2 \end{bmatrix} \begin{bmatrix} A(\rho) & A_h^1(\rho) & A_h^2(\rho) & E(\rho) \\ A(\rho) & A_h(\rho) & A_h^2(\rho) & E(\rho) \end{bmatrix} \right) Y(t) \\ & - \frac{(1-\dot h(t))}{\delta} Y(t)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ \star & R_2 & -R_2 & 0 \\ \star & \star & R_2 & 0 \\ \star & \star & \star & 0 \end{bmatrix} Y(t) \end{aligned} \tag{3.148}$$

and this leads to

$$\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) & h_{max}A(\rho)^TR_1 & A(\rho)^TR_2 \\ \star & \Psi_{22} & (1-\dot{h}(t))R_2/\delta & 0 & h_{max}A_h^1(\rho)^TR_1 & A_h^1(\rho)^TR_2 \\ \star & \star & \Psi_{33} & 0 & h_{max}A_h^2(\rho)^TR_1 & A_h^2(\rho)^TR_2 \\ \star & \star & \star & 0 & h_{max}E(\rho)^TR_1 & E(\rho)^TR_2 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & -(2\delta)^{-1}R_2 \end{bmatrix} \prec 0 \tag{3.149}$$

where $\Psi_{22} = -(1-\dot{h}(t))Q_1 - (1-\dot{h})R_2/\delta - R_1$ *and* $\Psi_{33} = -(1-\dot{h}_c(t))Q_2 - (1-\dot{h}(t))R_2/\delta$.

Finally, adding the same input/ouput constraint as for the case 1, yields

$$\begin{bmatrix} \Psi_{11}(\rho,\dot{\rho}) & P(\rho)A_h^1(\rho)+R_1 & P(\rho)A_h^2(\rho) & P(\rho)E(\rho) & h_{max}A(\rho)^TR_1 & A(\rho)^TR_2 & C(\rho)^T \\ \star & \Psi_{22} & (1-\dot{h}(t))R_2/\delta & 0 & h_{max}A_h^1(\rho)^TR_1 & A_h^1(\rho)^TR_2 & C_h^1(\rho)^T \\ \star & \star & \Psi_{33} & 0 & h_{max}A_h^2(\rho)^TR_1 & A_h^2(\rho)^TR_2 & C_h^2(\rho)^T \\ \star & \star & \star & -\gamma I & h_{max}E(\rho)^TR_1 & E(\rho)^TR_2 & F(\rho)^T \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -R_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -(2\delta)^{-1}R_2 & 0 \\ \star & \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{3.150}$$

Bounding $-(1-\dot{h}(t)) \leq -(1-\mu)$ *and* $-(1-\dot{h}(t)) \leq -(1-\mu_c)$ *leads to the proposed result. Finally considering that* $\dot{\rho}$ *belongs to the polytope* $\mathrm{hull}(U_\nu)$, *the dependence on* $\dot{\rho}$ *is relaxed.* □

This lemma involves two LMIs where coupling terms are present: $PA, RA, \ldots$. Hence similarly as for latter results, it is convenient to introduce a relaxation of such a result. This is given in the following section.

3.7.2 Associated Relaxation

Due to a high number of coupling terms, the relaxation of such a LMI is of interest while considering it for a design purpose. Indeed, since multiple coupling generally makes the linearization a difficult (impossible) task, the relaxation allows for a nice solution of the problem, even if the relaxation result and the original solution are not equivalent.

The following result is obtained by using a similar method as for previous results, the original Theorem 3.7.1 is relaxed using the projection lemma:

Theorem 3.7.2 *System (3.137) is delay-dependent stable with* $h(t) \in [0, h_{max}]$, $h_c(t) = h_c(t) + \theta(t)$, $\theta(t) \in [-\delta, \delta]$, $|\dot{h}(t)| < \mu$ *and* $|\dot{h}_c(t)| < \mu_c$ *such that* $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ *if there exists a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}^n_{++}$ *and symmetric matrices*

$Q_1, Q_2, R_1, R_2 \succ 0$, a matrix $X : U_\rho \to \mathbb{R}^{n\times n}$ and a scalar $\gamma > 0$ such that

$$\begin{bmatrix} -X(\rho)^H & P(\rho)+X(\rho)^T A(\rho) & X(\rho)^T A_h(\rho) & X(\rho)^T E(\rho) & 0 & X(\rho)^T & h_{max}R_1 & R_2 \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & C(\rho)^T & 0 & 0 & 0 \\ \star & \star & \Theta_{22} & 0 & C_h(\rho)^T & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I & F(\rho)^T & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \prec 0 \tag{3.151}$$

and

$$\begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) \\ \star & \Pi_{22}(\rho) \end{bmatrix} \prec 0 \tag{3.152}$$

hold for all $\rho \in U_\rho$ and where

$$\begin{aligned}
\Pi_{11}(\rho,\nu) &= \begin{bmatrix} -X(\rho)^H & P(\rho)+X(\rho)^T A(\rho) & X(\rho)^T A_h^1(\rho) & X(\rho)^T A_h^2(\rho) & X(\rho)^T E(\rho) \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & 0 \\ \star & \star & \Psi_{22} & (1-\mu)R_2/\delta & 0 \\ \star & \star & \star & \Psi_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} \\
\Pi_{12}(\rho) &= \begin{bmatrix} 0 & X(\rho)^T & h_{max}R_1 & R_2 \\ C(\rho)^T & 0 & 0 & 0 \\ C_h^1(\rho)^T & 0 & 0 & 0 \\ C_h^2(\rho)^T & 0 & 0 & 0 \\ F(\rho)^T & 0 & 0 & 0 \end{bmatrix} \\
\Pi_{22}(\rho) &= \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & -R_1 & 0 \\ \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \\
\Theta_{11}(\rho,\nu) &= -P(\rho) + Q_1 + Q_2 + \sum_{i=1}^{N} \frac{\partial P}{\partial \rho_i}\nu_i - R_1 \\
\Theta_{22} &= -(1-\mu)(Q_1+Q_2) - R_1 \\
\Psi_{22} &= -(1-\mu)(Q_1 + R_2/\delta) - R_1 \\
\Psi_{33} &= -(1-\mu_c)Q_2 - (1-\mu)R_2/\delta \\
A_h &= A_h^1 + A_h^2 \\
C_h &= C_h^1 + C_h^2
\end{aligned}$$

Proof: *The proof is similar as for other relaxations.* □

3.8 Chapter Conclusion

This chapter has developed preliminary results which will be used in the remaining of the thesis. First of all, fundamental definitions for the set of the delay and the parameters have been detailed and explained.

Secondly, a new method to relax polynomially parameter dependent LMIs have been developed. Indeed, it has the benefit of turning the initial LMI condition into a new LMI condition whose dependence is linear only with respect to the parameters, at the expense of an increase of the computational complexity through the addition of a 'slack' variable.

Then a novel relaxation for concave nonlinearity has been presented which finds its interest where the cone complementary algorithm cannot apply, i.e. when the involved matrices are not constant. This method will be applied successfully in Section 5.1.3.

The following section has been devoted to the computation of the bounds on parameter derivatives in the polytopic case and allows to deal easily with robust stability and synthesis in the LPV polytopic approach.

A simple Lyapunov-Krasovskii has been presented with its associated relaxation. This functional has proven its efficiency despite of its simplicity and this has motivated its use in this thesis. The associated relaxation finds its interest in the design problems which greatly simplifies the problem.

In order to improve latter results based on a simple functional, the following section has been devoted to a discretized version of this functional where the decision matrices are functions. Using this 'complete' version it is possible to refine the results until reach theoretical delay margin. Its associated relaxation allows to transfer the quality of the results from the stability analysis to design purposes

Finally, a new Lyapunov-Krasovskii functional has been provided in order to analyze the stability of a system with two delays which are coupled through an algebraic inequality. Such case occurs when a time-delay systems is observed or controlled by an observer or a controller with memory but implementing a delay different from the system one. This will be used in Sections and 4.1.2 and 5.1.6.

Chapter 4

Observation and Filtering of LPV time-delay systems

One of the objectives of systems theory is to provide tools on observation and filtering of systems. The objectives of observation and filtering is to estimate unmeasured signals or clean signals from eventual noises and/or disturbances provided that a model of the system is available. However, conceptual differences remain between the notion of filters and observers and will be emphasized in the introduction of this chapter.

An observer aims at estimating signals of a system by finding observer matrices such that the state estimation error is asymptotically stable (exponentially stable is the linear case). This means that, for every initial conditions of the observer, the observation error will converge and remains to zero provided that no disturbances are active, in other words, the autonomous linear differential equation governing the observation error is asymptotically (exponentially) stable. Moreover, it is important to note that a good observer should be able to observe whatever the value of the state of the system is and hence the observation error should be independent of the system state: this can be handled in the certain case only since in the uncertain case the observation error depends on the current state. However, it is possible to construct nonlinear observers which makes the error converges to zero even if the state is nonzero [Boutayeb and Darouach, 2003, Gu and Poon, 2001]. As a final remark, the use of observers is better suited for controlling purposes since the observer estimates sufficiently well the system state, allowing the use of a state-feedback.

On the other hand, the filtering of system does not require any stability of a 'filtering error' but aims at guaranteeing a minimal attenuation, in some norm sense, of a residual computed from the difference of a desired estimated signal and the estimate under action of disturbances. In this case, the quality of the estimation would depend on the current state of the system.

At first sight, it seems that filtering is less relevant than observation but actually each way has its own benefits and drawbacks. While many observation approaches work well for LTI and certain systems, when dealing with LPV systems, the problem is far more difficult. Moreover, the class of systems that can be treated by observation theories is not as wide as for filtering. Filtering approaches can address a large variety of systems and the resulting problem is generally more simple to handle and for this reason, only filtering of LPV systems is generally provided in the literature [Mohammadpour and Grigoriadis, 2006a,b, 2007a,b, 2008].

When dealing with time-delay systems, the diversity of observers and filters is a bit larger than for finite-dimensional linear dynamical systems. Indeed, it is possible to consider a complementary information on the delay when it is available. This gives rise to the notion of filters/observers with memory and memoryless observers/filters. It may seems uninteresting to design memoryless observers and filters but actually, for two reasons, it is important to consider them. First of all, implementing a delay in the observer/filter needs memory space which can be incompatible with embedded applications; secondly, the real-time estimation of the value of the current delay of a physical system remains a challenging open problem [Belkoura et al., 2007, 2008, Drakunov et al., 2006].

In this chapter, we will be interested in both observation and filtering of LPV time-delay systems. Observers that will be designed for the LPV case are based on an algebraic approach, initially developed for LTI time-delay systems [Darouach, 2001, 2005]. Reduced-order as well as full-order observers will be designed for LPV time-delay systems. Necessary and sufficient conditions for their existence will be provided in terms of algebraic matrix equalities and the stability of a functional differential equation. The computation of observer matrices will be performed through by computing the solution of LMIs. An example of filter design for uncertain LPV systems will also be introduced and is a generalization of the method presented in [Tuan et al., 2001, 2003] to time-delay systems and it will be shown that interesting performances are achieved.

It is important to point out that, in both filtering and observation case, only memory processes with exact delay value is addressed and generally no robustness analysis is given in presence of uncertainty on implemented delay. In [Sename and Briat, 2006, Verriest et al., 2002], a robustness analysis is performed a posteriori in the case of constant delay according to an application of the Rouché's theorem (see Section 2.2.2 and appendix F.7). In this section, robust filtering/observation with respect to delay uncertainty and parametric uncertainties will be addressed and therefore the designed processes will remain stable even in presence of (time-varying) delay-uncertainty provided that the delay implementation error remains in a ball whose radius is a priori fixed or maximized by an optimization algorithm based on LMIs. This problem has never been addressed in the literature and is one of the main points on this section. In [Briat et al., 2007c], an Luenberger observer has been developed for LPV time-delay systems using a free weighting approach [He et al., 2004]; the results are not presented here since we will focus on more interesting observer synthesis techniques.

Hereunder a non exhaustive bibliography on observation and filtering of time-delay systems and LPV systems is provided:

- For pioneering works on observation of delay systems see [Bhat and Koivo, 1976, Fattouh, 2000, Fattouh et al., 1998, Gressang and Lamont, 1975, H.-Hashemi and Leondes, 1979, Lee and Olbrot, 1981, Ogunnaike, 1981, Pearson and Fiagbedzi, 1989, Sename, 2001]

- Concerning observers for nonlinear delay systems see [Germani et al., 1998, 1999, 2001, 2002, Pepe, 2001]

- Recent works on observation of linear time-delay systems [Chen, 2007, Koenig and Marx, 2004, Koenig et al., 2004, 2006, Picard and Lafay, 1996, Picard et al., 1996, Sename, 1997, Sename and Briat, 2006, Sename et al., 2001, Verriest et al., 2002]

- Recent works on the filtering of time-delay systems [DeSouza et al., 1999, Fridman et al., 2003a,b, Zhang and Han, 2008]
- Filtering of LPV systems [Borges and Peres, 2006, Tuan et al., 2001, 2003]
- Filters for LPV time-delay systems [Mohammadpour and Grigoriadis, 2006a,b, 2007a, 2008, Wu et al., 2006]

4.1 Observation of Unperturbed LPV Time-Delay Systems

This section is devoted to the design of observers and filters for LPV time-delay systems without uncertainties. Several approaches will be provided depending on the type of filter/observer (with or without memory) and the knowledge of the delay (exactly or approximately known and unknown).

The observers designed in this section are based on the extension to the LPV case of the method of Darouach [2001, 2005]. Throughout this section on observers the following class of LPV time-delay system will be considered:

$$\begin{array}{rcl}\dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t)\\ z(t) & = & Tx(t)\\ y(t) & = & Cx(t)\end{array}\tag{4.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and $z \in \mathbb{R}^r$ are respectively the system state, the system control input, the system measurements, the system exogenous inputs and the signal to be estimated. In this framework, the observer aims at estimating as close as possible the signal $z(t)$ which is a linear combination of the state variables of the system. The matrix T is assumed to have full row-rank and whenever $\text{rank}(T) = n$ then $T = I$. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1$ and more specifically $\mathscr{H}_1^\circ$ recalled below:

$$\mathscr{H}_1^\circ := \left\{h \in \mathcal{C}^1(\mathbb{R}_+, [0, h_{max}]) : \ |\dot{h}| < \mu\right\}\tag{4.2}$$

The corresponding general observer is governed by the following expressions:

$$\begin{array}{rcl}\dot{\xi}(t) & = & M_0(\rho)\xi(t) + M_h(\rho)\xi(t-d(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-d(t)) + S(\rho)u(t)\\ \hat{z}(t) & = & \xi(t) + Hy(t)\end{array}\tag{4.3}$$

where $\xi \in \mathbb{R}^r$, $\hat{z} \in \mathbb{R}^r$ are respectively the observer state and the estimated output. The delay $d(t)$ is unconstrained at this point and precisions on its set will be provided in each forthcoming sections since it depends on the context and on the type of observer considered. The matrices $M_0(\rho)$, $M_h(\rho)$, $N_0(\rho)$, $M_H(\rho)$ and H are matrices of appropriate dimensions which define the observer. Note that H is a constant matrix as $\hat{z}$ is a linear combination of the observer state and the measurement vector y.

It is worth mentioning that when dealing with such observer it is difficult to consider a disturbance term on the measured output since during the design procedure, the measured output needs to be differentiated. If it would depend on the disturbance w, then a term $\dot{w}$ would appear in the equations and then the disturbance vector should be augmented in order to contain both w and $\dot{w}$ (e.g. $\tilde{w} = \text{col}(w, \dot{w})$). This is a straightforward generalization of the current method and is then not explored in the thesis.

Definition 4.1.1 *If $r = n$ then the observer is called full-order observer while if $T = T_r \in \mathbb{R}^{r\times n}$ such as* $\text{rank}(T_r) = r < n$ *then the observer is called reduced-order observer.*

The aim of the observer is to decouple the system state from the error $e(t) = z(t) - \hat{z}(t)$ as in [Darouach, 2001], that is we should have an equation of the form

$$\dot{e}(t) = f(e_t) + g(\eta(t)) \tag{4.4}$$

where $f(\cdot)$ is a functional and $g(\cdot)$ is a function gathering other signals (such as disturbances) excluding the state of the system. In this case, it is clear that if $f(\cdot)$ describes a stable vector field then the observation error has stable dynamics. Moreover, for every trajectories of the system, the error will have the same behavior in terms of rate of convergence, response time,... We will see that this ideal behavior can be only be achieved when the delay implemented in the observer is identical to the delay involved in the system and when the system is perfectly known (no uncertainties). Therefore such a behavior cannot be reached from a practical point of view.

It will be shown that when the observation error cannot be isolated from the state of the system and from this fact, only stable LPV time-delay system can be observed. Indeed, suppose that the error obeys the following dynamical model

$$\dot{e}(t) = f(e_t) + g(x_t) + h(\eta(t)) \tag{4.5}$$

where $f(\cdot)$, $g(\cdot)$ are functionals and $h(\cdot)$ is a function gathering other signals. From this expression even if $f(\cdot)$ is a stable vector field, then the error will remains bounded around 0 if and only if the other terms are bounded too (BIBO stability). However, if the system is unstable then the term may be such that $g(x_t) \to +\infty$ as $t \to +\infty$ and hence $e(t) \to +\infty$ as $t \to +\infty$. Such a behavior arises when dealing with observer involving a delay which is different from the system one or using memoryless observers. An immediate choice would be to consider the term $g(x_t)$ as a disturbance term and in many frameworks it would be correct, for instance in a pure stabilization or α-stabilization problems where it is assumed that the system is stable or controlled (i.e. $x(t)$ does not tends to $+\infty$ as t goes to ∞).

In the $\mathcal{H}_\infty$ problem where an observer minimizing the influence of the disturbances onto the observation error (in the $\mathcal{L}_2$ sense) is sought, we are faced to two possibilities:

1. either the vector of disturbances is augmented to contain both the initial disturbances vector $\eta(t)$ and the term $g(x_t)$ but in this case a loss of information occurs since the relation between the disturbances $\eta(t)$ and $x(t)$ is not taken into account; or

2. the system is augmented in order to contain both the state of the system and the observation error and in this case, the $\mathcal{H}_\infty$ analysis/synthesis is more tight.

This is the last approach that we will considered throughout this section on observers.

4.1.1 Observer with exact delay value - simple Lyapunov-Krasovskii functional case

In this section, the problem of observation of a LPV time-delay system with an observer involving a delay identical to the system one is solved; see [Darouach, 2001] for the LTI case. Even if this observer may be not implementable, the design approach is interesting and can be extended to more complicated cases. The observer to be designed is then given by the equations:

$$\begin{array}{rcl}\dot{\xi}(t) & = & M_0(\rho)\xi(t) + M_h(\rho)\xi(t-h(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-h(t)) + S(\rho)u(t) \\ \hat{z}(t) & = & \xi(t) + Hy(t)\end{array} \tag{4.6}$$

where $\xi \in \mathbb{R}^r$, $\hat{z} \in \mathbb{R}^r$ are respectively the observer state and the estimated output.

First of all, the delay value is assumed to be exactly known in real time. The following theorem provides a necessary and sufficient condition to the existence of such an observer.

Theorem 4.1.2 *There exists an LPV $\mathcal{H}_\infty$ observer with memory of the form (4.6) for system of the form (4.1) if and only if the following statements hold:*

1. *The autonomous error dynamical expression $\dot{e}(t) = M_0(\rho)e(t) + M_h(\rho)e(t-h(t))$ is asymptotically stable where $e(t) = z(t) - \hat{z}(t)$ and $h \in \mathscr{H}_1^\circ$.*
2. $(T - HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T - HC) = 0$
3. $(T - HC)A_h(\rho) - N_h(\rho)C - M_h(\rho)(T - HC) = 0$
4. $(T - HC)B(\rho) - S(\rho) = 0$
5. *The inequality $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$ holds for some $\gamma > 0$*

Proof: *First let $e(t) = z(t) - \hat{z}(t)$ be the estimation error. The latter equality reduces to*

$$e(t) = (T - HC)x(t) - \xi(t) \tag{4.7}$$

according to the definition of $\hat{z}(t)$ in (4.6). Computing the time derivative of $e(t)$ we get

$$\begin{array}{rcl}\dot{e}(t) & = & (T-HC)\dot{x}(t) - \dot{\xi}(t) \\ & = & (T-HC)\left[A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t)\right] \\ & & -\left[M_0(\rho)\xi(t) + M_h(\rho)\xi(t-h(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-h(t))\right. \\ & & \left.+S(\rho)u(t)\right] \\ & = & [(T-HC)A(\rho) - N_0(\rho)C]x(t) \\ & & +[(T-HC)A_h(\rho) - N_h(\rho)C]x(t-h(t)) \\ & & +[(T-HC)B(\rho) - S(\rho)]u(t) - M_0(\rho)\xi(t) - M_h(\rho)\xi(t-h(t)) \\ & & +(T-HC)E(\rho)w(t)\end{array} \tag{4.8}$$

Using $\xi(t) = (T - HC)x(t) - e(t)$ obtained from (4.7) we get

$$\begin{array}{rcl}\dot{e}(t) & = & [(T-HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T-HC)]x(t) \\ & & +[(T-HC)A_h(\rho) - N_h(\rho)C - M_h(\rho)(T-HC)]x(t-h(t)) \\ & & +[(T-HC)B(\rho) - S(\rho)]u(t) + M_0(\rho)e(t) + M_h(\rho)e(t-h(t)) \\ & & +(T-HC)E(\rho)w(t)\end{array} \tag{4.9}$$

According to the discussion at the beginning of Section 4.1, we aim to obtain an error e whose dynamical model is independent of the the control input, the current and delayed state of the system. Hence by imposing

$$(T-HC)A(\rho)x(t) - N_0(\rho)C - M_0(\rho)(T-HC) = 0 \tag{4.10}$$

$$(T-HC)A_h(\rho)x(t-h(t)) - N_h(\rho)C - M_h(\rho)(T-HC) = 0 \tag{4.11}$$

$$(T-HC)B(\rho) - S(\rho) = 0 \tag{4.12}$$

the error dynamical model reduces to

$$\dot{e}(t) = M_0(\rho)e(t) + M_h(\rho)e(t - h(t)) + (T - HC)E(\rho)w(t) \tag{4.13}$$

and is actually independent of the system state and control input.

Finally, if the latter dynamical model defines stable dynamics then it is possible to find a $\gamma > 0$ *such that* $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$. *This concludes the proof.* $\square$

A theorem providing necessary and sufficient conditions for the existence of an observer of the form (4.7) for systems (4.1) has been developed. It is worth noting that such a design can be extended to $\mathcal{H}_2$, $\mathcal{L}_\infty$-$\mathcal{L}_\infty$ problems and so on. However, such a result is not constructive and then Theorem 4.1.2 cannot be used for synthesis purposes. It can be divided in two parts

1. the first one involves nonlinear algebraic equations (statements 2 to 4) which are 'static'
2. the second part involves dynamic related conditions related to the stability of a system and its worst-case energy gain

The first step of the solution is to explicitly define the set of all matrices satisfying statements 2 to 5. This is performed in the following lemma where it considered that the matrix $H(\rho)$ depends on the parameters while it should be constant. This condition will be relaxed when the LMI conditions for gain computation will be provided.

Lemma 4.1.3 *There exists a solution* $M_0(\rho), M_h(\rho), N_0(\rho), N_h(\rho), S(\rho), H(\rho)$ *to equations (4.10), (4.11) and (4.12) if and only if the following rank equality holds*

$$\operatorname{rank}\begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \\ CA(\rho) & CA_h(\rho) \\ TA(\rho) & TA_h(\rho) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} T & 0 \\ 0 & T \\ C & 0 \\ 0 & C \\ CA(\rho) & CA_h(\rho) \end{bmatrix} \tag{4.14}$$

for all $\rho \in U_\rho$.

Proof: *Equation (4.12) is explicit since it suffices to find* H *then* S *is obtained by the explicit expression*

$$S(\rho) = (T - HC)B(\rho) \tag{4.15}$$

On the other hand, the two equalities (4.10) and (4.11) are nonlinear due to terms

$$M_0(\rho)(T - HC) \qquad M_h(\rho)(T - HC)$$

However rewriting them into the form

$$(T - H(\rho)C)A(\rho)x(t) + (M_0(\rho)H(\rho) - N_0(\rho))C - M_0(\rho)T = 0 \tag{4.16}$$

$$(T - H(\rho)C)A_h(\rho)x(t) + (M_h(\rho)H(\rho) - N_h(\rho))C - M_h(\rho)T = 0 \tag{4.17}$$

shows that the change of variable

$$\begin{array}{rcl} K_0(\rho) & = & N_0(\rho) - M_0(\rho)H(\rho) \\ K_h(\rho) & = & N_h(\rho) - M_h(\rho)H(\rho) \end{array} \tag{4.18}$$

linearizes the expressions into

$$\begin{array}{rcl}(T-H(\rho)C)A(\rho)x(t)-K_0(\rho)C-M_0(\rho)T & = & 0\\ (T-H(\rho)C)A_h(\rho)x(t)-K_h(\rho)C-M_h(\rho)T & = & 0\end{array} \tag{4.19}$$

It is important to note that the change of variable is bijective and hence no conservatism is introduced. Indeed, the set of matrices $(M_0(\rho), M_h(\rho), K_0(\rho), K_h(\rho), H(\rho))$ *defines in a unique way the set* $(M_0(\rho), M_h(\rho), N_0(\rho), N_h(\rho), H(\rho))$ *due to the change of variable (4.18).*

Rewriting equalities (4.19) in a more compact matrix expression leads to

$$\nabla(\rho)\Gamma(\rho) = \Lambda(\rho) \tag{4.20}$$

where

$$\begin{array}{rcl}\nabla(\rho) & = & \left[\begin{array}{ccccc} M_0(\rho) & M_h(\rho) & K_0(\rho) & K_h(\rho) & H(\rho)\end{array}\right] : U_\rho \to \mathbb{R}^{r\times 2r+3m}\\ \Gamma(\rho) & = & \left[\begin{array}{cc} T & 0\\ 0 & T\\ C & 0\\ 0 & C\\ CA(\rho) & CA_h(\rho)\end{array}\right]\\ \Lambda(\rho) & = & \left[\begin{array}{cc} TA(\rho) & TA_h(\rho)\end{array}\right]\end{array} \tag{4.21}$$

According to [Darouach, 2001, Koenig et al., 2006, Lancaster and Tismenetsky, 1985, Mitra and Mitra, 1971], there exist solutions with $H(\rho)$ *to this expression if and only if* $\operatorname{rank}\left[\begin{array}{c}\Gamma(\rho)\\ \Lambda(\rho)\end{array}\right] = \operatorname{rank}\Gamma(\rho)$ *which is exactly (4.14). This concludes the proof.* □

Whenever lemma 4.1.3 is satisfied then it is confirmed that there exists at least one solution to equations (4.10), (4.11) and (4.12). The number of solution is either 1 or is infinite. We are interested in the case of an infinite number of solutions since it is not guaranteed that the unique solution engenders a stable error dynamical model.

A sufficient condition for an infinite number of solutions is that the number of unknown variables (the number of coefficients in the unknown matrices) exceeds the number of equations (the number of coefficients in matrices of dimension equals to the order of the observer). Hence, it suffices that the following condition

$$2\dim(z)\dim(x) \le \dim(z)^2 + 3\dim(z)\dim(y) \tag{4.22}$$

holds. From this inequality it is possible to give more relevant conditions for the existence of an infinite number of solutions, indeed we must have

$$\begin{array}{rcl}\dim(y) & \ge & \frac{2}{3}(\dim(x)-\dim(z))\\ \dim(z) & \ge & \dim(x) - \frac{3}{2}\dim(y)\end{array} \tag{4.23}$$

The first inequality indicates the minimal number of sensors that must be employed for a given system dimension and observer order such that such an observer may exist. The second inequality provides the minimal observer order that can be used for some given system dimension and output dimension. It is worth noting that the problem may be unsolvable since no consideration on the stability of the error is taken into account.

When the number of solution is infinite, the objective (and interest is to parametrize the set of solution. The following lemma provides such a parametrization provided that lemma 4.1.3 is satisfied.

Lemma 4.1.4 *Under conditions of theorem 4.1.3, the observer matrices are given by the expressions* $M_0 = \Theta - L\Xi$, $M_h = \Upsilon - L\Omega$ *and* $H = \Phi - L\Psi$ *where* L *is a free matrix of appropriate dimensions and*

$$\Theta = TAU - \Lambda\Gamma^+\Delta_0 \begin{bmatrix} C \\ CA \end{bmatrix} U \tag{4.24}$$

$$\Xi = -(I - \Gamma\Gamma^+)\Delta_0 \begin{bmatrix} C \\ CA \end{bmatrix} U \tag{4.25}$$

$$\Upsilon = TA_hU - \Lambda\Gamma^+\Delta_h \begin{bmatrix} C \\ CA_h \end{bmatrix} U \tag{4.26}$$

$$\Omega = -(I - \Gamma\Gamma^+)\Delta_h \begin{bmatrix} C \\ CA_h \end{bmatrix} U \tag{4.27}$$

$$\Phi = \Lambda\Gamma^+\Delta_H \tag{4.28}$$

$$\Psi = (I - \Gamma\Gamma^+)\Delta_H \tag{4.29}$$

$$S = FB \tag{4.30}$$

$$N_0 = K_0 + M_0H \tag{4.31}$$

$$N_h = K_h + M_hH \tag{4.32}$$

$$F = T - HC \tag{4.33}$$

$$U \text{ is defined s.t. } \begin{bmatrix} T \\ \bar{T} \end{bmatrix}^{-1} = \begin{bmatrix} U & V \end{bmatrix} \tag{4.34}$$

$$\tag{4.35}$$

where $\bar{T}$ *is a full column rank matrix such that* $\begin{bmatrix} T \\ \bar{T} \end{bmatrix}$ *is nonsingular and*

$$\Delta_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_m & 0 \\ 0 & 0 \\ 0 & I_m \end{bmatrix} \Delta_h = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_m & 0 \\ 0 & I_m \end{bmatrix} \Delta_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_m \end{bmatrix}$$

Proof: *Provided that lemma 4.1.3 is verified, all the solutions of equation* $\nabla(\rho)\Gamma(\rho) = \Lambda(\rho)$ *are given by the expression (see Appendix A.8 or [Darouach, 2001, Skelton et al., 1997]):*

$$\nabla_s(\rho) = \Lambda(\rho)\Gamma^+(\rho) - L(\rho)(I - \Gamma(\rho)\Gamma^+(\rho)) \tag{4.36}$$

where $L(\rho)$ *is a free variable giving the parametrization of the set of solutions and is considered as a generalized observer gain.*

It is of interest to express these relations as functions of the generalized gain $L(\rho)$ *which leads to*

$$\begin{aligned} \begin{bmatrix} K_0(\rho) & H(\rho) \end{bmatrix} &= \nabla_s(\rho)\Delta_0 \\ \begin{bmatrix} K_h(\rho) & H(\rho) \end{bmatrix} &= \nabla_s(\rho)\Delta_h \\ H(\rho) &= \nabla_s(\rho)\Delta_H \end{aligned} \tag{4.37}$$

where Δ_0, Δ_h and Δ_H are given by the expressions

$$\Delta_0 \quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \qquad \Delta_h = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \qquad \Delta_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} \tag{4.38}$$

Hence (4.10) and (4.11) are rewritten into the form:

$$M_0(\rho)T \quad = \quad TA(\rho) - \begin{bmatrix} K_0(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} \tag{4.39}$$

$$M_h(\rho)T \quad = \quad TA_h(\rho) - \begin{bmatrix} K_h(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA_h(\rho) \end{bmatrix} \tag{4.40}$$

Since T is a full row rank matrix then there exists a full row rank matrix $\bar{T}$ such that

$$\det \begin{bmatrix} T \\ \bar{T} \end{bmatrix} \neq 0 \tag{4.41}$$

Hence this matrix in invertible and we denote its inverse by $\begin{bmatrix} U & V \end{bmatrix}$. Then by right multiplying expressions (4.39) and (4.40) by $\begin{bmatrix} U & V \end{bmatrix}$ we get

$$\begin{aligned} M_0(\rho) \quad &= \quad TA(\rho)U - \begin{bmatrix} K_0(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U \\ M_h(\rho) \quad &= \quad TA_h(\rho)U - \begin{bmatrix} K_h(\rho) & H(\rho) \end{bmatrix} \begin{bmatrix} C \\ CA_h(\rho) \end{bmatrix} U \end{aligned}$$

which are explicit formulae for observer matrices $M_0(\rho)$ and $M_h(\rho)$. then $M_0(\rho) = \Theta(\rho) - L(\rho)\Xi(\rho)$, $M_h(\rho) = \Upsilon(\rho) - L(\rho)\Omega(\rho)$ and $H(\rho) = \Phi(\rho) - L(\rho)\Psi(\rho)$ with matrices defined in theorem 4.1.4. This concludes the proof. $\square$

The problem of finding five distinct matrices ($M_0(\cdot)$, $M_h(\cdot)$, $N_0(\cdot)$, $N_h(\cdot)$, $H(\rho)$) under the algebraic equality constraints (4.10), (4.11) and (4.12) has been turned into a problem of finding a free 'generalized' gain $L(\rho)$ which parametrizes the set of all solutions to equations (4.10), (4.11) and (4.12).

This transformation is the keypoint of this algebraic approach and makes the final problem to be the 'good' choice of such a generalized gain. It is clear that some elements in the set of all observer matrices would give unstable error dynamics. Hence a 'good' choice is synonym to a choice giving good convergence properties, good disturbances rejection. We have chosen in this thesis to consider the $\mathcal{L}_2$-induced norm of the transfer from the disturbances $w(t)$ to the observation error $e(t)$ as a criterium to minimize for the choice of $L(\rho)$ (i.e. we aim at finding $L(\rho)$ such that $||e||_{\mathcal{L}_2} \leq \gamma ||e||_{\mathcal{L}_2}$ where $\gamma > 0$ is as small as possible). It is clear that other performances criteria could be used such as $\mathcal{H}_2$ or $\mathcal{L}_\infty$ induced norm. Such a search is difficult to perform analytically and it will be shown a bit later that such an optimization problem can be cast as a SDP.

Now noting that by considering lemma 4.1.4 the state estimation error dynamics are governed by the expression

$$\dot{e}(t) = (\Theta(\rho) - L(\rho)\Xi(\rho))e(t) + (\Upsilon(\rho) - L(\rho)\Omega(\rho))e_h(t) + FE(\rho)w(t) \tag{4.42}$$

with $F = T - HC = T - (\Phi - L\Psi)C$.

It is important to point out that if $FE = 0$ then the observer totally decouples the state estimation error e from the exogenous inputs w and thus the state estimation error is autonomous. Observers having this property are called *unknown input observers* and some additional material can be found in [Koenig and Marx, 2004, Koenig et al., 2004, Sename, 1997, Sename et al., 2001] and references therein.

In the following we will consider that $FE \neq 0$ and the objective is to minimize the impact of the disturbances $w(t)$ onto the error $e(t)$ (in the $\mathcal{L}_2$ sense) by an appropriate choice of the matrix $L(\rho)$. Note that if there exists $L(\rho)$ such that $||FE|| = 0$ or is close to 0, the algorithms would find out it.

Finally, according to the latter results on the family of observers with infinite cardinal, the following theorem provides a constructive sufficient condition on the existence of an optimal observer minimizing the $\mathcal{L}_2$-induced norm of the transfer from $w(t)$ to $e(t)$:

Theorem 4.1.5 *There exists a parameter dependent observer of the form (4.6) for LPV time-delay system (4.1) such that theorem 4.1.2 is satisfied for all $h \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^r_{++}$, a matrix function $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, constant matrices $Q, R \in \mathbb{S}^r_{++}$, $X \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and a positive scalar $\gamma > 0$ such that the following matrix inequality*

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ U_{21}(\rho) & U_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ U_{31}(\rho) & R & -Q_\mu - R & \star & \star & \star & \star \\ U_{41} & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0 \tag{4.43}$$

holds for all $(\rho,\nu) \in U\rho \times U_\nu$ with

$$\begin{array}{rcl} U_{21}(\rho) & = & \Theta(\rho)^T X - \Xi(\rho)^T \bar{L}(\rho)^T + P(\rho) \\ U_{31}(\rho) & = & \Upsilon(\rho)^T X - \Omega(\rho)^T \bar{L}(\rho)^T \\ U_{22}(\rho,\nu) & = & \dfrac{\partial P(\rho)}{\partial \rho} - P(\rho) + Q - R \\ U_{41}(\rho) & = & (\rho)E(\rho)^T(T^T X - C^T \bar{H}^T) \end{array}$$

and

$$\bar{L}(\rho) = (X^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \tag{4.44}$$

Moreover, the gain is given by $L(\rho) = X^{-T}\bar{L}(\rho)$ and we have $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

Proof: *Since the dynamical model of the observation error is a LPV time-delay system, in order to derive constructive sufficient conditions for its stability, it is possible to consider lemma 3.5.2 which consider such systems by providing a relaxation to the simple Lyapunov-Krasovskii functional. Substituting the model of the estimation error (4.42) into LMI of lemma 3.5.2 where the matrices C, C_h and F are respectively set to I, 0 and 0 (in order to minimize the impact of the disturbances w onto the observation error e only) leads to*

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ V_{21}(\rho) & V_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ V_{31} & R & -Q_\mu - R & \star & \star & \star & \star \\ V_{41} & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0 \tag{4.45}$$

with

$$\begin{array}{rcl} V_{21}(\rho) & = & (\Theta(\rho) - L(\rho)\Xi(\rho))^T X + P(\rho) \\ V_{31}(\rho) & = & V_{31}(\rho) = (\Upsilon(\rho) - L(\rho)\Omega(\rho))^T X \\ V_{41} & = & E(\rho)^T [T - (\Phi(\rho) - L(\rho)\Psi(\rho))C]^T X \\ V_{22}(\rho,\nu) & = & \dfrac{\partial P}{\partial \rho} - P(\rho) + Q - R \end{array}$$

By considering the change of variable $\bar{L}(\rho) = X^T L(\rho)$, the problem is linearized and results in the following LMI:

$$\begin{bmatrix} -(X+X^T) & \star & \star & \star & \star & \star & \star \\ W_{21}(\rho) & V_{22}(\rho,\nu) & \star & \star & \star & \star & \star \\ W_{31}(\rho) & R & -Q_\mu - R & \star & \star & \star & \star \\ W_{41}(\rho) & 0 & 0 & -\gamma I_q & \star & \star & \star \\ 0 & I_r & 0 & 0 & -\gamma I_r & \star & \star \\ X & 0 & 0 & 0 & 0 & -P(\rho) & \star \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} \prec 0 \tag{4.46}$$

with

$$\begin{array}{rcl} W_{21}(\rho) & = & \Theta(\rho)^T X - \Xi(\rho)^T \bar{L}(\rho)^T + P(\rho) \\ W_{31}(\rho) & = & V_{31}(\rho) = \Upsilon(\rho)^T X - \Omega(\rho)^T \bar{L}(\rho)^T \\ W_{41} & = & E(\rho)^T [X^T (T - \Phi(\rho)C) + \bar{L}(\rho)\Psi(\rho)C]^T \end{array}$$

Actually the problem is still not solved yet since in the reconstruction of the observer, the matrix H may depend on ρ. Indeed, it is worth noting that in the definition of the observer (4.3), H is a constant matrix but in the construction procedure provided in Lemma 4.1.4, H is allowed to be parameter varying which is an aberrant result. Hence, an extra constraint is needed in order to enforce H as a constant matrix while using Lemma 4.1.4. This is developed in the following. Since from lemma 4.1.4, H satisfies the relation

$$H = \Phi(\rho) - L(\rho)\Psi(\rho) \tag{4.47}$$

which implies

$$\begin{array}{rcl} L(\rho)\Psi(\rho) & = & \Phi(\rho) - H \\ \bar{L}(\rho)\Psi(\rho) & = & X^T\Phi(\rho) - \bar{H} \end{array}$$

with $\bar{H} = X^T H$, $\bar{L}(\rho) = X^T L(\rho)$ *and since* $\Psi(\rho)$ *is a full column rank matrix then the solution of the equality is given by*

$$\bar{L}(\rho) = (X^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \tag{4.48}$$

for any $Z(\rho)$ *of appropriate dimensions (see Appendix A.8). This expression will guarantee that for any matrix* $Z(\rho)$*, the resulting* H *will be parameter independent. Moreover by replacing the new expression of* $\bar{L}$ *into the expressions*

$$\begin{array}{c} X^T\Theta(\rho) - \bar{L}(\rho)\Xi(\rho) \\ X^T\Upsilon(\rho) - \bar{L}(\rho)\Omega(\rho) \end{array}$$

of LMI (4.46) then we ensure that H *is a constant in the expression of matrices* $M_0(\rho)$ *and* $M_h(\rho)$*. Finally, the problem reformulated in a LMI optimization problem (4.43) where the matrix* $Z(\rho)$ *is the new parametrizing (decision) matrix.* □

Remark 4.1.6 *It is important to note that the choice of the structure of the matrices* $P(\rho)$ *and* $Z(\rho)$ *is crucial in such a problem. Actually, according to [Apkarian and Adams, 1998], the idea is to 'mimic' the dependence of the system on the parameters but no complete theory is available to choose the structure of parameter dependent matrices.*

On the other hand, it is possible to derive a detectability test by eliminating the matrix $Z(\rho)$ *from (4.43) using the projection lemma (see Appendix E.18). However, this test will only provide a sufficient optimal condition which is independent of the controller. Following Appendix A.9, it is possible to construct an optimal gain* $Z(\rho)$ *from the existence condition obtained using the projection lemma. It is important to notice that the the optimal gain in non-unique according to Appendix A.9. However, the resulting controller may depend on the derivative of the parameters making the observer non-implementable in practice.*

Finally, the analysis of the detectability of the system for given structure of $P(\rho)$ *and* $Z(\rho)$ *is a difficult problem due to the time-varying nature of the parameters and the delay. Hence, no criteria as rank conditions are allowed in this case.*

To conclude on this remark, a rigorous analysis of the choice of the structure of $Z(\rho)$ *or the impact of the choice of* $Z(\rho)$ *on the existence of the controller is a very difficult problem in the case of LPV time-delay systems with time-varying delays. The development of the solution of such a problem needs new technical tools which are, to my best knowledge, unavailable at this time.*

This section ends with the following example.

Example 4.1.7 *Let us consider the system proposed in [Mohammadpour and Grigoriadis, 2007a] with* $D_{21} = 0$ *which is the transfer from* w *to* y*:*

$$\begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x_h(t) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\ y(t) & = & \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x(t) \\ z(t) & = & x(t) \end{array} \tag{4.49}$$

The matrices $Z(\rho)$ *and* $P(\rho)$ *are chosen to be polynomial of degree 2. For simulation purpose, the delay is assumed to be constant and set to* $h = 0.5 < h_{max} = 0.8$*. A step*

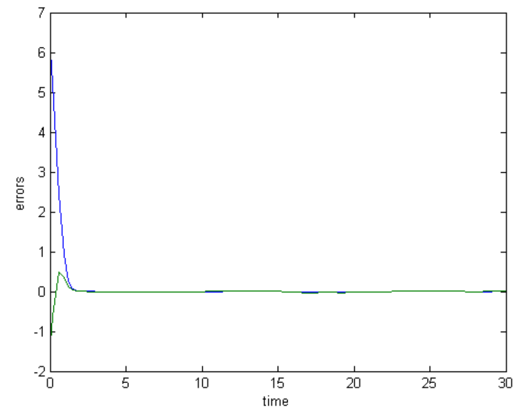

Figure 4.1: Evolution of the observation errors

disturbance $w(t)$ of magnitude 10 is applied on the system at time $t = 15s$ and the parameter trajectory is given by $\rho(t) = \sin(t)$. Applying Theorem 4.1.5, we compute an observer for which we have $||e||_{\mathcal{L}_2} \leq 0.01||w||_{\mathcal{L}_2}$. Figure 4.1.7 shows the evolution of the observation errors.

We can see that the errors converge to 0 and remains close to even in presence of a disturbance.

Since heavy symbolic computation are necessary to compute such an observer (e.g. pseudo-inverse of parameter dependent matrices...) the solutions for matrices are rational functions with high degrees but by analyzing the zeroes and the poles of each coefficient, it appears that several pairs of zeroes/poles are very near. Hence using a least mean square approximation of these polynomial coefficients we get the following observer matrices (energy error between initial and approximants) less than 10^{-6}):

$$M_0(\rho) = \begin{bmatrix} -0.836\rho^2 - 0.836\rho - 0.667 & -0.078\rho^2 - 0.072\rho + 0.1345 \\ -0.0376\rho^2 - 0.0376\rho - 0.361 & -0.396\rho^2 - 0.406\rho - 0.800 \end{bmatrix}$$

$$M_h(\rho) = \begin{bmatrix} -0.009\rho^2 - 0.0002\rho + 0.00822 & -0.007\rho^2 - 0.0071\rho + 0.014 \\ 0.016\rho^2 - 0.00001\rho - 0.0162 & 0.0134\rho^2 + 0.0134\rho - 0.27 \end{bmatrix}$$

$$N_0(\rho) = \begin{bmatrix} -0.073\rho^2 - 0.063\rho + 0.326 & 0.146\rho^2 + 0.148\rho + 0.620 \\ 0.076\rho^2 + 0.058\rho - 0.684 & -0.152\rho^2 - 0.156\rho - 1.054 \end{bmatrix}$$

$$N_h(\rho) = \begin{bmatrix} 0.001\rho^2 + 0.001\rho + 0.040 & -0.001\rho^2 + 0.019\rho + 0.046 \\ -0.001\rho^2 - 0.27\rho - 0.077 & 0.002\rho^2 - 0.035\rho - 0.088 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.106 & 1.788 \\ 0.798 & 0.404 \end{bmatrix}$$

As a conclusion of the approach, observers designed with this approach lead to interesting results due to their good performances. As the model of the system is exact such observers can be designed on unstable systems and the delay-margin of the observation error can be larger than the delay-margin of the system. However, such properties are not of interest since in practice the system is not known exactly: uncertainties on the delay and on the coefficient of the system are generally encountered. These problems are (partially) answered in the following sections.

4.1.2 Observer with approximate delay value

From the dynamical equations of the observer, it is clear that the exact knowledge of the delay is a crucial condition to the design of the observer of the latter sections. However, estimating or measuring the delay in real time is a challenging open problem [Belkoura et al., 2007, 2008, Drakunov et al., 2006] and the delay is known with infinite precision. Therefore, it seems convenient to consider the case where the delay is approximately known and the design of an observer with approximate delay value is exposed in what follows.

The considered observer is given by:

$$\begin{array}{rcl}\dot{\xi}(t) & = & M_0(\rho)\xi(t) + M_h(\rho)\xi(t-d(t)) + N_0(\rho)y(t) + N_h(\rho)y(t-d(t)) \\ \hat{z}(t) & = & \xi(t) + Hy(t)\end{array} \tag{4.50}$$

where $d(t)$ is the delay implemented in the observer. The idea is to impose a relationship between the real and implemented delays:

$$d(t) = h(t) + \varepsilon(t)$$

where $\varepsilon(t)$ denotes a bounded error ($\varepsilon(t) \in [-\delta, \delta]$).

Whenever the delays are locally equal (in time), then the error dynamical model is identical to (4.13). On the other hand, if the delays are different then we have the following extended model:

$$\begin{array}{rcl}\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} & = & \mathcal{A}(\rho)\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} + \mathcal{A}_h(\rho)\begin{bmatrix}x(t-h(t))\\ e(t-h(t))\end{bmatrix} + \mathcal{A}_d(\rho)\begin{bmatrix}x(t-d(t))\\ e(t-d(t))\end{bmatrix} + \mathcal{E}(\rho)w(t) \\ \mathcal{A}(\rho) & = & \begin{bmatrix}A(\rho) & 0\\ (T-HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T-HC) & M_0(\rho)\end{bmatrix} \\ \mathcal{A}_h(\rho) & = & \begin{bmatrix}A_h(\rho) & 0\\ (T-HC)A_h(\rho) & 0\end{bmatrix} \\ \mathcal{A}_d(\rho) & = & \begin{bmatrix}0 & 0\\ -M_h(\rho)(T-HC) - N_h(\rho)C & M_h(\rho)\end{bmatrix} \\ \mathcal{E}(\rho) & = & \begin{bmatrix}E(\rho)\\ (T-HC)E(\rho)\end{bmatrix}\end{array} \tag{4.51}$$

where we have assumed without loss of generality that the control input is 0 (i.e. $u(t) \equiv 0$) since the solution $S(\rho)$ of the observer gain is trivial.

From this model, it is obvious that only asymptotically stable systems can be observed in such a framework since the delayed state of the system enters in the observer model and cannot be removed by vanishing the coefficient. Indeed, if the system is unstable then $x(t) \to +\infty$ as $t \to +\infty$ and thus $e(t) \to +\infty$ as $t \to +\infty$. On the second hand, the observer will not be able in this case to observe uniformly z since the delayed states of the system $x(t-d(t))$ and $x(t-h(t))$ affect the observation error e.

Conditions of lemmas 4.1.3 and 4.1.4 are supposed to be fulfilled in this case and we have

$$\begin{array}{rcl}(T-HC)A(\rho) - N_0(\rho)C - M_0(\rho)(T-HC) & = & 0\\ (T-HC)A_h(\rho) - M_h(\rho)(T-HC) - N_h(\rho)C & = & 0\end{array}$$

Then matrices $\mathcal{A}(\rho)$ and $\mathcal{A}_d(\rho)$ in model (4.51) can be rewritten as

$$\begin{aligned} \mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ 0 & M_0(\rho) \end{bmatrix} \\ \mathcal{A}_d(\rho) &= \begin{bmatrix} 0 & 0 \\ -(T-HC)A_h(\rho) & M_h(\rho) \end{bmatrix} \end{aligned} \tag{4.52}$$

Similarly as for the latter section of observer, it is possible to provide nonconstructive necessary and sufficient conditions taking the form of the following theorem:

Theorem 4.1.8 *There exists an LPV/$\mathcal{H}_\infty$ observer with memory of the form (4.50) for system of the form (4.3) if and only if the following statements hold:*

1. *The autonomous error dynamical expression $\dot{\eta}(t) = \mathcal{A}(\rho)\eta(t) + \mathcal{A}_h(\rho)e(t-h(t)) + \mathcal{A}_d(\rho)\eta(t-d(t))$ is asymptotically stable where $\eta(t) = \mathrm{col}(x(t), e(t))$ with $e(t) = z(t) - \hat{z}(t)$, $d(t) = h(t) + \varepsilon(t)$, $|\varepsilon(t)| \leq \delta$ and $h \in \mathscr{H}_1^\circ$.*

2. $(T-HC)A(\rho)x(t) - N_0(\rho)C - M_0(\rho)(T-HC) = 0$

3. $(T-HC)A_h(\rho)x(t-h(t)) - N_hC - M_h(\rho)(T-HC) = 0$

4. *The inequality $||e||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ holds for some $\gamma > 0$*

Due to the form of the dynamical model of the observation error, we can easily recognize that the general structure considered in Section 3.7 where the problem of the stability of system with two correlated delays is addressed.

Theorem 4.1.9 *There exists a parameter dependent observer of the form (4.50) such that theorem 4.1.8 holds for all $h \in \mathscr{H}_1^\circ$, $d(t) = h(t) + \varepsilon(t)$ with $\varepsilon(t) \in [-\delta, \delta]$ is satisfied if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^r$, a matrix function $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, constant matrices $Q_i, R_i \in \mathbb{S}_{++}^{r+n}$, $i = 1, 2$, $X_1 \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times r}$, $X_3 \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and a positive scalar $\gamma > 0$ such that the following LMIs*

$$\begin{bmatrix} -X^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_d(\rho)+\tilde{\mathcal{A}}_h(\rho) & \bar{\mathcal{E}}(\rho) & 0 & X^T & h_{max}R_1 & R_2 \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & \mathcal{I}^T & 0 & 0 & 0 \\ \star & \star & \Theta_{22} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \prec 0 \tag{4.53}$$

and

$$\begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) \\ \star & \Pi_{22}(\rho) \end{bmatrix} \prec 0 \tag{4.54}$$

hold for all $(\rho, \nu) \in U_\rho \times U_\nu$ *and where*

$$\begin{aligned}
\Pi_{11}(\rho,\nu) &= \begin{bmatrix} -X^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{A}}_d(\rho) & \bar{\mathcal{E}}(\rho) \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & 0 \\ \star & \star & \Psi_{22} & (1-\mu)R_2/\delta & 0 \\ \star & \star & \star & \Psi_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} \\
\Pi_{12}(\rho) &= \begin{bmatrix} 0 & X^T & h_{max}R_1 & R_2 \\ \mathcal{I}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\Pi_{22}(\rho) &= \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & -R_1 & 0 \\ \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \\
X &= \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} \\
\Theta_{11}(\rho,\nu) &= -P(\rho)+Q_1+Q_2+\sum_{i=1}^{N}\frac{\partial P}{\partial \rho_i}(\rho)\nu_i - R_1 \\
\Theta_{22} &= -(1-\mu)(Q_1+Q_2)-R_1 \\
\Psi_{22} &= -(1-\mu)(Q_1+R_2/\delta)-R_1 \\
\Psi_{33} &= -(1-\mu_c)Q_2-(1-\mu)R_2/\delta \\
\mathcal{I} &= \begin{bmatrix} 0 & I_r \end{bmatrix} \\
\bar{L}(\rho) &= (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(\rho)(I-\Psi(\rho)\Psi(\rho)^+) \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} X_1^TA(\rho) & 0 \\ X_2^TA(\rho) & X_3^T\Theta(\rho)-\bar{L}(\rho)\Xi(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} X_1^TA_h(\rho) & 0 \\ X_2^TA_h(\rho)+X_3^T(T-\Phi(\rho)C)A_h(\rho)+\bar{L}(\rho)\Psi(\rho)CAh(\rho) & 0 \end{bmatrix} \\
\tilde{\mathcal{A}}_d(\rho) &= \begin{bmatrix} 0 & 0 \\ -X_3^T(T-\Phi(\rho)C)-\bar{L}(\rho)\Psi(\rho)C & X_3^T\Upsilon(\rho)-\bar{L}(\rho)\Omega(\rho) \end{bmatrix} \\
\bar{\mathcal{E}}(\rho) &= \begin{bmatrix} X_1^TE(\rho)^T \\ (X_2^TT-\bar{H}C)E(\rho)^T \end{bmatrix}
\end{aligned}$$

Moreover, the gain is given by $L(\rho) = X_3^{-T}\bar{L}(\rho)$ *and we have* $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

Proof: *Let us consider LMIs (3.140) and (3.141) of lemma 3.7.1. Let us define the* *matrices*

$$\begin{array}{rclcl}\tilde{\mathcal{A}}(\rho) & = & X^T\mathcal{A}(\rho) & = & \begin{bmatrix} X_1^TA(\rho) & 0 \\ X_2^TA(\rho) & X_3^T\Theta(\rho)-\bar{L}(\rho)\Xi(\rho)\end{bmatrix} \\ \tilde{\mathcal{A}}_h(\rho) & = & X^T\mathcal{A}_h(\rho) & = & \begin{bmatrix} X_1^TA_h(\rho) & 0 \\ X_2^TA_h(\rho)+X_3^T(T-\Phi(\rho)C)A_h(\rho)+\bar{L}(\rho)\Psi(\rho)CAh(\rho) & 0\end{bmatrix} \\ \tilde{\mathcal{A}}_d(\rho) & = & X^T\mathcal{A}_d(\rho) & = & \begin{bmatrix} 0 & 0 \\ -X_3^T(T-\Phi(\rho)C)-\bar{L}(\rho)\Psi(\rho)C & X_3^T\Upsilon(\rho)-\bar{L}(\rho)\Omega(\rho)\end{bmatrix} \\ \tilde{\mathcal{E}}(\rho) & = & X^T\mathcal{E}(\rho) & = & \begin{bmatrix} X_1^TE(\rho) \\ X_2^TE(\rho)+X_3^T(T-\Phi(\rho)C)+\bar{L}(\rho)\Psi(\rho)C\end{bmatrix}\end{array}$$

Now substituting these expressions in the LMIs (3.140) and (3.141) of lemma 3.7.1 we get LMIs (4.53) and (4.54). Since the matrix H has to be chosen independent of the parameter ρ it suffices to parametrize $\bar{L}$ by $Z(\rho)$ as

$$\bar{L}(\rho) = (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(\rho)(I-\Psi(\rho)\Psi(\rho)^+)$$

for some $Z(\rho)$ of appropriate dimensions. Now denoting $\bar{\mathcal{E}}(\rho) = \begin{bmatrix} X_1^TE(\rho)^T \\ (X_2^TT-\bar{H}C)E(\rho)^T\end{bmatrix}$ concludes the proof. □

4.1.3 Memoryless Observer

As a final design technique, we consider here the case where the delay cannot or is not known and therefore no information on the delay can be used in the observer. This motivates the choice of the following observer:

$$\begin{array}{rcl}\dot{\xi}(t) & = & M(\rho)\xi(t)+N(\rho)y(t) \\ \hat{z}(t) & = & \xi(t)+Hy(t)\end{array} \tag{4.55}$$

In this case the extended system containing both the dynamical model of the system and the observer is then given by

$$\begin{array}{rcl}\begin{bmatrix}\dot{x}(t) \\ \dot{e}(t)\end{bmatrix} & = & \mathcal{A}(\rho)\begin{bmatrix} x(t) \\ e(t)\end{bmatrix} + \mathcal{A}_h(\rho)\begin{bmatrix} x(t-h(t)) \\ e(t-h(t))\end{bmatrix} + \mathcal{E}(\rho)w(t) \\ \mathcal{A}(\rho) & = & \begin{bmatrix} A(\rho) & 0 \\ (T-HC)A(\rho)-M(\rho)(T-HC)-N(\rho)C & M(\rho)\end{bmatrix} \\ \mathcal{A}_h(\rho) & = & \begin{bmatrix} A_h(\rho) & 0 \\ (T(HC)A_h(\rho) & 0\end{bmatrix} \\ \mathcal{E}(\rho) & = & \begin{bmatrix} E(\rho) \\ (T-HC)E(\rho)\end{bmatrix}\end{array} \tag{4.56}$$

From this expression, it is possible to provide the following theorem:

Theorem 4.1.10 *There exists an LPV/$\mathcal{H}_\infty$ observer with memory of the form (4.55) for system of the form (4.1) if and only if the following statements hold:*

1. *The unforced extended dynamical system $\dot{\zeta}(t) = \mathcal{A}(\rho)(\rho)\zeta(t)+\mathcal{A}(\rho)_h(\rho)\zeta(t-h(t))$ is asymptotically stable where $\zeta(t) = \mathrm{col}(x(t), e(t))$ and $e(t) = z(t)-\hat{z}(t)$*

2. $(T - HC)A(\rho) - M(\rho)(T - HC) - N(\rho)C = 0$

3. *The inequality* $||e||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$ *holds for some* $\gamma > 0$

The next results are the memoryless counterparts of Lemmas 4.1.3 and 4.1.4 dealing with observers with memory.

Lemma 4.1.11 *There exists a solution* $M(\rho), N(\rho), H$ *to the equation of statement 2 if and only if the following rank equality holds*

$$\text{rank}\begin{bmatrix} T \\ C \\ CA(\rho) \\ TA(\rho) \end{bmatrix} = \text{rank}\begin{bmatrix} T \\ C \\ CA(\rho) \end{bmatrix} \tag{4.57}$$

Proof: *The proof is similar as for lemma 4.1.3.* □

In the case when lemma 4.1.11 is verified then it is possible to find matrices $M(\rho)$ and $N(\rho)$ such that equation of theorem 4.1.10, statement 2 is verified.

Lemma 4.1.12 *Under condition of Lemma 4.1.11, the observer matrices are parametrized with respect to a free matrix* $L(\rho)$ *according to the following expressions*

$$\begin{array}{rcl}
M(\rho) & = & \Theta(\rho) - L(\rho)\Xi(\rho) \\
H & = & \Phi(\rho) - L(\rho)\Omega(\rho) \\
\Theta(\rho) & = & TA(\rho)U - \Lambda(\rho)\Gamma(\rho)^+\Delta_M \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U \\
\Xi(\rho) & = & -(I - \Gamma(\rho)\Gamma(\rho)^+)\Delta_M \begin{bmatrix} C \\ CA(\rho) \end{bmatrix} U \\
\Phi(\rho) & = & \Lambda(\rho)\Gamma(\rho)^+\Delta_H \\
\Psi(\rho) & = & -(I - \Gamma(\rho)\Gamma(\rho)^+)\Delta_H \\
S(\rho) & = & (T - HC)B(\rho) \\
N(\rho) & = & K(\rho) + M(\rho)H \\
K(\rho) & = & [\Lambda(\rho)\Gamma(\rho)^+ + L(\rho)s(I - \Gamma(\rho)\Gamma(\rho)^+)]\,\Delta_K
\end{array}$$

and

$$\Delta_M = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \qquad \Delta_K = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \qquad \Delta_H = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

Proof: *The proof is similar as for lemma 4.1.4* □

Whenever lemma 4.1.11 is satisfied and according to matrix definitions of lemma 4.1.12, system (4.56) rewrites

$$\begin{array}{rcl}\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} & = & \mathcal{A}(\rho)\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}+\mathcal{A}_h(\rho)Z\begin{bmatrix}x(t-h(t))\\ e(t-h(t))\end{bmatrix}+\mathcal{B}(\rho)u(t)+\mathcal{E}(\rho)w(t)\\ \mathcal{A}(\rho) & = & \begin{bmatrix}A(\rho) & 0\\ 0 & \Theta(\rho)-L(\rho)\Xi(\rho)\end{bmatrix}\\ \mathcal{A}_h(\rho) & = & \begin{bmatrix}A_h(\rho)\\ \left[T-\Phi(\rho)C+L(\rho)\Omega(\rho)C\right]A_h(\rho)\end{bmatrix}\\ \mathcal{B}(\rho) & = & \begin{bmatrix}B(\rho)\\ 0\end{bmatrix}\\ \mathcal{E}(\rho) & = & \begin{bmatrix}E(\rho)\\ \left[T-\Phi(\rho)C+L(\rho)\Omega(\rho)\right]E(\rho)\end{bmatrix}\\ Y & = & \begin{bmatrix}I_n & 0\end{bmatrix}\end{array} \tag{4.58}$$

Finally we have the following theorem:

Theorem 4.1.13 *There exists a parameter dependent observer of the form (4.55) such that theorem 4.1.10 for all $h \in \mathscr{H}_1^\circ$ is satisfied if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}^r_{++}$, a matrix function $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, constant matrices $Q, R \in \mathbb{S}^{r+n}_{++}$, $X_1 \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times r}$, $X_3 \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and a positive scalar $\gamma > 0$ such that the following LMI*

$$\begin{bmatrix} -X^H & P(\rho)+X^T\tilde{\mathcal{A}}(\rho) & X^T\tilde{\mathcal{A}}_h(\rho) & X^T\tilde{\mathcal{E}}(\rho) & 0 & X^T & h_{max}Y^TR\\ \star & \Psi'_{22}(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & 0\\ \star & \star & -(1-\mu)Q-R & 0 & 0 & 0 & 0\\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0\\ \star & \star & \star & \star & -\gamma I_r & 0 & 0\\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}Y^TR\\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.59}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where

$$\begin{array}{rcl}\Psi'_{22}(\rho,\nu) & = & \dfrac{\partial}{\partial\rho}P(\rho)\nu - P(\rho) + Y^T(Q-R)Y\\ X & = & \begin{bmatrix}X_1 & X_2\\ 0 & X_3\end{bmatrix}\\ Y & = & \begin{bmatrix}I_n & 0\end{bmatrix}\\ \mathcal{I} & = & \begin{bmatrix}0 & I_r\end{bmatrix}\\ \bar{L}(\rho) & = & (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(I-\Psi(\rho)\Psi(\rho)^+)\end{array}$$

Moreover the generalized observer gain $L(\rho)$ is given by the relation $L(\rho) = X_3^{-T}\bar{L}(\rho)$ and we have $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$.

Proof: *Due to the structure of $\mathcal{A}_h(\rho)$ it is clear that such a problem falls into the framework of Section 3.5.3 which considers the stability of time-delay systems in which the delay acts on only a specific subpart of the system state (i.e. the state of the system and not the state of* *the observer in this case). Hence injecting the extended system into LMI (3.109) we get with*

$$\begin{bmatrix} -X^H & P(\rho)+X^T\mathcal{A}(\rho) & X^T\mathcal{A}_h(\rho) & X^T\mathcal{E}(\rho) & 0 & X^T & h_{max}Y^TR \\ \star & \Psi'_{22}(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}Y^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.60}$$

with

$$\begin{aligned} \Psi'_{22}(\rho,\nu) &= \partial_\rho P(\rho)\nu - P(\rho) + Y^T(Q-R)Y \\ R &\in \mathbb{S}^n_{++} \\ Z &= \begin{bmatrix} I_n & 0 \end{bmatrix} \\ \mathcal{I} &= \begin{bmatrix} 0 & I_r \end{bmatrix} \end{aligned}$$

Choosing $X = \begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix}$ *then we have the following relations:*
Expanding the relations we get

$$\begin{aligned} \tilde{\mathcal{A}}(\rho) &= X^T\mathcal{A}(\rho) = \begin{bmatrix} X_1^TA(\rho) & 0 \\ X_2^TA(\rho) & X_3^T\Theta(\rho)-\bar{L}(\rho)\Xi(\rho) \end{bmatrix} \\ \tilde{\mathcal{A}}_h(\rho) &= X^T\mathcal{A}_h(\rho) = \begin{bmatrix} X_1^TA_h(\rho) \\ X_2^TA_h(\rho)+X_3^T(T-\Phi(\rho)C)+\bar{L}(\rho)\Omega(\rho)CA_h(\rho) \end{bmatrix} \\ \tilde{\mathcal{E}}(\rho) &= X^T\mathcal{E}(\rho) = \begin{bmatrix} X_1^TE(\rho) \\ X_2^TE(\rho)+X_3^T(T-\Phi(\rho)C)+\bar{L}(\rho)\Omega(\rho)E(\rho) \end{bmatrix} \end{aligned}$$

where $\bar{L}(\rho) = X_3^TL(\rho)$.

An interesting fact of such a Lyapunov-Krasovskii functional of the form (3.79) is the embedding of an information on the structure of the system (the delay does not act on some part of the state) and allows to reduce the number of decision variables.

Finally, since a constant H *matrix is sought (as in proof of theorem 4.1.5, then by choosing* $\bar{L}$ *such that*

$$\bar{L}(\rho) = (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(\rho)(I-\Psi(\rho)\Psi(\rho)^+)$$

where $Z(\rho)$ *is a free matrix with appropriate dimension and* $\bar{H} = X_3^TH$*. This completes the proof.* □

4.2 Filtering of uncertain LPV Time-Delay Systems

This section is devoted to the filtering of LPV time-delay systems, where we are interested in finding a LPV filter of the form

$$\begin{aligned} \dot{x}_F(t) &= A_F(\rho)x(t) + A_{Fh}(\rho)x(t-d(t)) + B_F(\rho)y(t) \\ z_F(t) &= C_F(\rho)x(t) + C_{Fh}(\rho)x(t-d(t)) + D_F(\rho)y(t) \end{aligned} \tag{4.61}$$

for systems of the form

$$\begin{array}{rcl}\dot{x}(t) & = & (A(\rho)+\Delta A(\rho,t))x(t)+(A_h(\rho)+\Delta A_h(\rho,t))x(t)+(E(\rho)+\Delta E(\rho,t))w(t)\\ z(t) & = & (C(\rho)x(t)+\Delta C(\rho,t))x(t)+(C_h(\rho)x(t)+\Delta C_h(\rho,t))x(t-h(t))\\ & & +(F(\rho)x(t)+\Delta F(\rho,t))w(t)\\ y(t) & = & (C_y(\rho)x(t)+\Delta C_y(\rho,t))x(t)+(C_{yh}(\rho)x(t)+\Delta C_{yh}(\rho,t))x(t-h(t))\\ & & +(F_y(\rho)x(t)+\Delta F_y(\rho,t))w(t)\end{array} \tag{4.62}$$

where $x \in \mathbb{R}^n,\ x_F \in \mathbb{R}^r,\ w \in \mathbb{R}^m,\ y \in \mathbb{R}^p,\ z, z_F \in \mathbb{R}^t$ are respectively the system state, the filter state, the system measurements, the system exogenous inputs, the signal to be estimated and its estimate. The time-varying delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ and the filter delay $d(t)$ is unconstrained at this time. When $r < n$ the filter is said to be a *reduced-order filter* while if $r = n$ it is a *full-order filter*. We will consider in the following only full-order filters (i.e. $r = n$). It is possible to generalize the results to the case of reduced-order filters by considering, for instance, the approach of [Tuan et al., 2001].

The uncertain terms are assumed to obey

$$\begin{bmatrix} \Delta A(\rho,t)) & \Delta A_h(\rho,t)) & \Delta E(\rho,t)) \\ \Delta C(\rho,t)) & \Delta C_h(\rho,t)) & \Delta F(\rho,t)) \\ \Delta C_y(\rho,t)) & \Delta C_{yh}(\rho,t)) & \Delta F_y(\rho,t)) \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ G_z(\rho) \\ G_y(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_x(\rho) & H_{xh}(\rho) & H_w(\rho) \end{bmatrix} \tag{4.63}$$

where all matrices are of appropriate dimensions provided that the uncertain terms are all defined.

4.2.1 Design of robust filters with exact delay-value - simple Lyapunov-Krasovskii functional

This section is devoted to the design of filter with exact delay-value (i.e. $d(t) = h(t)$ for all $t \geq 0$). Even if such a filter are difficult to realize they allow to give a lower bound on the best achievable $\mathcal{H}_\infty$ norm. Indeed, since such filters can be considered as full-information filters using any other filters, it is not possible to reach better performances.

In this case, the extended system describing the evolution of the system and the filter is given by

$$\begin{array}{rcl}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_F(t) \end{bmatrix} & = & (\mathcal{A}(\rho)+\Delta\mathcal{A}(\rho,t))\begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + (\mathcal{A}_h(\rho)+\Delta\mathcal{A}_h(\rho,t))\begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix}\\ & & +(\mathcal{E}(\rho)+\Delta\mathcal{E}(\rho,t))w(t)\\ e(t) & = & z(t)-z_F(t)\\ & = & (\mathcal{C}(\rho)+\Delta\mathcal{C}(\rho,t))\begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + (\mathcal{C}_h(\rho)+\Delta\mathcal{C}_h(\rho,t))\begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix}\\ & & +(\mathcal{F}(\rho)+\Delta\mathcal{F}(\rho,t))w(t)\end{array} \tag{4.64}$$

where

$$
\begin{array}{rcl}
\mathcal{A}(\rho) & = & \begin{bmatrix} A(\rho) & 0 \\ B_F(\rho)C_y(\rho) & A_F(\rho) \end{bmatrix} \\
\Delta\mathcal{A}(\rho,t) & = & \begin{bmatrix} \Delta A(\rho,t) & 0 \\ B_F(\rho)\Delta C_y(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_x(\rho) & 0 \end{bmatrix} \\
\mathcal{A}_h(\rho) & = & \begin{bmatrix} A_h(\rho) & 0 \\ B_F(\rho)C_{yh}(\rho) & A_{Fh}(\rho) \end{bmatrix} \\
\Delta\mathcal{A}_h(\rho,t) & = & \begin{bmatrix} \Delta A_h(\rho,t) & 0 \\ B_F(\rho)\Delta C_{yh}(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_{xh}(\rho) & 0 \end{bmatrix} \\
\mathcal{E}(\rho) & = & \begin{bmatrix} E(\rho) \\ B_F(\rho)F_y(\rho) \end{bmatrix} \\
\Delta\mathcal{E}(\rho,t) & = & \begin{bmatrix} \Delta E(\rho,t) & 0 \\ B_F(\rho)\Delta E(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} \Delta(t)H_w(\rho) \\
\mathcal{C}(\rho) & = & \begin{bmatrix} C(\rho) - D_F(\rho)C_y(\rho) & -C_F(\rho) \end{bmatrix} \\
\Delta\mathcal{C}(\rho,t) & = & \begin{bmatrix} \Delta C(\rho,t) - D_F(\rho)\Delta C_y(\rho,t) & 0 \end{bmatrix} = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t) \begin{bmatrix} H_x(\rho) & 0 \end{bmatrix} \\
\mathcal{C}_h(\rho) & = & \begin{bmatrix} C_h(\rho) - D_F(\rho)C_{yh}(\rho) & -C_{Fh}(\rho) \end{bmatrix} \\
\Delta\mathcal{C}_h(\rho,t) & = & \begin{bmatrix} \Delta C_h(\rho,t) - D_F(\rho)\Delta C_{yh}(\rho,t) & 0 \end{bmatrix} = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t) \begin{bmatrix} H_{xh}(\rho) & 0 \end{bmatrix} \\
\mathcal{F}(\rho) & = & F(\rho) - D_F(\rho)F_y(\rho) \\
\Delta\mathcal{F}(\rho,t) & = & \Delta F(\rho) - D_F(\rho)\Delta F_y(\rho) = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t)H_w(\rho)
\end{array}
$$

This leads to the following theorem which an application of the relaxation theorem of the simple Lyapunov-Krasovskii developed in Section 3.5.

Theorem 4.2.1 *There exists a full-order filter of the form (4.61) with $d(t) = h(t)$, $h(t) \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}^{2n}_{++}$, symmetric matrices $\tilde{Q}, \tilde{R} \in \mathbb{S}^{2n}_{++}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, matrix functions $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ and scalars $\gamma, \varepsilon > 0$ such that the LMI*

$$
\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \tilde{\mathcal{G}}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{4.65}
$$

holds for all $\rho \in U_\rho$ with

$$\begin{aligned}
\Psi(\rho,\nu) &= \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}\tilde{R} \\ \star & \tilde{\Psi}_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)\tilde{Q}-\tilde{R} & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} \\
\Psi_{22}(\rho,\nu) &= \partial_\rho \tilde{P}(\rho)\nu - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} \\
\tilde{P}(\rho) &= \tilde{X}^T P(\rho)\tilde{X} \\
\tilde{Q} &= \tilde{X}^T Q\tilde{X} \\
\tilde{R} &= \tilde{X}^T R\tilde{X} \\
\hat{X} &= \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) &= \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T &= \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T &= \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\hat{X}_2 &= X_2 X_4^{-1} X_3 = U^T \Sigma V \qquad \text{(SVD)}
\end{aligned}$$

$$\mathcal{H}(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho)^T \\ 0 \\ \hline H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad \tilde{\mathcal{G}}(\rho)^T = \left[\begin{array}{c} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho)G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline G_z(\rho) - D_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

where $\hat{X}_3 = U\Sigma V$ and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *Substitute the model (4.64) into LMI (3.95) we get*

$$\begin{bmatrix} -X^H & P(\rho)+X^T\bar{A}(\rho) & X^T\bar{A}_h(\rho) & X^T\bar{E}(\rho) & 0 & X^T & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & \bar{C}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & \bar{C}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \bar{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{4.66}$$

with

$$\begin{array}{rcl} \Psi_{22}(\rho,\nu) &=& \partial_\rho P(\rho)\nu - P(\rho) + Q - R \\ \bar{A}(\rho) &=& \mathcal{A}(\rho)+\Delta\mathcal{A}(\rho) \\ \bar{A}_h(\rho) &=& \mathcal{A}_h(\rho)+\Delta\mathcal{A}_h(\rho) \\ \bar{E}(\rho) &=& \mathcal{E}(\rho)+\Delta\mathcal{E}(\rho) \\ \bar{C}(\rho) &=& \mathcal{C}(\rho)+\Delta\mathcal{C}(\rho) \\ \bar{C}_h(\rho) &=& \mathcal{C}_h(\rho)+\Delta\mathcal{C}_h(\rho) \\ \bar{F}(\rho) &=& \mathcal{F}(\rho)+\Delta\mathcal{F}(\rho) \end{array}$$

The latter inequality can be rewritten in the following form

$$\mathcal{M}(\rho,\nu)+\mathcal{D}^T\mathcal{G}(\rho)^T\Delta(t)\mathcal{H}(\rho)+\mathcal{H}(\rho)^T\Delta(t)^T\mathcal{G}(\rho)\mathcal{D}\prec 0$$

where

$$\begin{array}{rcl} \mathcal{D} &=& \mathrm{diag}(X,I,\dots,I) \\ \mathcal{M}(\rho,\nu) &=& \begin{bmatrix} -X^H & P(\rho)+X^T\mathcal{A}(\rho) & X^T\mathcal{A}_h(\rho) & X^T\mathcal{E}(\rho) & 0 & X^T & h_{max}R \\ \star & \Psi_{22}(\rho,\nu) & R & 0 & \mathcal{C}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & \mathcal{C}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \end{array}$$

$$\mathcal{H}(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho)^T \\ 0 \\ \hline H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad \mathcal{G}(\rho)^T = \left[\begin{array}{c} G_x(\rho) \\ B_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline G_z(\rho)-D_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$$

Then invoking the bounding lemma (see Appendix E.14), we get the following equivalent matrix inequality

$$\begin{bmatrix} \mathcal{M}(\rho,\nu)+\varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{4.67}$$

where $\varepsilon > 0$ *is unknown variable to be designed. Since the latter matrix inequality is nonlinear, it cannot be solved efficiently in a reasonable time. The remaining of the proof is devoted to the linearization of such an inequality.*

Let us define the matrix $\tilde{X} = \begin{bmatrix} I_n & 0 \\ 0 & X_4^{-1}X_3 \end{bmatrix}$ *then we have*

$$\begin{array}{rclcl}
\tilde{\mathcal{A}}(\rho) & = & \tilde{X}^T X^T \mathcal{A}(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1^T A(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) & = & \tilde{X}^T X^T \mathcal{A}_h(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1^T A_h(\rho)+\tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho)+\tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) & = & \tilde{X}^T X^T \mathcal{E}(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1 E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T & = & \tilde{X}^T \mathcal{C}(\rho) & = & \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T & = & \tilde{X}^T \mathcal{C}_h(\rho) & = & \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\tilde{G}_1(\rho) & = & \tilde{X}^T X^T \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} & = & \begin{bmatrix} \hat{X}_1^T G_x(\rho)+\tilde{B}_F(\rho)G_y(\rho) \\ \hat{X}^T G_x+\tilde{B}_F(\rho)G_y(\rho) \end{bmatrix} \\
\hat{X} & = & \tilde{X}^T X \tilde{X} & = & \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_2X_4^{-1}X_3 \\ X_3^T X_4^{-1}X_3 & X_3^T X_4^{-1}X_3 \end{bmatrix} \\
\tilde{A}_F(\rho) & = & X_3^T A_F(\rho)X_4^{-1}X_3 \\
\tilde{A}_{Fh}(\rho) & = & X_3^T A_{Fh}(\rho)X_4^{-1}X_3 \\
\tilde{B}_F(\rho) & = & X_3^T B_F(\rho) \\
\tilde{C}_F(\rho) & = & C_F(\rho)X_4^{-1}X_3 \\
\tilde{C}_{Fh}(\rho) & = & C_{Fh}(\rho)X_4^{-1}X_3
\end{array}$$

Then perform a congruence transformation on (4.67) with respect to $\operatorname{diag}(\tilde{X},\tilde{X},\tilde{X},I_q,I_r,\tilde{X},\tilde{X},I)$ *we get LMI (4.65). Now let us focus on the computation of the filter matrices. Note that*

$$\begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ \tilde{C}_F(\rho) & \tilde{C}_{Fh}(\rho) & \tilde{D}_F(\rho) \end{bmatrix} = \begin{bmatrix} X_3^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} X_4^{-1}X_3 & 0 & 0 \\ 0 & X_4^{-1}X_3 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{4.68}$$

Thus it suffices to construct back the matrix X *in order to compute the observer gain. A singular values decomposition (SVD, see Appendix A.6) on* $\hat{X}_3$ *allows to compute the matrices* X_3 *and* X_4 *which are necessary to construct the filter matrices. Indeed, we have* $\hat{X}_2 = U^T\Sigma V$ *and hence*

$$\begin{array}{rcl} X_2 & = & U^T \\ X_4 & = & \Sigma^{-1} \\ X_3 & = & V \end{array}$$

and finally we have

$$\begin{aligned}\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} &= \begin{bmatrix} V^T & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ \tilde{C}_F(\rho) & \tilde{C}_{Fh}(\rho) & \tilde{D}_F(\rho) \end{bmatrix} \begin{bmatrix} \Sigma V & 0 & 0 \\ 0 & \Sigma V & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)V^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)V^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)V^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)V^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}\end{aligned}$$

□

4.2.2 Design of robust memoryless filters

This last section is devoted to the synthesis of robust memoryless filters. The resulting synthesis condition are based on the application of the reduced Lyapunov-Krasovskii functional introduced in Section **??** which applies on system where the delay acts only on a subpart of the state. In this case, only the state of the system is affected by the delay.

Theorem 4.2.2 *There exists a full-order filter of the form (4.61) with $A_{Fh} = 0$ and $C_{Fh} = 0$ with $h(t) \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}^{2n}_{++}$, symmetric matrices $\tilde{Q}, \tilde{R} \in \mathbb{S}^{2n}_{++}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, matrix functions $\tilde{A}_F : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ and scalars $\gamma, \varepsilon > 0$ such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{4.69}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where

$$\begin{aligned}
\Psi(\rho,\nu) &= \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}Z^TR \\ \star & \tilde{\Psi}'_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}Z^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \\
\tilde{\Psi}'_{22}(\rho,\nu) &= \partial_\rho\tilde{P}(\rho)\nu - \tilde{P}(\rho) + Z^T(Q(\rho)-R)Z \\
\hat{X} &= \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} \hat{X}_1^TA(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^TA(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} \hat{X}_1^TA_h+\tilde{B}_F(\rho)C_{yh}(\rho) \\ \hat{X}_2^TA_h+\tilde{B}_F(\rho)C_{yh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) &= \begin{bmatrix} \hat{X}_1E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^TE(\rho)+\tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T &= \begin{bmatrix} C(\rho)^T - C_y(\rho)^TD_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T &= C_h(\rho)^T - C_{hy}(\rho)^TD_F(\rho)^T \\
\hat{X}_3 &= X_2X_4^{-1}X_3 = U^T\Sigma V \qquad \text{(SVD)}
\end{aligned}$$

$$\mathcal{H}(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho) \\ \hline H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] \qquad \mathcal{G}(\rho)^T = \left[\begin{array}{c} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho) G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho) G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ \hline G_z(\rho) - \tilde{D}_F(\rho) G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right]$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)V^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)V^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

4.2.3 Example

We will show the effectiveness of the approach compared to existing ones through the following example. Let us consider the following system Mohammadpour and Grigoriadis [2007a]:

$$\begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0 & 1+0.2\rho \\ -2 & -3+0.1\rho \end{bmatrix} x + \begin{bmatrix} 0.2\rho & 0.1 \\ -0.2+0.1\rho & -0.3 \end{bmatrix} x_h \\ & & + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w \\ z & = & \begin{bmatrix} 0.3 & 1.5 \\ -0.45 & 0.75 \end{bmatrix} x + \begin{bmatrix} 0.5\rho & -0.5 \end{bmatrix} w \\ y & = & \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1+0.1\rho \end{bmatrix} w \end{array} \tag{4.70}$$

where $\rho(t) = sin(t) \in [-1, 1]$ and $\dot{\rho}(t) \in [-1, 1]$.

All the parameter dependent matrices are expressed onto a basis formed by the functions

$$f_0(\rho) = 1 \qquad f_1(\rho) = \rho \tag{4.71}$$

We use theorems 4.2.1 and 4.2.2 with an uniform gridding of 11 points over the whole parameter space and the results are verified over a denser grid (around 100 points).

Results of Mohammadpour and Grigoriadis [2007a] are depicted in Figure 4.2. In figure 4.3, the evolution of the worst case performance for the delayed filter computed with Theorem 4.2.2 and the memoryless one computed with 4.2.1. As a first analysis, the delayed filter gives better performance than the memoryless one which seems obvious since the information on the delay is used in the delayed filter. As a comparison with the results in Mohammadpour and Grigoriadis [2007a], our results are less conservative and then improves the existing ones (see result of Mohammadpour and Grigoriadis [2007a] in figure 4.2 and proposed results in figure 4.3). It is possible to see that for small delay values both solutions leads to very similar results. The main difference appears for larger delay values for which the worst case disturbance gain is drastically different.

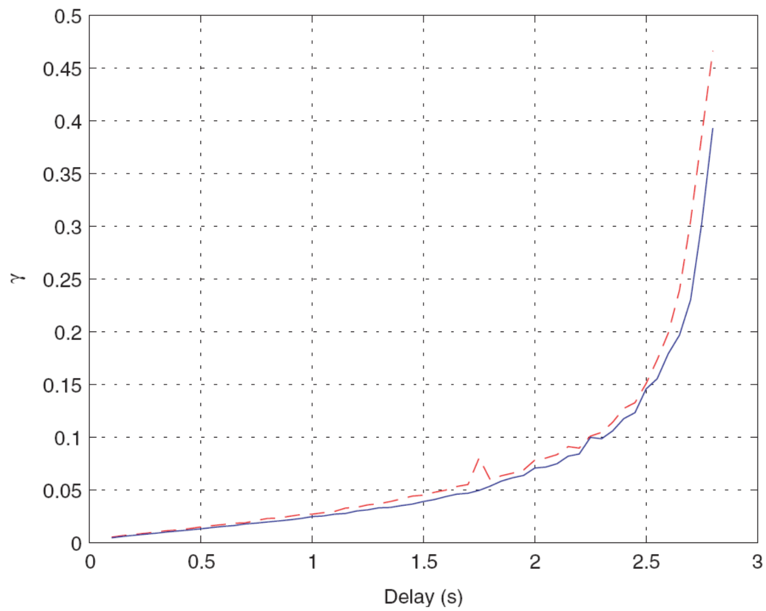

Figure 4.2: Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain) in Mohammadpour and Grigoriadis [2007a]

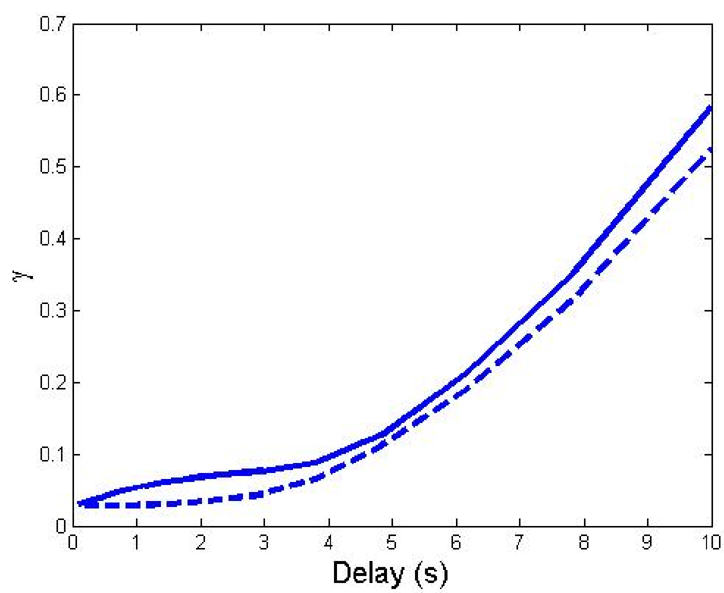

Figure 4.3: Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain)

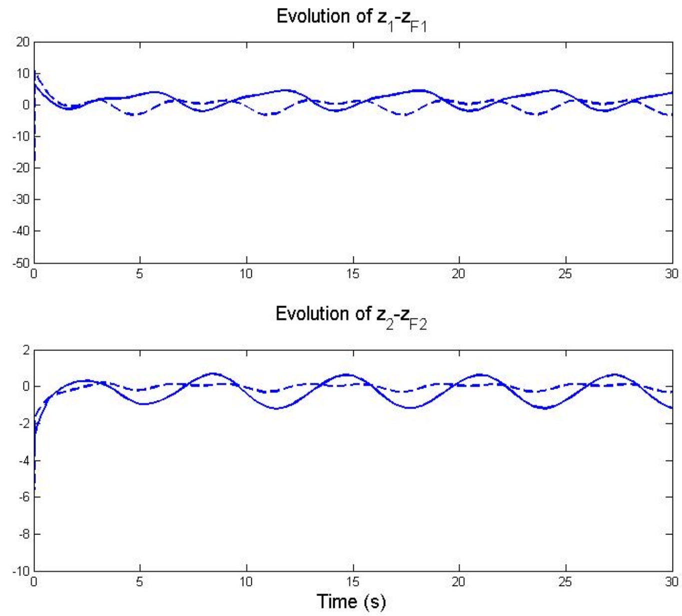

Figure 4.4: Evolution of $z(t) - z_F(t)$ for the delayed filter (dashed) and the memoryless filter (plain)

Figure 4.4 shows the evolution of the error $z(t) - z_F(t)$ for a delay $h = 3$ and a step disturbance of amplitude 20. We can easily see that the delayed filter gives better results than the memoryless one. Consider now system (4.70) with uncertainties defined by:

$$\begin{array}{l} G_x = G_y = 0.1 I_2 \quad G_z = 0 \quad H_x = H_{x_h} = I_2 \\ H_w = \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \end{array} \tag{4.72}$$

according to the notation (4.63). The evolution of the worst-case performance $\mathcal{L}_2$-gain is depicted in figure 4.5.

Figure 4.6 shows the evolution of the error $z(t) - z_F(t)$ for a delay $h = 4.5$, $\Delta(t) = sin(10t)I_2$ and a step disturbance of amplitude 20. The delayed filter achieves a $\mathcal{L}_2$ performance gain of $\gamma_{del} = 0.59$ and the memoryless of $\gamma_{ml} = 0.78$.

4.3 Chapter Conclusion

This chapter has been devoted to the design of observer and filters for both unperturbed and uncertain LPV time-delay systems.

Three types of observers have been synthesized: observers with memory, with both exact and approximate delay value knowledge, and memoryless observers. They have been developed using an algebraic approach generalized from [Darouach, 2001] to the LPV framework. The matrices of the observers are chosen such that the system state and the control input do not affect the evolution of the observation error and that the disturbances are attenuated in the $\mathcal{L}_2$ sense. The set of observers that decouple the error from the system state and the

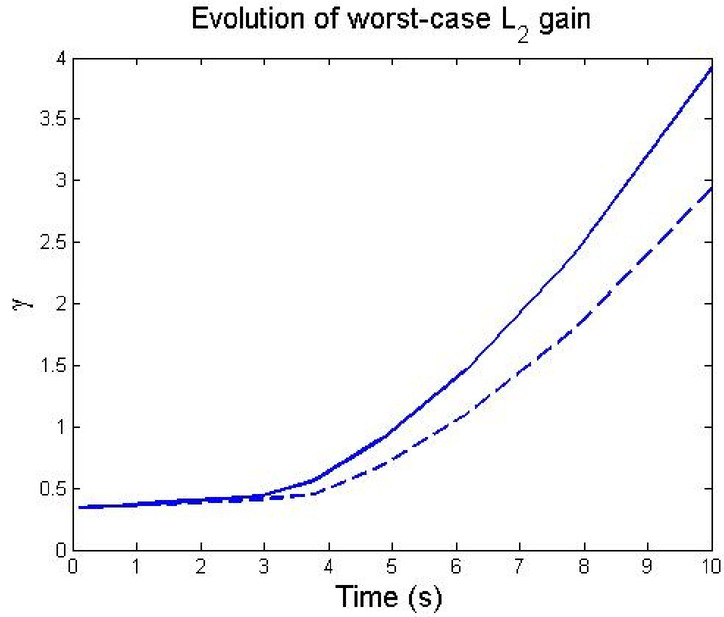

Figure 4.5: Evolution of the worst case $\mathcal{L}_2$ gain for the delayed filter (dashed) and the memoryless filter (plain)

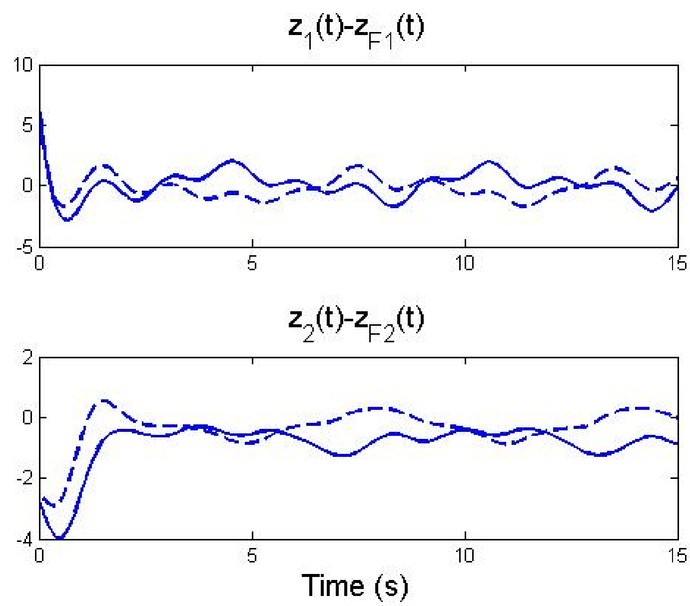

Figure 4.6: Evolution of $z(t) - z_F(t)$ for the delayed filter (dashed) and the memoryless filter (plain)

control input is parametrized through an algebraic equation involving a free parameter to be chosen. This parameter is chosen as a solution of a LMI optimization problem where the $\mathcal{L}_2$ gain from the disturbances to the observation error is minimized. This approach is better suited for certain systems since the matrices acting on the system state in the observation error dynamical model can be set to zero by an appropriate choice of the observer matrices. However, in the uncertain case, these matrices cannot be set to zero due to uncertain terms and, inevitably, the observer has much lower performances. Such observers are developed in Appendix J.

On the second hand, the problem of designing filters for uncertain systems has been addressed in the second section of the chapter. Two types of filters have been considered: memoryless filters and filters with memory (exact delay value knowledge). The design of filters is more simple and direct than observer design, since it is not sought to obtain an estimation error which is independent of the state variables of the system. The constructive existence conditions are given in terms of a convex optimization problem involving LMIs where the $\mathcal{L}_2$ attenuation gain from the disturbances to the filtering error is minimized. Other filters are developed in Appendix J.

Chapter 5

Control of LPV Time-Delay Systems

This chapter is devoted to the control of (uncertain) LPV time-delay systems. Despite of its apparent simplicity the control of LPV time-delay systems is still an open problem. Indeed, in the LMI based approaches, conservatism induced by relaxations (as bounding techniques, model transformations, relaxations of nonlinear terms. . .) is a cause for bad results. There exists a lot of approaches to the control of time-delay systems and a great similarity is the relaxation of coupled terms. Coupled terms (e.g. KR, KP where P, K, R are decision variables) arise very often in any design concerning time-delay systems and prevent the linearization of BMIs into LMIs. For instance, in the descriptor approach [Fridman, 2001], the coupled terms KP_1 and KP_2 (when considering a state-feedback control law) must be relaxed and then the relaxation $P_2 = \varepsilon P_1$ is usually performed where ε is an a priori chosen scalar. This type of relaxation is also needed when the design is done using the free-weighting approach [He et al., 2004]. Most of the approaches are done using the same procedure as follows:

1. Elaborate a stability/performance test based on some method for the open-loop system
2. Substitute the closed-loop system
3. Linearize by congruence, relaxation and change of variable to obtain LMIs.

In this chapter we will propose another strategy by adding a step into this methodology:

1. Develop a stability/performances test for an open loop system
2. If the obtained conditions give rise to coupled terms then they are relaxed using for instance the Finsler's lemma (see Appendix E.16) in order to remove these coupled terms.
3. Substitute the closed-loop system expression in the relaxed version of the stability/performances LMI test.
4. Linearize immediately or by the use of congruence transformations.

It will be shown in this chapter that this methodology gives rise to good results, not only for LTI systems but also for LPV system. Since the results for uncertain LTI systems are numerous compared to those for LPV systems, the methods will be compared with both LTI and LPV methods.

It is worth mentioning that even if the relaxed version of the test without coupled terms is not equivalent to the original test, the conservatism is generally not worse than with classical relaxations and is a good point of our methods. Moreover, our relaxation allows for the use of discretized versions of 'complete' Lyapunov-Krasovskii functionals in the control design framework by relaxing the high number of involved coupled-terms (actually the number of coupled-terms is of close to the order of discretization).

A new approach for the control of time-delay systems with time-varying delays is developed in this chapter. This method allows to find a memoryless controller where the gains of the controller are smoothly scheduled by the delay value or an approximate one. Due to the similarity with gain-scheduled controller synthesis in the LPV framework, this type of controllers is referred to as 'delay-scheduled' controllers. Such a controller is hence midway between memoryless and with memory since it embeds an information on the delay value with any delay terms in the control law expression.

Several of our results have been detailed in the following papers:

- [Briat et al., 2007a] a delay-scheduled state-feedback strategy is designed based on a specific model transformation. In this paper, the computed controller is LPV depends on the value of the delay in a LFT fashion. A paper version [Briat et al., 2007b] is still under review at IEEE Transactions on Automatic Control (2nd round).
- [Briat et al., 2008b] an enhanced delay-scheduled controller approach is developed where the model transformation has been improved and the controller is not in LFT form. The results are then less conservative. A journal version is in revision at Systems and Control Letters.
- A full-block $\mathcal{S}$-procedure approach is provided in [Briat et al., 2008c] where the control of uncertain time-delay systems is solved.
- LPV control for time-delay systems is detailed in [Briat et al., 2008d] where a projection approach is used to provide constructive sufficient conditions for a stabilizing controller.

Some key references of modern control techniques for time-delay systems are recalled below (see also Chapter 2):

Robust control of LTI time-delay systems: [Fu et al., 1998], [Souza and Li, 1999], [Ivanescu et al., 2000], [Moon et al., 2001], [Fridman and Shaked, 2002a], [Wu, 2003], [Suplin et al., 2004], [Jiang and Han, 2005], [Fridman, 2006a], [Suplin et al., 2006], [Fridman, 2006b], [Fridman and Shaked, 2006], [Jiang and Han, 2006], [Xu et al., 2006], [Chen, 2007].

LPV control of LPV/LTI time-delay systems: [Wu and Grigoriadis, 2001], [Wu, 2001], [Zhang and Grigoriadis, 2005]

The first section will be concerned to the synthesis of state-feedback control laws, both memoryless and with memory state-feedback controllers will be explored. Moreover, the uncertainty on the delay-knowledge using state-feedback with memory will be taken into account through a specific Lyapunov-Krasovskii functional. Finally the synthesis of delayed-scheduled state-feedback will be solved. The last section will be devoted to the synthesis of dynamic output feedback controllers. Both observer-based and full-block controllers will be synthesized and compared.

5.1 State-Feedback Control laws

In this section the stabilization of general uncertain LPV time-delay systems of the form

$$\begin{array}{rcl}\dot{x}(t) & = & (A(\rho)+\Delta A(\rho,t))x(t)+(A_h(\rho)+\Delta A_h(\rho,t))x(t-h(t)) \\ & & +(B(\rho)+\Delta B(\rho,t))u(t)+(E(\rho)+\Delta E(\rho,t))w(t) \\ z(t) & = & C(\rho)x(t)+C_h(\rho)x(t-h(t))+D(\rho)u(t)+F(\rho)w(t)\end{array} \tag{5.1}$$

using control a control law of the form

$$u(t) = K(\rho)x(t) + K_h(\rho)x(t-d(t)) \tag{5.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ and $h(t) \in \mathscr{H}_1^\circ$ are respectively the state of the system, the control input, the disturbances, the controlled outputs and the system delay. The set $\mathscr{H}_1^\circ$ is given by

$$\mathscr{H}_1^\circ := \left\{ h \in \mathcal{C}^1(\mathbb{R}_+,[0,h_{max}]) : \ |\dot{h}| < \mu \right\} \tag{5.3}$$

The controller delay $d(t) = h(t) + \varepsilon(t)$, $\varepsilon(t) \in [-\delta, \delta]$ is not defined a priori and may admit fast variations. The uncertain terms are given as:

$$\left[\begin{array}{cccc} \Delta A(\rho) & \Delta A_h(\rho) & \Delta B(\rho) & \Delta E(\rho) \end{array}\right] = G(\rho)\Delta(t) \left[\begin{array}{cccc} H_A(\rho) & H_{A_h}(\rho) & H_B(\rho) & H_E(\rho) \end{array}\right]$$

where matrices $G(\rho), H_A(\rho), H_{A_h}(\rho), H_B(\rho), H_E(\rho)$ are full rank matrices and $\Delta(t)^T\Delta(t) \preceq I$. Whenever $K_h(\cdot) = 0$, the controller is said to be *memoryless* while when $K_h(\cdot) \neq 0$, it is said to be *with memory*.

Problem 5.1.1 *The problem is to find a control law of the form (5.2) which asymptotically stabilizes system (5.1) for all $h \in \mathscr{H}_1^\circ$ and all $\Delta(t)$ such that $\Delta(t)^T\Delta(t) \preceq I$ and minimizes $\gamma > 0$ such that*

$$||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

In the following the following main problems will be addressed:

1. The stabilization of the system by a memoryless state-feedback control law of the form $u(t) = K(\rho)x(t)$.
2. The stabilization of the system by a state-feedback control law with memory of the form $u(t) = K(\rho)x(t) + K_h(\rho)x(t-h(t))$ with exact delay value.
3. The stabilization of the system by a state-feedback control law with memory of the form $u(t) = K(\rho)x(t) + K_h(\rho)x(t-h(t))$ with approximate delay value.
4. The stabilization of the system using delay-scheduled controllers.

In this chapter, the following terminology will be used: by *relaxed* we mean that the initial stability and performances test has been modified in order to remove all potential coupled terms. The term *simple* means that the Lyapunov-Krasovskii functional is 'simple' in the sense that the structure of the functional involves constant matrices only. This term has to be put in parallel with the term *complete*. Finally, the term *discretized* means that a *complete* Lyapunov-Krasovskii functional involving function as decision variables has been discretized in order to get tractable stability conditions (see [Gu et al., 2003]).

5.1.1 Memoryless State-Feedback Design - Relaxed Simple Lyapunov-Krasovskii functional

The most simple state-feedback controller that can be designed is the memoryless one (i.e. $K_h(\cdot) = 0$), where only the instantaneous state $x(t)$ is used to compute the control input $u(t)$. In such a case, the closed-loop system is governed by

$$\begin{array}{rcl} \dot{x}(t) & = & (A(\rho) + \Delta A(\rho,t) + [B(\rho) + \Delta B(\rho,t)]K(\rho))x(t) \\ & & +(A_h(\rho) + \Delta A_h(\rho,t))x(t-h(t)) + (E(\rho) + \Delta E(\rho,t))w(t) \\ z(t) & = & (C(\rho) + D(\rho)K(\rho))x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t) \end{array} \tag{5.4}$$

The solution is given using the relaxed version of the LMI condition obtained from the simple Lyapunov-Krasovskii functional as described in Section 3.5.2. Hence by a substitution of the closed-loop into the relaxed LMI, a simple existence test of the controller can be provided in terms of parameter dependent LMIs.

Theorem 5.1.2 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes system (5.1) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}^n_{++}$, constant matrices $Y \in \mathbb{R}^{n\times n}, \tilde{Q}, \tilde{R} \in \mathbb{S}^n_{++}$, a matrix function $V : U_\rho \to \mathbb{R}^{m\times n}$, a scalar $\gamma > 0$ and scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the LMIs*

$$\begin{bmatrix} \tilde{U}_{11}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} & 0 \\ \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 & Y^TH_{A_h}(\rho)^T \\ \star & \star & -\gamma I_p F(\rho)^T & 0 & 0 & H_E(\rho)^T & \\ \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} & 0 \\ \star & \star & \star & \star & \star & \tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{5.5}$$

and

$$\mathrm{Ker}[\mathcal{U}_1(\rho)]^T \Xi(\rho,\nu) \mathrm{Ker}[\mathcal{U}_1(\rho)] \prec 0 \tag{5.6}$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Xi(\rho,\nu)$ is defined by

$$\begin{bmatrix} \tilde{U}_{11}(\rho) & \tilde{P}(\rho) + A(\rho)Y & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} & 0 \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & Y^TC(\rho)^T & 0 & 0 & Y^TH_A(\rho)^T \\ \star & \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 & Y^TH_{A_h}(\rho)^T \\ \star & \star & \star & -\gamma I_p F(\rho)^T & 0 & 0 & H_E(\rho)^T & \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & \tilde{R} & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\varepsilon(\rho)I \end{bmatrix}$$

$$\begin{array}{rcl} \tilde{U}_{11}(\rho) & = & -(Y + Y^T) + \varepsilon(\rho)G(\rho)G(\rho)^T \\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho \tilde{P}(\rho)\nu \\ \tilde{U}_{33} & = & -(1-\mu)\tilde{Q} - \tilde{R} \\ \mathcal{U}_1(\rho) & = & \begin{bmatrix} B(\rho)^T & 0 & 0 & 0 & D(\rho)^T & 0 & 0 & H_B(\rho)^T \end{bmatrix} \\ \mathcal{U}_2 & = & \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

In this case a suitable controller is given by the expression:

$$\begin{array}{rcl} K(\rho) & = & -\tau\mathcal{U}_1(\rho)\Psi(\rho,\dot{\rho})\mathcal{U}_2^T(\mathcal{U}_2\Psi(\rho,\dot{\rho})\mathcal{U}_2^T)^{-1}Y^{-1} \\ \tau & > & 0 \qquad \text{such that } \Psi(\rho,\dot{\rho}) = (\tau\mathcal{U}_1(\rho)^T\mathcal{U}_1(\rho) - \Xi(\rho,\dot{\rho}))^{-1} \succ 0 \end{array} \tag{5.7}$$

or by solving the LMI

$$\tilde{\Xi}(\rho,\nu) + \bar{B}(\rho)K(\rho)Y\bar{C} + [\bar{B}(\rho)K(\rho)Y\bar{C}]^T \prec 0 \tag{5.8}$$

in $K(\rho)$. Moreover with such a control law, the closed-loop system satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$.

Proof: *The proof is based on an application of lemma 3.5.2. Substituting matrices of the closed-loop system (5.4) into LMI (3.95), we get*

$$\bar{\Psi}(\rho,\nu) + \bar{\mathcal{G}}(\rho)^T\Delta(t)\bar{\mathcal{H}}(\rho) + (\star)^T \prec 0 \tag{5.9}$$

where $\bar{\Psi}(\rho,\nu)$ is defined by

$$\begin{bmatrix} -X^H & U_{12}(\rho) & X^TA_h(\rho) & X^TE(\rho) & 0 & X^T & h_{max}R \\ \star & U_{22}(\rho,\nu) & R & 0 & U_{25}(\rho)^T & 0 & 0 \\ \star & \star & U_{33}(\rho) & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix}$$

with

$$\begin{array}{rcl} U_{12}(\rho) & = & P(\rho) + X^T(A(\rho) + B(\rho)K(\rho)) \\ U_{25}(\rho) & = & (C(\rho) + D(\rho)K(\rho)) \\ U_{22}(\rho,\nu) & = & -P(\rho) + Q - R + \partial_\rho P(\rho)\nu \\ U_{33} & = & -(1-\mu)Q - R \\ \bar{\mathcal{G}}(\rho) & = & \begin{bmatrix} G(\rho)^TX & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \bar{\mathcal{H}}(\rho) & = & \begin{bmatrix} 0 & H_A(\rho) + H_B(\rho)K(\rho) & H_{A_h}(\rho) & H_E(\rho) & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Due to the structure of LMI (5.9) it is possible to apply the bounding lemma (see Appendix E.14) and hence we obtain the following LMI

$$\bar{\Psi}(\rho,\nu) + \varepsilon(\rho)\bar{\mathcal{G}}(\rho)^T\bar{\mathcal{G}}(\rho) + \varepsilon(\rho)^{-1}\bar{\mathcal{H}}(\rho)^T\bar{\mathcal{H}}(\rho)^T \prec 0$$

which involves an additional scalar function $\varepsilon(\rho)$.

Then perform a congruence transformation with respect to matrix $\mathrm{diag}(Y,Y,Y,I,I,Y,Y)$ *where $Y = X^{-1}$ and using the change of variable $V(\rho) = K(\rho)Y$ we get the inequality:*

$$\Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) + \varepsilon(\rho)^{-1}\mathcal{H}(\rho)^T\mathcal{H}(\rho) \prec 0 \tag{5.10}$$

where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} -(Y+Y^T) & \tilde{U}_{12}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

in which

$$\begin{array}{rcl}
\tilde{U}_{12}(\rho) & = & \tilde{P}(\rho)+A(\rho)Y+B(\rho)V(\rho) \\
\tilde{U}_{25}(\rho) & = & [C(\rho)Y+D(\rho)V(\rho)]^T \\
\tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho)+\tilde{Q}-\tilde{R}+\partial_\rho\tilde{P}(\rho)\nu \\
\tilde{U}_{33} & = & -(1-\mu)\tilde{Q}-\tilde{R} \\
\tilde{P}(\rho) & = & Y^TP(\rho)Y \\
\tilde{Q} & = & Y^TQY \\
\tilde{R} & = & Y^TRY \\
\mathcal{G}(\rho) & = & \left[\begin{array}{ccccccc} G(\rho)^T & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\
\mathcal{H}(\rho) & = & \left[\begin{array}{ccccccc} 0 & H_A(\rho)Y+H_B(\rho)V(\rho) & H_{A_h}(\rho)Y & H_E(\rho) & 0 & 0 & 0 \end{array}\right]
\end{array}$$

The latter inequality is not a LMI due to the term $\varepsilon(\rho)^{-1}\mathcal{H}(\rho)^T\mathcal{H}(\rho)$ but using the Schur complement formula (see Appendix E.15) we get the following equivalent LMI formulation:

$$\left[\begin{array}{cc} \Psi(\rho,\nu)+\varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho) \end{array}\right] \prec 0 \tag{5.11}$$

Now rewrite the latter LMI as

$$\Xi(\rho,\nu)+\mathcal{U}_1(\rho)^TV(\rho)\mathcal{U}_2+(\star)^T \prec 0 \tag{5.12}$$

where $\Xi(\rho,\nu)$ is defined by

$$\left[\begin{array}{cccccccc}
\tilde{U}_{11}(\rho) & \tilde{U}_{12}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} & 0 \\
\star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & Y^TC(\rho)^T & 0 & 0 & Y^TH_A(\rho)^T \\
\star & \star & \tilde{U}_{33} & 0 & Y^TC_h(\rho)^T & 0 & 0 & Y^TH_{A_h}(\rho)^T \\
\star & \star & \star & -\gamma I_pF(\rho)^T & 0 & 0 & H_E(\rho)^T & \\
\star & \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} & 0 \\
\star & \star & \star & \star & \star & \star & \tilde{R} & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\varepsilon(\rho)I
\end{array}\right]$$

with

$$\begin{array}{rcl}
\tilde{U}_{11}(\rho) & = & -(Y+Y^T)+\varepsilon(\rho)G(\rho)G(\rho)^T \\
\tilde{U}_{12}(\rho) & = & \tilde{P}(\rho)+A(\rho)Y \\
\mathcal{U}_1(\rho) & = & \left[\begin{array}{cccccccc} B(\rho)^T & 0 & 0 & 0 & D(\rho)^T & 0 & 0 & H_B(\rho)^T \end{array}\right] \\
\mathcal{U}_2 & = & \left[\begin{array}{cccccccc} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\end{array}$$

Since $V(\rho)$ is a free variable then the projection lemma applies (see appendix E.18) and we get conditions of theorem 5.1.2. The controller can be constructed using either

$$\Xi(\rho,\nu)+\mathcal{U}_1(\rho)^TV(\rho)\mathcal{U}_2+(\star)^T \prec 0$$

or by applying the algebraic relations given in Appendix A.9. □

The latter theorem is a theorem stating the existence of a parameter dependent matrix gain $K(\rho)$ such that system (5.4) is asymptotically stable and $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$. The advantage of such a result is the possibility of constructing the controller from algebraic equations involving only known matrices computed from the solution of the LMI problem. The advantage of such a construction is that the computed controller will fit exactly the predicted performances. On the other hand, such a controller may depend on the parameter derivative (as emphasized by the relaxation (5.7)) making the controller (in most of the cases) unimplementable in practice. Three solutions are offered to overcome this difficulty:

1. Choose a constant matrix P which removes the parameter derivative term but increasing the conservatism of the approach by tolerating arbitrarily fast varying parameters (quadratic stability).

2. Construct the controller using SDP (5.8) but in this case a specific structure must be affected to the controller (for instance polynomial in ρ) which may result in a deterioration of performances. Moreover, since the structure of the controller is chosen by the designer after solving for the other matrices (i.e. $\tilde{P}, \tilde{Q}, \tilde{R}, Y$), then the SDP may have no solution if the controller is not sufficiently complex.

This nonequivalence is a consequence of the parameter varying nature of the matrices involved in the LMIs and the will of considering robust stability. The following result solves this problem of non-equivalence between the set of LMIs of Theorem 5.1.2 and the SDP (5.8) by providing a global approach where only one LMI has to be solved.

Theorem 5.1.3 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes system (5.1) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}^n_{++}$, a matrix function $V(\rho) : U_\rho \to \mathbb{R}^{m\times n}$, constant matrices $Y \in \mathbb{R}^{n\times n}, \tilde{Q}, \tilde{R} \in \mathbb{S}^n_{++}$, a matrix function $V : U_\rho \to \mathbb{R}^{m\times n}$, a scalar $\gamma > 0$ and a scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{5.13}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} -Y^H & \tilde{U}_{12}(\rho) & A_h(\rho)Y & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & C_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

$$\begin{array}{rcl} \tilde{U}_{12}(\rho) & = & \tilde{P}(\rho) + A(\rho)Y + B(\rho)V(\rho) \\ \tilde{U}_{25}(\rho) & = & [C(\rho)Y + D(\rho)V(\rho)]^T \\ \mathcal{G}(\rho) & = & \begin{bmatrix} G(\rho)^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathcal{H}(\rho) & = & \begin{bmatrix} 0 & H_A(\rho)Y + H_B(\rho)V(\rho) & H_{A_h}(\rho) & H_E(\rho) & 0 & 0 & 0 \end{bmatrix} \\ \tilde{U}_{22}(\rho,\nu) & = & -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho P(\rho)\nu \\ \tilde{U}_{33} & = & -(1-\mu)\tilde{Q} - \tilde{R} \end{array}$$

In this case, a suitable control law is given by $u(t) = K(\rho)x(t)$ *where* $K(\rho) = V(\rho)Y^{-1}$ *and the closed-loop system (5.4) satisfies*

$$||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

Proof: The proof follows the same lines as for the proof of Theorem 5.1.2 but stops just before the application of the projection lemma. □

As for theorem 5.1.2, the structure of the controller is fixed by the designer through the choice of the structure of $V(\rho)$ and may result in conservative results if the structure is sufficiently complex or not. On the other hand, Theorem 5.1.3 is easier to use since the controller synthesis is made in one step only while the number of steps for controller computation using Theorem 5.1.2 is two. The interest of Theorem 5.1.2 is to provide in the first step (the solution of the projected inequalities) the minimal γ that can be expected using this Lyapunov-Krasovskii functional (modulo the conservatism induced by the relaxation) whatever the structure of the controller is. Hence, this result may be used to tune the complexity of the controller using Theorem 5.1.3.

Remark 5.1.4 *Another result has been developed in [Briat et al., 2008c] for uncertain LTI time-delay systems using the full-block $\mathcal{S}$-procedure approach [Scherer, 2001, Wu, 2003]. The results of [Briat et al., 2008c] can be extended to the LPV framework by authorizing a parameter dependent Lyapunov function and parameter dependent scalings.*

5.1.2 Memoryless State-Feedback Design - Relaxed Discretized Lyapunov-Krasovskii functional

The Lyapunov-Krasovskii functional used to derive conditions of Theorems 5.1.2 and 5.1.3 is simple in the sense that the decision matrices are in finite number and is small. Thus latter results can be enhanced and completed by considering more a complex functional, where functions have to be found. Due to the difficulty to find numerically such functions, the functions are then approximated by piecewise constant functions and the obtained functional is called 'the discretized version' of such a functional. On each interval where the functions are constant, are associated decision matrices and hence it seems obvious that such an approach would result in less conservative results than by using a simple Lyapunov-Krasovskii functional due to a greater number of decision matrices.

The following result is obtained by the use of the relaxation of the discretized Lyapunov-Krasovskii functional described in Theorem 3.6.4 of Section 3.6.2. The methodology is usual: substitute the closed-loop system in the LMI and then turn the BMI problem into a LMI one through the use of congruence transformation.

Theorem 5.1.5 *There exists a state-feedback control law of the form* $u(t) = K(\rho)x(t)$ *which asymptotically stabilizes system (5.1) with* $h(t) \in \mathscr{H}_1^\circ$ *if there exist a continuously differentiable matrix function* $\tilde{P} : U_\rho \to \mathbb{S}^n_{++}$*, constant matrices* $Y \in \mathbb{R}^{n\times n}, \tilde{Q}_i, \tilde{R}_i \in \mathbb{S}^n_{++}$*,* $i = 0, \ldots, N-1$*, a matrix function* $V : U_\rho \to \mathbb{R}^{m\times n}$*, a scalar* $\gamma > 0$ *and a scalar function* $\varepsilon : U_\rho \to \mathbb{R}_{++}$ *such that the LMIs*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G}(\rho) & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{5.14}$$

holds for all $(\rho, \rho_h, \nu) \in U_\rho \times U_{\rho_h} \times U_\nu$ *and where*

$$\Psi(\rho,\nu) = \left[\begin{array}{cccc|ccc} -Y^H & \tilde{U}_{12}(\rho) & 0 & Y & \bar{h}_1\tilde{R}_0 & \dots & \bar{h}_1\tilde{R}_{N-1} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{U}_{23}(\rho) & 0 & 0 & \dots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \dots & 0 \\ \star & \star & \star & -\tilde{P}(\rho) & -\bar{h}_1\tilde{R}_0 & \dots & -\bar{h}_1\tilde{R}_{N-1} \\ \hline \star & \star & \star & \star & & -\operatorname{diag}_i \tilde{R}_i & \end{array}\right]$$

$$\tilde{U}_{22} = \begin{bmatrix} \tilde{U}'_{11} & \tilde{R}_0 & 0 & 0 & \dots & 0 & 0 \\ \star & \tilde{N}_1^{(1)} & \tilde{R}_1 & 0 & \dots & 0 & 0 \\ \star & \star & \tilde{N}_2^{(1)} & \tilde{R}_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \tilde{R}_{N-1} & 0 \\ & & & & & \tilde{N}^{(2)} & 0 \\ \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{bmatrix}$$

and

$$\begin{aligned}
\tilde{U}'_{11} &= \partial_\rho \tilde{P}(\rho)\dot{\rho} - \tilde{P}(\rho) + \tilde{Q}_0 - \tilde{R}_0 \\
\tilde{N}_i^{(1)} &= -(1 - i\mu_N)\tilde{Q}_{i-1} + (1 + i\mu_N)\tilde{Q}_i - \tilde{R}_{i-1} - \tilde{R}_i \\
\tilde{N}^{(2)} &= -(1-\mu)\tilde{Q}_{N-1} - \tilde{R}_{N-1} \\
\tilde{U}_{12}(\rho) &= \begin{bmatrix} \tilde{P}(\rho) + A(\rho)Y + B(\rho)V(\rho) & 0 & \dots & 0 & A_h(\rho)Y & \dots & 0 & E(\rho) \end{bmatrix} \\
\tilde{U}_{23}(\rho) &= \begin{bmatrix} C(\rho)Y + D(\rho)V(\rho) & 0 & \dots & 0 & C_h(\rho) & F(\rho) \end{bmatrix}^T \\
\mathcal{G}(\rho) &= \left[\begin{array}{c|cccccc|c|ccc} 0 & G(\rho)^T & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{array}\right] \\
\mathcal{H}(\rho) &= \left[\begin{array}{c|cccccc|c|ccc} 0 & H_A(\rho)Y + H_B(\rho)V(\rho) & 0 & \dots & 0 & H_{A_h}(\rho)Y & H_E(\rho) & 0 & 0 & \dots & 0 \end{array}\right]
\end{aligned}$$

Proof: *The proof is similar as for the proof of theorem 5.1.3 but using lemma 3.6.4.* □

It is important to emphasize that without the use of any relaxation which decouples the matrix A from the decision matrices $P(\rho), R_i$, the controller synthesis directly using Theorem 3.6.2 would be a difficult problem. Indeed, in this case and similarly as in other methods (see for instance [Fridman and Shaked, 2002b]), the only available relaxation would be $P(\rho) = \varepsilon_i(\rho)R_i$ for some positive scalar functions $\varepsilon_i(\rho)$. Such a relaxation would greatly increase the conservatism of the approach until removing all the contributions of the use of the discretized Lyapunov-Krasovskii.

The choice of $P(\rho)$ is arbitrary and, based on the experience of different authors, it is important to 'mimic' the system dependence on the parameters, e.g. if $A(\rho) = A_2\rho^2 + A_1\rho + A_0$ then $P(\rho)$ must be at least of order 2. Moreover, the controller dependence must be close to this order to.

5.1.3 Memoryless State-Feedback Design - Simple Lyapunov-Krasovskii functional

Part of this thesis pertains on the difference between a priori (before the substitution of closed-loop system matrices in the stability/performances) and a posteriori relaxations (after substitution) in the design of controllers, observers and filters for (uncertain) LTI and LPV time-delay systems. It is shown that, in most of the cases, a priori relaxation leads to better results than a posteriori relaxations. This section is about the use of the original simple stability/performances test without any a priori relaxation and we will show that it is possible to define a relevant relaxation scheme associated with an algorithm whose structure is close to the D-K iteration technique used in μ-synthesis [Gahinet et al., 1995]. The D-K iteration algorithm is a very simple algorithm used to solve BMI problems where the coupled variables are found alternatively.

The proposed approach developed in this section is based on the properties of the adjoint system of a time-delay system [Bensoussan et al., 2006, Suplin et al., 2006]. The interest of adjoint systems is to allow for the computation of controllers without any congruence transformation on the matrix inequalities. While this is not very interesting for finite dimensional linear systems, it is of great importance for time-delay systems for which a large number of decision matrices are involved in the stability conditions. Indeed, linearizing congruence transformations on matrix inequalities may not exist in time-delay system framework (for instance there exists no linearizing congruence transformation for state-feedback design using LMI (3.80) of Lemma 3.5.1).

5.1.3.1 About adjoint systems of LPV systems

The first property of adjoint system of a LTI system is that the stability of the adjoint is equivalent to the stability of the original system. Moreover, the $\mathcal{H}_\infty$-norm is also preserved by considering the adjoint. However, does that statement hold when the system is time-varying (LTV or LPV) ? Actually, this is not a trivial equation since the outputs are computed by integrating time-varying matrices and then for a given input, the outputs of the original and the adjoint systems are different. Thus they have different energies.

However, in the light of the use of the dualization lemma (see Appendix E.13) for LTV/LPV systems expressed under LFT forms, it turns out that the $\mathcal{L}_2$-induced norm is preserved by considering the adjoint. However, the worst-case signal (the signal for which the $\mathcal{L}_2$-induced norm is effectively attained) will be different for the original and the adjoint system.

Let us consider the system

$$\begin{array}{rcl} \dot{x}(t) & = & \mathcal{A}(\rho)x(t) + \mathcal{B}(\rho)w_1(t) \\ z_1(t) & = & \mathcal{C}(\rho)x(t) + \mathcal{D}(\rho)w_1(t) \end{array} \tag{5.15}$$

which is rewritten in the LFT form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B_0w_0(t) + B_1w_1(t) \\ z_0(t) & = & C_0x(t) + D_{00}w_0(t) + D_{01}w_1(t) \\ z_1(t) & = & C_1x(t) + D_{10}w_0(t) + D_{11}w_1(t) \\ w_0(t) & = & \Theta(\rho)z_0(t) \end{array} \tag{5.16}$$

For such a system, the adjoint is given by the expression:

$$\begin{array}{rcl} \dot{\tilde{x}}(t) & = & A^T\tilde{x}(t) + C_0^T\tilde{w}_0(t) + C_1^T\tilde{w}_1(t) \\ \tilde{z}_0(t) & = & B_0^T\tilde{x}(t) + D_{00}^T\tilde{w}_0(t) + D_{10}^T\tilde{w}_1(t) \\ \tilde{z}_1(t) & = & B_1^T\tilde{x}(t) + D_{01}^T\tilde{w}_0(t) + D_{11}^T\tilde{w}_1(t) \\ \tilde{w}_0(t) = \Theta(\rho)^T\tilde{z}_0(t) & & \end{array} \tag{5.17}$$

Since every LPV system can be turned into an equivalent 'LFT' system, this approach is very general to demonstrate that the $\mathcal{L}_2$-norm of (5.16) and (5.17) coincides. The following results shows the identity:

Theorem 5.1.6 *Let us consider system (5.15) and (5.16), then the following statements are equivalent:*

1. *The LPV system is quadratically stable if and only if there exist $P \in \mathbb{S}^n_{++}$, $F \in \mathbb{S}^{2n_0}$ and a scalar $\gamma > 0$ such that following LMIs*

$$\begin{bmatrix} I & 0 & 0 \\ A & B_0 & B_1 \\ \hline 0 & I & 0 \\ C_0 & D_{00} & D_{01} \\ \hline 0 & 0 & I \\ C_1 & D_{10} & D_{11} \end{bmatrix}^T \left[\begin{array}{cc|c|cc} 0 & P & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & F & 0 & 0 \\ \hline 0 & 0 & 0 & -\gamma I & 0 \\ 0 & 0 & 0 & 0 & \gamma^{-1} I \end{array}\right] \begin{bmatrix} I & 0 & 0 \\ A & B_0 & B_1 \\ \hline 0 & I & 0 \\ C_0 & D_{00} & D_{01} \\ \hline 0 & 0 & I \\ C_1 & D_{10} & D_{11} \end{bmatrix} \prec 0 \quad (5.18)$$

$$\begin{bmatrix} \Theta(\rho) \\ I \end{bmatrix}^T F \begin{bmatrix} \Theta(\rho) \\ I \end{bmatrix} \succ 0 \quad (5.19)$$

holds for all $\rho \in U_\rho$. In this case, the system satisfies $||z||_{\mathcal{L}_2} < \gamma ||w||_{\mathcal{L}_2}$.

2. *LPV system is quadratically asymptotically stable if and only if there exist $\tilde{P} \in \mathbb{S}^n_{++}$, $\tilde{F} \in \mathbb{S}^{2n_0}$ and a scalar $\gamma > 0$ such that following LMIs*

$$\begin{bmatrix} A^T & C_0^T & C_1^T \\ I & 0 & 0 \\ \hline B_0^T & D_{00}^T & D_{10}^T \\ 0 & I & 0 \\ \hline B_1^T & D_{01}^T & D_{11}^T \\ 0 & 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c|cc} 0 & \tilde{P} & 0 & 0 & 0 \\ \tilde{P} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{F} & 0 & 0 \\ \hline 0 & 0 & 0 & -\gamma^{-1} I & 0 \\ 0 & 0 & 0 & 0 & \gamma I \end{array}\right] \begin{bmatrix} A^T & C_0^T & C_1^T \\ I & 0 & 0 \\ \hline B_0^T & D_{00}^T & D_{10}^T \\ 0 & I & 0 \\ \hline B_1^T & D_{01}^T & D_{11}^T \\ 0 & 0 & I \end{bmatrix} \succ 0 \quad (5.20)$$

$$\begin{bmatrix} -I \\ \Theta(\rho)^T \end{bmatrix}^T F \begin{bmatrix} -I \\ \Theta(\rho)^T \end{bmatrix} \succ 0 \quad (5.21)$$

holds for all $\rho \in U_\rho$. In this case, the system satisfies $||z||_{\mathcal{L}_2} < \gamma ||w||_{\mathcal{L}_2}$.

Moreover, we have the following relations between the matrices:

$$\begin{aligned} \tilde{P} &= P^{-1} \\ \tilde{F} &= F^{-1} \end{aligned}$$

Proof: *Statement 1 can be obtained by applying the full-block $\mathcal{S}$-procedure on LFT system (5.16). See Appendix E.12, Section 1.3.4.4 or [Scherer, 2001] for more details. Statement can be proved by applying the dualization lemma (see Appendix E.13 or [Scherer, 2001]) on LMIs (5.18) and (5.19).* □

Actually, it is difficult to see that it suffices to replace the original system matrices by adjoint matrices into the matrix inequality (5.18) to obtain (5.20). This motivates the introduction of the following corollary where we have assumed that we have $\Theta(\rho)^T\Theta(\rho) \leq I$ and $F = \text{diag}(-I_{n_0}, I_{n_0})$:

Corollary 5.1.7 *Let us consider system (5.15) and (5.16), then the following statements are equivalent:*

1. *The LPV system is quadratically stable if there exist $P \in \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that following LMIs*

$$\begin{bmatrix} PA + A^TP & PB_0 & PB_1 & C_1^T & C_0^T \\ \star & -I & 0 & D_{10}^T & D_{00}^T \\ \star & \star & -\gamma I & D_{11}^T & D_{01}^T \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -I \\ \end{bmatrix} \prec 0 \tag{5.22}$$

2. *The LPV system is quadratically stable if there exist $\tilde{P} \in \mathbb{S}^n_{++}$ and a scalar $\gamma > 0$ such that following LMIs*

$$\begin{bmatrix} PA^T + AP & PC_0^T & PC_1^T & B_1 & B_0 \\ \star & I & 0 & D_{01} & D_{00} \\ \star & \star & -\gamma I & D_{11}^T & D_{10} \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -I \\ \end{bmatrix} \prec 0 \tag{5.23}$$

Proof: *The proof is done by expanding the inequalities and the above matrix inequalities are obtained modulo Schur complement arguments (see Appendix E.15).* □

From this result, it is possible to conclude that the $\mathcal{L}_2$-induced norm is identical for a time-varying system and its adjoint since the same LMI structure is feasible for both system. Roughly speaking, it suffices to substitute the adjoint system in the original stability condition. This means that we can strongly think that it is also the case for time-delay systems. This has been done in the case of LTI system in [Suplin et al., 2006] and this means that it is also the case for uncertain LTI time-delay systems with constant uncertainties. In the case of time-varying uncertainties it has been shown in [Wu, 2003] using the dualization lemma (see Appendix E.13) that the delay-independent stability with $\mathcal{L}_2$ performances is preserved by considering the adjoint system. Although the dualization lemma provides an efficient and strongly theoretical way to deal correctly with adjoint systems, the rank condition (see Appendix E.13) is unfortunately rarely satisfied while considering time-delay systems and this makes the use of the adjoint a difficult problem in the context of LPV time-delay systems.

5.1.3.2 LPV Control of LPV time-delay systems using adjoint

One of our papers [Briat et al., 2008c], provides a solution to the state-feedback stabilization problem of uncertain time-delay systems and it is shown that adjoint of delay systems may involve delayed uncertainties and delayed loop inputs creating then difficulties and leading to some conservatism when the delayed state is affected by uncertainties. A solution is provided using the projection lemma (see Appendix E.18) and the cone-complementary algorithm [Ghaoui et al., 1997] used here to relax a non-convex (even concave) term in a matrix inequality similarly as in [Chen and Zheng, 2006]. Since this paper only deals with uncertain systems and not LPV, this will not be explained here but such an approach can be generalized to the LPV framework by introducing parameter dependent matrices in the Lyapunov-Krasovskii functionals and by authorizing the scalings (separators) to be parameter dependent. On the other hand, this makes the cone complementary algorithm unapplicable since this algorithm can only be applied on constant matrices while we are in presence of parameter dependent matrices.

In what follows, we propose a method to solve this problem which has been proposed in [Briat et al., 2008d]. The idea of the method is the following: first of all the LMI (3.80) of Lemma 3.5.1 (Section 3.5.1), obtained from a simple parameter dependent Lyapunov-Krasovskii functional, is considered. This LMI has two coupled terms $P(\rho)A(\rho)$ and $RA(\rho)$ which means that if the closed-loop is substituted into, then exact linearization by congruence transformations is not possible (i.e. $P(\rho)(A(\rho)+B(\rho)K(\rho))$ and $R(A(\rho)+B(\rho)K(\rho))$). Since it is not wished to simplify the relation by fixing $P(\rho)=\alpha(\rho)R$ for instance, another path is considered. This path is the use of the projection lemma whose action is to remove the controller matrix from the inequalities. If the projection lemma were applied directly on the original system then it would lead to a projection with respect to a basis of the kernel of matrices

$$\begin{array}{rcl} M_1 & = & \begin{bmatrix} I & 0 & 0 & 0 & 0 \end{bmatrix} \\ M_2(\rho) & = & \begin{bmatrix} B(\rho)^T P(\rho) & 0 & 0 & D(\rho)^T & h_{max}B(\rho)^T R \end{bmatrix} \end{array}$$

since we have an inequality of the form

$$\Psi_o(\rho,\dot{\rho}) + M_2(\rho)^T K(\rho) M_1 + (\star)^T \prec 0.$$

From the expression of $M_2(\rho)$ we can see that

$$\mathrm{Ker}[M_2(\rho)] = \mathrm{diag}(P(\rho)^{-1}, I, I, I, h_{max}^{-1}R^{-1})\mathrm{Ker}\begin{bmatrix} B(\rho)^T & 0 & 0 & D(\rho)^T & B(\rho)^T \end{bmatrix}$$

and hence a congruence transformation with respect to $\mathrm{diag}(P(\rho)^{-1}, I, I, I, h_{max}^{-1}R^{-1})$ has to be performed and leads to nonlinear terms in the resulting conditions. Moreover, these nonlinear terms cannot be relaxed since the kernel $Z(\rho)$ surrounds the matrix $\mathcal{NL}(\cdot)$ containing these nonlinear terms

$$Z(\rho)^T \mathcal{NL}\left\{X(\rho), Q, R, X(\rho)QX(\rho), X(\rho)RX(\rho), R^{-1}, X(\rho)R, \rho, \dot{\rho}\right\} Z(\rho) \prec 0$$

where $X(\rho) = P(\rho)^{-1}$,$Z(\rho) = \mathrm{Ker}\begin{bmatrix} B(\rho)^T & 0 & 0 & D(\rho)^T & B(\rho)^T \end{bmatrix}$ Such a configuration prevents any congruence transformations in order to linearize the inequality and the high number of nonlinear terms indicates that considering the original system with this stability/performance test is not a good idea.

Let us consider now the adjoint system instead: in this case, the projection must be done with respect to a basis of the kernel of matrices

$$\begin{array}{rcl} M_1(\rho) & = & \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix} \\ M_2(\rho) & = & \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix} \end{array}$$

since we have inequality

$$\Psi_a(\rho,\dot{\rho}) + M_2(\rho)^T K(\rho) M_1 + (\star)^T \prec 0$$

We can see in this case that no congruence transformation is needed and it is possible to project immediately: this is the interest of the use of the adjoint. After that, since the matrix

$P(\rho)$ and R are nonsingular then there exist an infinite number of values for $\mathrm{Ker}[M_1(\rho)]$ and moreover this set can be given explicitly. The next step of the approach resides in the choice of a 'good' kernel basis for $M_1(\rho)$. It is shown that a good basis is given by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -h_{max}^{-1}R^{-1}P(\rho) & 0 & 0 & 0 \end{bmatrix}$$

and such a choice limits the number of nonlinearities: there is only one nonlinearity of the form $-h_{max}^2 P(\rho)R^{-1}P(\rho)$ which is a concave nonlinearity meaning that it is difficult to relax. The remaining of the approach consists in relaxing exactly this concave nonlinearity by a BMI involving a 'slack' variable (see Section 3.3) which is more simple to solve than the 'rational' matrix inequality involving the matrix R and its inverse. This approach leads then to the following theorem:

Theorem 5.1.8 *There exists a state-feedback control law of the form $u(t) = K(\rho)x(t)$ which asymptotically stabilizes nominal system (5.1) (with $C_h(\cdot) = 0$ and $\Delta = 0$) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^n$, constant matrices $Q, R \in \mathbb{S}_{++}^n$, a scalar $\gamma > 0$ and scalar function $\varepsilon : U_\rho \to \mathbb{R}_{++}$ such that the matrix inequalities*

$$\begin{bmatrix} Q - R + \partial_\rho P(\rho)\nu - h_{max}^{-2} P(\rho)R^{-1}P(\rho) & R & 0 & E(\rho) \\ \star & -(1-\mu)Q - R & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{5.24}$$

$$\mathrm{Ker}[U(\rho)]^T \Psi(\rho,\nu) \mathrm{Ker}[U(\rho)] \prec 0 \tag{5.25}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where $\Psi(\rho,\nu)$ is defined by

$$\begin{bmatrix} A(\rho)P(\rho) + (\star)^T + Q - R + \partial_\rho P(\rho)\nu & P(\rho)A_h(\rho)^T + R & P(\rho)C(\rho)^T & E & h_{max}A(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix}$$

$$U(\rho) = \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix}$$

Moreover, in this case a suitable control law can be computed by solving the following SDP in $K(\rho)$

$$\Psi(\rho,\nu) + U(\rho)^T K(\rho) V(\rho) + (\star)^T \prec 0 \tag{5.26}$$

with $V(\rho) = \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix}$ and the closed-loop system satisfies $||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *The proof is based on an application of lemma 3.5.2 which considers the stability and $\mathcal{L}_2$ performances for general time-delay systems through a simple Lyapunov-Krasovskii* *functional. Substituting matrices of the closed-loop system (5.4) into LMI (3.95) with* $C_h(\cdot) = 0$ *and* $\Delta = 0$ *we get:*

$$\begin{bmatrix} A_{cl}(\rho)P(\rho) + (\star)^T + Q - R + \partial_\rho P(\rho)\nu & P(\rho)A_h(\rho)^T + R & P(\rho)C_{cl}(\rho)^T & E(\rho) & h_{max}A_{cl}(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C_{cl}(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{5.27}$$

with $A_{cl}(\rho) = A(\rho) + B(\rho)K(\rho)$ *and* $C_{cl}(\rho) = C(\rho) + D(\rho)K(\rho)$*. The latter inequality can be rewritten as*

$$\Psi(\rho,\nu) + U(\rho)^T K(\rho)V(\rho) + V(\rho)^T K(\rho)^T U(\rho) \prec 0 \tag{5.28}$$

where $\Psi(\rho,\nu)$ *is defined by*

$$\begin{bmatrix} A(\rho)P(\rho) + (\star)^T + Q - R + \partial_\rho P(\rho)\nu & P(\rho)A_h(\rho)^T + R & P(\rho)C(\rho)^T & E & h_{max}A(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix}$$

$$\begin{aligned} U(\rho) &= \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix} \\ V(\rho) &= \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix} \end{aligned}$$

Hence the projection lemma applies and we get the following underlying matrix inequalities:

$$\begin{aligned} \text{Ker}[U(\rho)]^T\Psi(\rho,\nu)\text{Ker}[U(\rho)] &\prec 0 \\ \text{Ker}[V(\rho)]^T\Psi(\rho,\nu)\text{Ker}[V(\rho)] &\prec 0 \end{aligned} \tag{5.29}$$

While $\text{Ker}[U(\rho)]$ *cannot be computed exactly in the general case,* $\text{Ker}[V(\rho)]$*can since it involves unknown decision matrices* $P(\rho)$ *and* R *whose properties are knwon. The whole null-space of* $V(\rho)$ *is spanned by*

$$\text{Ker}[V(\rho)] = \begin{bmatrix} P_1(\rho) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ P_2(\rho) & 0 & 0 & 0 \end{bmatrix}$$

where $P_1(\rho)$ *and* $P(\rho)$ *are such that* $P(\rho)P_1(\rho) + h_{max}P_2(\rho)R = 0$*. Since the matrices* $P(\rho)$ *and* R *are positive definite and hence nonsingular then there exists an infinite number of couple of solutions* $(P_1(\rho), P_2(\rho))$*. Choosing* $P_1(\rho) = I$ *and* $P_2(\rho) = -h_{max}^{-1}R^{-1}P(\rho)$ *we get the following basis for the nullspace of* $V(\rho)$*:*

$$\text{Ker}[V(\rho)] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -h_{max}^{-1}R^{-1}P(\rho) & 0 & 0 & 0 \end{bmatrix} \tag{5.30}$$

Finally applying the projection lemma we get inequality $\mathrm{Ker}[V(\rho)]^T\Psi(\rho,\nu)\mathrm{Ker}[V(\rho)] \prec 0$ *which is equivalent to*

$$\begin{bmatrix} Q - R + \partial_\rho P(\rho)\nu - h_{max}^{-2}P(\rho)R^{-1}P(\rho) & R & 0 & E(\rho) \\ \star & -(1-\mu)Q - R & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{5.31}$$

And we obtain a sufficient condition for the existence of a stabilizing controller. The computation of the controller can be done by SDP. Indeed, after solving the existence conditions, the variables $P(\rho), Q, R, \gamma$ *are known and hence the matrix inequality (5.28) is linear in* $K(\rho)$ *and is a LMI problem.* $\square$

It is worth mentioning that matrix inequality (5.24) is strongly nonconvex due to the term $-h_{max}^{-2}P(\rho)R^{-1}P(\rho)$ which is a concave nonlinearity. In [Briat et al., 2008c, Chen and Zheng, 2006] such a nonlinearity is relaxed by considering the inverse of matrix P (which is parameter independent in their case) and hence such a problem can be solved using the cone complementary algorithm [El-Ghaoui and Gahinet, 1993]. However, in the present case, such a relaxation scheme cannot be considered due to the parameter dependence of $P(\rho)$ which is a matrix function. Hence, new relaxation schemes should be developed.

The first one is just mentioned for completeness but will not be detailed deeper due to its (too high) conservatism. It proposes to bound the concave function by an hyperplane (which is a linear function). This is done using a completion of the squares (see Section 3.3) and we get

$$-h_{max}^{-2}P(\rho)R^{-1}P(\rho) \preceq -2P(\rho) + h_{max}^2 R \tag{5.32}$$

Actually this method is too conservative since it corresponds to a linearization of the nonlinearity around a certain point and hence the approximation is correct in a neighborhood of the linearization point only. This motivates the development and use of a more complex relaxation which is described in Section 3.3. Such a relaxation turns the rational nonlinearity into a bilinear nonlinearity in which the bilinearities occur with an introduced 'slack' variable.

Theorem 5.1.9 *There exists a state-feedback control law of the form* $u(t) = K(\rho)x(t)$ *which asymptotically stabilizes nominal system (5.1) (with* $C_h(\cdot) = 0$ *and* $\Delta = 0$*) for all* $h(t) \in \mathscr{H}_1^\circ$ *if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}_{++}^n$*, a matrix function* $\Lambda : U_\rho \to \mathbb{R}^{n\times n}$*, constant matrices* $Q, R \in \mathbb{S}_{++}^n$*, a scalar* $\gamma > 0$ *and scalar function* $\varepsilon : U_\rho \to \mathbb{R}_{++}$ *such that the matrix inequalities*

$$\begin{bmatrix} Q - R + \partial_\rho P(\rho)\nu + \Lambda(\rho)^T P(\rho) + P(\rho)\Lambda(\rho) & R & 0 & E(\rho) & \Lambda(\rho)^T R \\ \star & -(1-\mu)Q - R & 0 & 0 & 0 \\ \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\star & -h_{max}^2 R \end{bmatrix} \prec 0 \tag{5.33}$$

$$\mathrm{Ker}[U(\rho)]^T\Psi(\rho,\nu)\mathrm{Ker}[U(\rho)] \prec 0 \tag{5.34}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *where* $\Psi(\rho,\nu)$ *is defined by*

$$\begin{bmatrix} A(\rho)P(\rho) + (\star)^T + Q - R + \partial_\rho P(\rho)\nu & P(\rho)A_h(\rho)^T + R & P(\rho)C(\rho)^T & E & h_{max}A(\rho)R \\ \star & -(1-\mu)Q - R & 0 & 0 & h_{max}A_h(\rho)R \\ \star & \star & -\gamma I & F(\rho) & h_{max}C(\rho)R \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix}$$

$$U(\rho) \;=\; \begin{bmatrix} B(\rho)^T & 0 & D(\rho)^T & 0 & 0 \end{bmatrix}$$

Moreover, in this case a suitable control law can be computed by solving the following SDP in $K(\rho)$

$$\Psi(\rho,\nu) + U(\rho)^T K(\rho) V(\rho) + (\star)^T \prec 0 \tag{5.35}$$

with $V(\rho) = \begin{bmatrix} P(\rho) & 0 & 0 & 0 & h_{max}R \end{bmatrix}$ *and the closed-loop system satisfies* $||z||_{\mathcal{L}_2} \le \gamma ||w||_{\mathcal{L}_2}$.

Proof: *The relaxation is done using Theorem 3.3.4 on the matrix inequality*

$$\begin{bmatrix} \Phi_{11} - h_{max}^2 P(\rho)R^{-1}P(\rho) & \Phi_{12}(\rho) \\ \star & \Phi_{22} \end{bmatrix} \prec 0$$

where

$$\begin{bmatrix} \Phi_{11}(\rho) & \Phi_{12}(\rho) \\ \star & \Phi_{22} \end{bmatrix} = \left[\begin{array}{c|ccc} Q - R + \partial_\rho P(\rho)\nu & R & 0 & E(\rho) \\ \hline \star & -(1-\mu)Q - R & 0 & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{array}\right]$$

and $\eta(\rho) = \Lambda(\rho)$, $\alpha(\rho) = P(\rho)$ *and* $\beta = h_{max}^{-2}R$. $\square$

Although this approach preserves the nonlinearity of the problem, the numerical difficulty is reduced due to the fact that the problem is bilinear only (while before it was rational). Hence simple algorithmic tools can be used to solve it and then provide local optimal solutions. One of the interest of this 'slack' variable is to decouple Lyapunov matrices allowing to solve them simultaneously while in the first nonlinear problem $P(\rho)$ and R needed to be solved separately without taking into account that matrices R and R^{-1} appear in the same inequality. So, even if the problem is still nonlinear, the nonlinearities are much 'nicer'.

The following algorithm describes how to this nonlinear optimization problem:

Algorithm 5.1.10 *1. Generate an initial symmetric constant matrix* Λ_0 *such that* $\Lambda_0^T P + P\Lambda_0 \prec 0$, *choose a common structure for* $P(\rho)$ *and* $\Lambda(\rho)$ *e.g.* $Z(\rho) = Z_0 + Z_1\rho + Z\rho^2$ *with* $Z(\rho) = \{P(\rho), \Lambda(\rho)\}$.

2. Solve the optimization problem

$$\begin{array}{l} \min\ \gamma \\ \text{such that } P(\rho), Q, R \succ 0,\ \gamma > 0 \\ (5.33) \text{ and } (5.34) \end{array}$$

If the problem is unfeasible then go to step 1.

3. Solve the optimization problem

$$\begin{array}{l}\min_{\gamma,\Lambda(\rho),Q}\gamma\\ \text{such that } Q\succ 0,\ \gamma>0\\ (5.33)\text{ and }(5.34)\end{array}$$

4. If stopping criterion is satisfied then stop else go to step 2.

Although this algorithm does not guarantee any global convergence, if the stabilization problem is feasible it turns out that it is easy to find an initial feasible point Λ_0 which can be defined here by $\Lambda_0 = -\varepsilon I$ with $\varepsilon > 0$. Moreover, experiments seem to emphasize that a small number of iterations are sufficient to converge to a local optimum. Advantages of such an approach is to deal directly with initial bounded real lemma without any relaxation at the expense of a larger computational complexity. For more details on this relaxation, the readers should refer to Section 3.3.

Example 5.1.11 *In this example we will compare the proposed method expressed through Theorem 5.1.9 with an existent one proposed in [Zhang and Grigoriadis, 2005]. Let us consider the system*

$$\begin{array}{rcl}\dot{x}(t) & = & \begin{bmatrix} 0 & 1+\phi sin(t) \\ -2 & -3+\delta sin(t)\end{bmatrix}x(t) + \begin{bmatrix}\phi sin(t) & 0.1\\ -0.2+\delta sin(t) & -0.3\end{bmatrix}x_h(t)\\ & & +\begin{bmatrix}0.2\\0.2\end{bmatrix}w(t)+\begin{bmatrix}\phi sin(t)\\ 0.1+\delta sin(t)\end{bmatrix}u(t)\\ z(t) & = & \begin{bmatrix}0 & 10\\ 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\0.1\end{bmatrix}u(t)\end{array} \tag{5.36}$$

which is borrowed from [Wu and Grigoriadis, 2001] and has been modified by [Zhang and Grigoriadis, 2005].

Case $\phi = 0.2$ ***and*** $\delta = 0.1$***:***

Choosing $\rho(t) = sin(t)$ *as parameter, it can be easily deduced that* $\rho, \dot{\rho} \in [-1, 1]$*. The parameter space is gridded over* $N_p = 40$ *points uniformly spaced.*

Choosing, as in [Zhang and Grigoriadis, 2005], $h_M = 0.5$, $\mu = 0.5$, $P(\rho) = P_c$ *and* $\Lambda(\rho) = \Lambda_c$ *(quadratic stability), we find* $\gamma^* = 1.8492$ *in 4 iterations of the algorithm for which the initial point has been randomly chosen. It is important to note that the first iteration gives a maximal bound on* γ *of 1.89 which is also a better result than those obtained before (See [Wu and Grigoriadis, 2001, Zhang and Grigoriadis, 2005]), for instance in Zhang and Grigoriadis [2005], an optimal value* $\gamma = 3.09$ *is found. In our case, the resulting a controller is given by* $K(\rho) = K_0 + K_1\rho + K_2\rho^2$ *where* $K_0 = \begin{bmatrix} -5.9172 & -16.3288 \end{bmatrix}$, $K_1 = \begin{bmatrix} -53.1109 - 32.4388 \end{bmatrix}$ *and* $K_2 = \begin{bmatrix} -8.4071 & 3.0878 \end{bmatrix}$.

It is worth noting that after computing the controller, the $\mathcal{L}_2$*-induced norm achieved is now* $\gamma_r = 2.2777$ *corresponding to an increase of* 23.17%*. Better performances should be obtained while considering a more complex controller form but we are limited by the fact that we do not consider rational controllers.*

The values of each coefficient of the gain $K(\rho)$ *w.r.t. parameter values are represented at the top of figure 5.1. The bottom of figure 5.1 describes the gain computed by the method of [Zhang and Grigoriadis, 2005].*

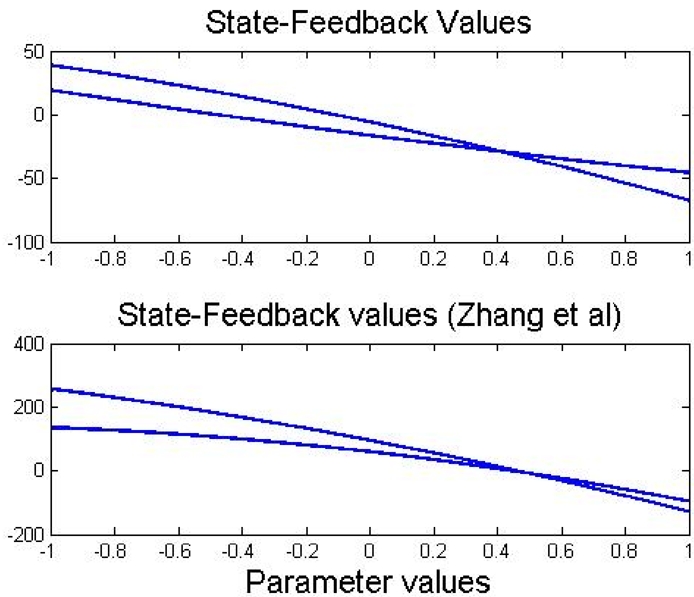

Figure 5.1: Simulation 1 - Gains controller evolution with respect to the parameter value - theorem 5.1.9 (top) and method of [Zhang and Grigoriadis, 2005]

Note that despite of lower controller gain values, we obtain better results than in the previous approaches, this is a great advantage of the proposed method.

For simulation purposes let $h(t) = 0.5|\sin(t)|$ *and* $\rho(t) = sin(t)$ *and we will differentiate two cases: the stabilization with non-zero initial conditions and zero inputs and the stabilization with zero initial conditions and non-zero inputs.*

Simulation 1: Stabilization ($x(0) \neq 0$ ***and*** $w(t) = 0$***):***

We obtain results depicted in figures 5.2-5.4. We can see that the rate of convergence is very near but using our method the necessary input energy to make the system converge to 0 is less than in the case of [Zhang and Grigoriadis, 2005].

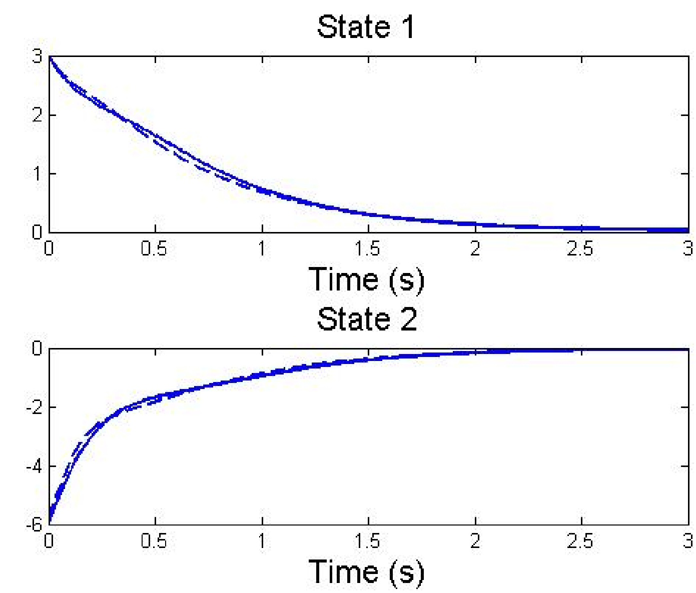

Figure 5.2: Simulation 1 - State evolution - theorem 5.1.9 in full and [Zhang and Grigoriadis, 2005] in dashed

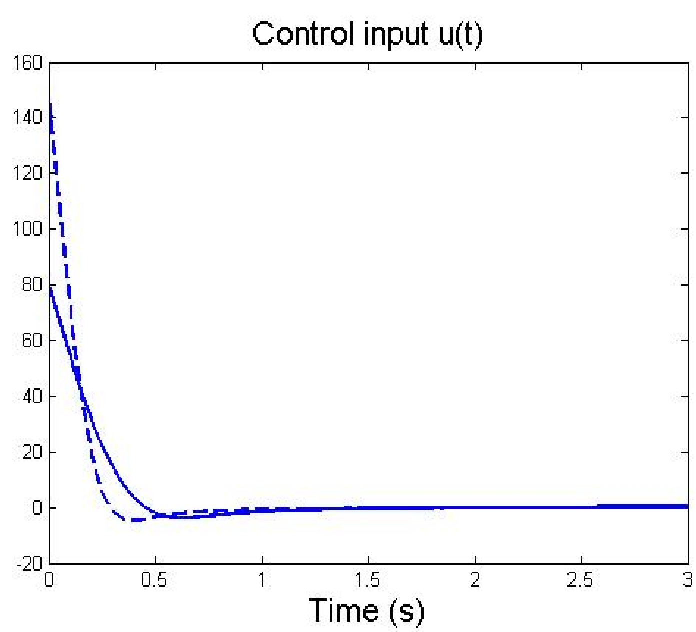

Figure 5.3: Simulation 1 - Control input evolution - theorem 5.1.9 in full and [Zhang and Grigoriadis, 2005] in dashed

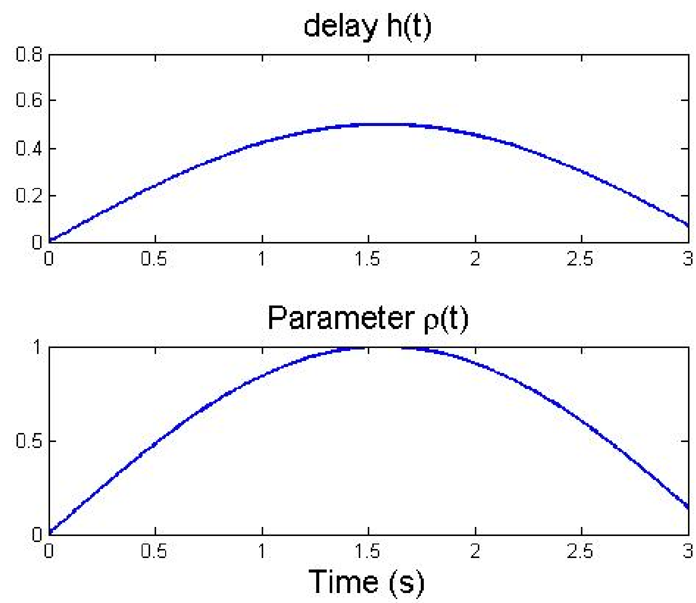

Figure 5.4: Simulation 1 - Delay and parameter evolution

Simulation 2: Disturbance attenuation (**$x(0) = 0$ **and** $w(t) \neq 0$**):

We consider here a unit step disturbance and we obtain the following results depicted in Figures 5.5-5.7. We can see that our control input has smaller bounds and that the second state is less affected by the disturbance than by using method of [Zhang and Grigoriadis, 2005]. Remember than the control output z contains the control input and the second state only, this is the reason why the first state is more sensitive to the disturbance than in [Zhang and Grigoriadis, 2005].

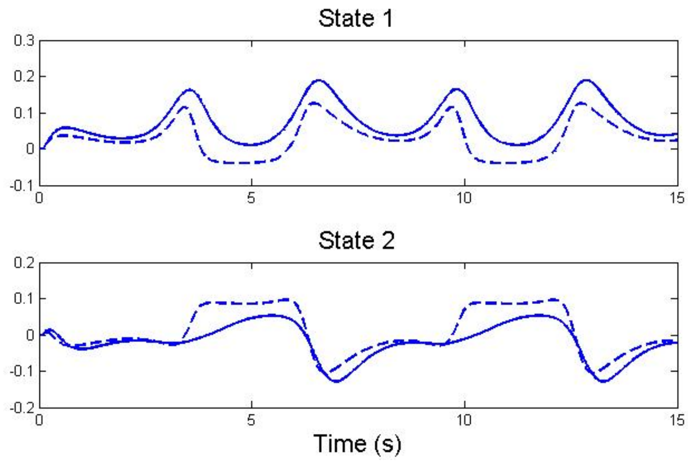

Figure 5.5: Simulation 2 - State evolution - theorem 5.1.9 in full and [Zhang and Grigoriadis, 2005] in dashed

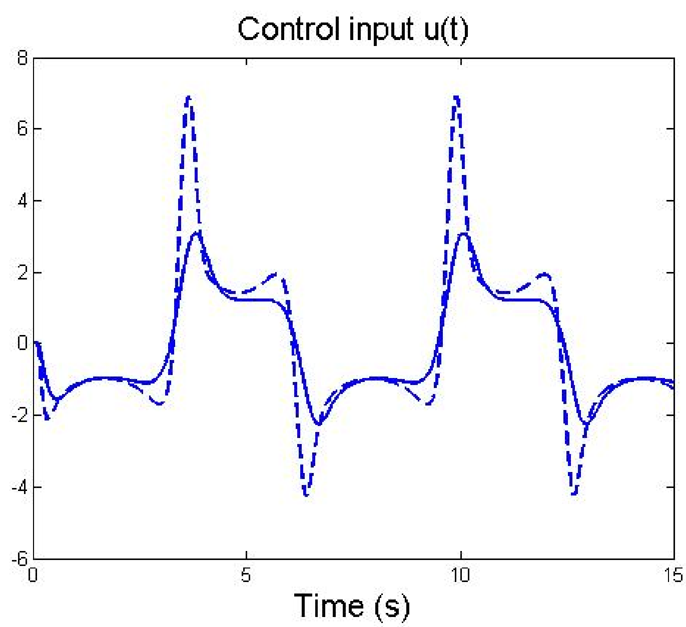

Figure 5.6: Simulation 2 - Control input evolution - theorem 5.1.9 in full and [Zhang and Grigoriadis, 2005] in dashed

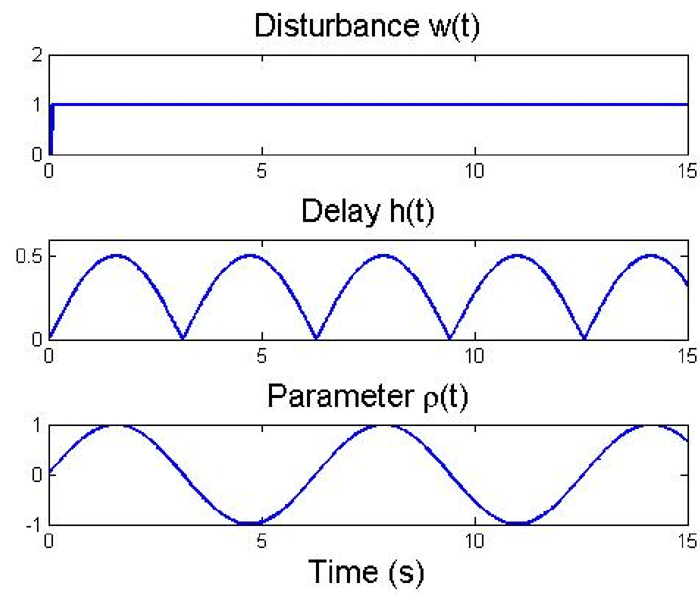

Figure 5.7: Simulation 2 - Delay and parameter evolution

Then we check, the delay upper bound for which a parameter dependent stabilizing controller exists and guarantees $\gamma^* < 10$ with $\mu = 0.5$ and we find $h_M = 79.1511$, for $\gamma^* < 2$ we find $h_M = 1.750$. In [Zhang and Grigoriadis, 2005], the delay upper bound for which a stabilizing controller exist is $h_M = 1.65$. This shows that our result is less conservative.

Case** $\phi = 2$ **and** $\delta = 1$**:

Using the results of [Zhang and Grigoriadis, 2005] no solution is found. With lemma 5.1.9, we find that there exists a controller such that the closed-loop system has a $\mathcal{L}_2$-induced norm lower than $\gamma = 6.4498$.

5.1.4 Memoryless state-feedback - Polytopic approach

Let us consider the polytopic LPV time-delay system:

$$\begin{array}{rcl} \dot{x}(t) & = & \sum_{i=1}^{N} \left(A_i x(t) + A_{hi} x(t-h_i(t)) + B_i u(t) + E_i w(t)\right) \\ z(t) & = & \sum_{i=1}^{N} \left(C_i x(t) + C_{hi} x(t-h_i(t)) + D_i u(t) + F_i w(t)\right) \end{array} \quad (5.37)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ and $h(t) \in \mathscr{H}_1^\circ$ are respectively the state of the system, the control input, the disturbances, the controlled outputs and the delay of the system. The goal is to stabilize the system with a LPV polytopic state-feedback control law of the form:

$$u(t) = \sum_{i=1}^{N} K_i x(t) \quad (5.38)$$

where the K_i are the gains to be designed. The parameters λ_i are assumed to evolve within a unitary polytope (unit simplex) characterized by

$$\Lambda := \left\{\lambda_i(t) \in [0,1],\ \lambda_i(t) \geq 0,\ \sum_i \lambda_i(t) = 1\right\} \tag{5.39}$$

When robust stability is addressed it is convenient to define the set in which the parameters derivative evolve

$$U_s := \left\{\dot{\lambda}_i(t),\ \sum_{i=1} \dot{\lambda}_i(t) = 0\right\} \subset \mathbb{R}^N \tag{5.40}$$

The idea of the approach is to define a parameter dependent Lyapunov-Krasovskii functional similar to those used before. Then we use a relaxation in order to remove coupled terms and we substitute the closed-loop system into the relaxed stability/performances conditions. Since the whole polytopic approach is based on the linear dependence on the parameters, it is not possible here to consider only the vertices since there are quadratic terms in $\lambda(t)$ in the LMIs due to the terms $B(\lambda)K(\lambda)$ and $D(\lambda)K(\lambda)$. We provide here a solution based on the linearizing result introduced in Section 3.2 and more precisely given in Corollary 3.2.2. This will turn the quadratic dependence into a linear one and then the latter LMI can be verified only at the vertices in order to provide a finite set of LMIs characterizing a sufficient condition for the stabilization of the polytopic LPV system. Finally, as the terms in $\dot{\lambda}$ are linear, a polytopic relaxation can then be directly applied on these terms. This approach gives rise to the following result:

Theorem 5.1.12 *There exists a state-feedback control law of the form $u(t) = \sum_{i=1}^{N} K_i x(t)$ which asymptotically stabilizes the system (5.37) for all $h \in \mathscr{H}_1^{\circ}$ if there exist matrices $\tilde{P}_i, \tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n, Y \in \mathbb{R}^{n\times n}, V_i \in \mathbb{R}^{m\times n}$ and a scalar $\gamma > 0$ such that the parameter dependent LMI:*

$$\Omega_0 + \sum_{i=1}^{N} \lambda_i \Omega_i + \sum_{i,j=1}^{N,N} \lambda_i \lambda_j \Omega_{ij} \prec 0 \tag{5.41}$$

holds for all λ_i *such that* $\sum_{i=1}^{N} \lambda_i = 1$, $\lambda_i(t) \geq 0$, $\dot{\lambda} \in U_s$ *and where*

$$\begin{aligned}
\Omega_0 &= \begin{bmatrix} -Y^H & 0 & 0 & 0 & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}^0(\dot{\lambda}) & \tilde{R} & 0 & 0 & 0 & 0 \\ \star & \star & \tilde{U}_{33}^0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_p & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} \\
\Omega_i &= \begin{bmatrix} 0 & \tilde{P}_i + A_i Y & A_{hi} Y & E_i & 0 & 0 & 0 \\ \star & \tilde{U}_{22}^i & 0 & 0 & [C_i Y]^T & 0 & 0 \\ \star & \star & 0 & 0 & C_{hi}^T & 0 & 0 \\ \star & \star & \star & 0 & F_i^T & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}_i & 0 \\ \star & \star & \star & \star & \star & \star & 0 \end{bmatrix} \\
\Omega_{ij} &= \begin{bmatrix} 0 & B_i V_j & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & [D_i V_j]^T & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 \end{bmatrix}
\end{aligned} \tag{5.42}$$

with

$$\begin{aligned}
\tilde{U}_{22}(\dot{\lambda})^0 &= \sum_{i=1}^{N} \dot{\lambda}_i(t) P_i + \tilde{Q} - \tilde{R} \\
\tilde{U}_{22}^i &= -P_i + \tilde{Q} - \tilde{R} \\
\tilde{U}_{33} &= -(1-\mu)\tilde{Q} - \tilde{R}
\end{aligned}$$

In this case, the controller matrices are given by $K_i = V_i Y^{-1}$ *and the closed-loop system satisfies* $||z||_{\mathcal{L}_2} \geq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *Consider the following Lyapunov-Krasovskii functional*

$$\begin{aligned}
V(x_t, \dot{x}_t) &= V^1(x_t) + V^2(x_t) + V^3(\dot{x}_t) \\
V^1(x_t) &= \textstyle\sum_{i=1}^{N} x(t)^T P_i x(t) \\
V^2(x_t) &= \textstyle\int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta \\
V^3(x_t) &= \textstyle\int_{-h_{max}}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T (h_{max} R) x(\eta) d\eta d\theta
\end{aligned} \tag{5.43}$$

Since the form is very similar to the Lyapunov-Krasovskii functionals developed in Section 3.5, get the following LMI

$$\Omega_0 + \sum_{i=1}^{N} \lambda_i \Omega_i + \sum_{i,j=1}^{N,N} \lambda_i \lambda_j \Omega_{ij} \prec 0 \tag{5.44}$$

which is the polytopic LPV counterpart of LMI (3.109) on which a congruence transformation with respect to the matrix $\text{diag}(Y, Y, Y, I, I, Y, Y)$ *with* $Y = X^{-1}$ *and the change of variable* $V_i = K_i Y$ *have been performed.* □

We are confronted here in a particular type of parameter dependent LMI. It is possible to use a gridding approach to verify its negative definiteness condition but in the polytopic framework it is of habit to relax the parameter dependent LMI and express it as a finite set of LMIs. Actually, this is possible exactly if the dependence on the parameter is affine. In the current case, the relation is quadratic and in this case a common but conservative relaxation is to fix all the terms Ω_{ij} to be negative semidefinite and hence a relaxation is to consider only vertices of the polytope. However, in our case the matrices Ω_{ij} cannot be defined as negative semidefinite due to their structure and hence this relaxation cannot work. This can be viewed by considering the scalar case, $B_i = b_i$, $V_j = v_j$ and $D_i = d_i$. After removing the zero columns/lines of Ω_{ij} we get the matrix

$$\begin{bmatrix} 0 & b_i v_j & 0 \\ b_i v_j & 0 & d_i v_j \\ 0 & d_i v_j & 0 \end{bmatrix}$$

The characteristic polynomial is given by $\chi(s) = s^3 - v_j^2(b_i^2 + d_i^2)s$ and exhibit zeroes at values $\{-v_j\sqrt{b_i^2 + d_i^2}, 0, v_j\sqrt{b_i^2 + d_i^2}\}$. Since one of them is positive hence the matrix $\Omega_{ij} \npreceq 0$ and this shows that in the general case Ω_{ij} cannot be negative semidefinite.

On the other hand, relaxations like SOS-relaxation, polynomial optimization and linearization approaches can be employed and induce a small degree of conservatism. The reader should refer to sections 3.2 and 1.3.3.1 to get more explanations on these approaches. We have chosen to illustrate here only one of them which we call the linearization approach which is detailed in Section 3.2.

The principle is to turn the initial parameter dependent LMI into a new LMI involving 'slack' variables which has a linear dependence on the parameters. The new LMI is not equivalent to the first one but is usually very near which makes this relaxation a useful tool for linearization of polynomially parameter dependent LMIs. We obtain the following result:

Theorem 5.1.13 *There exists a state-feedback control law of the form* $u(t) = \sum_{i=1}^{N} K_i x(t)$ *which asymptotically stabilizes the system (5.37) for all* $h \in \mathscr{H}_1^\circ$ *if there exist matrices* $\tilde{P}_i, \tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^n, Y \in \mathbb{R}^{n\times n}, V_i \in \mathbb{R}^{m\times n}$, *a scalar* $\gamma > 0$ *and a matrix* $\mathcal{Z}$ *such that the LMIs*

$$\tilde{\mathcal{K}} + \mathcal{Z}^T\Pi(\lambda) + \Pi(\lambda)^T\mathcal{Z} < 0 \tag{5.45}$$

hold for all $(\lambda, \dot{\lambda}) \in \Lambda \times U_s$ *and where*

$$\Pi(\lambda) = \begin{bmatrix} -\lambda_1 I & I & 0 & \dots & 0 \\ -\lambda_2 I & 0 & I & \dots & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -\lambda_{N-1} I & 0 & 0 & \dots & I \end{bmatrix}$$

$$\tilde{\mathcal{K}} = \begin{bmatrix} \Omega_0 & \Omega_1/2 & \dots & \Omega_{N-1}/2 \\ \star & \Omega_{11} & \dots & \Omega_{1(N-1)}/2 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \Omega_{(N-1)(N-1)} \end{bmatrix}$$

with $\mathcal{K}_0 = \Omega_0 + \Omega_N + \Omega_{NN}$, $\mathcal{K}_i = \Omega_i - \Omega_N + 2\Sigma_{iN} - 2\Omega_{NN}$, $\mathcal{K}_{ij} = \Omega_{ij} - 2\Sigma_{Ni} + \Omega_{NN}$, $\Omega_{ij} = (\Omega_{ij} + \Omega_{ji})/2$ *and* $\Sigma_{ij} = (\Omega_{ij} + \Omega_{ji})/2$.

In this case suitable controller matrices are given by $K_i = V_i Y^{-1}$ *and the closed-loop system satisfies* $||z||_{\mathcal{L}_2} \geq \gamma ||w||_{\mathcal{L}_2}$

Proof: *This is a straightforward application of Corollary 3.2.2 to LMI (5.41).* □

In some cases, system (5.37) is a polytopic parametrization of a physical system and the polytopic formulation is not natural in the sense that physical parameters are hidden in this parametrization. In such a case, it may be difficult to determine the set U_s exactly. The method exposed in Section 3.4 allows to compute systematically these bounds through a simple linear algebraic approach.

5.1.5 Hereditary State-Feedback Controller Design - exact delay value case

We consider in this section the design of state-feedback control laws embedding a delayed information:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \tag{5.46}$$

It is clear that such a control law should lead to better results than by considering a control law based on the current state only. We will consider first that the delay used in the controller is identical to the delay involved in the system dynamical model. The next section will be devoted to the case when the delay of the controller and the system are different.

The approach of this section is similar as for memoryless state-feedback control laws (see Theorem 5.1.3 for a similar proof). First of all, a generic stability/performances result based on a Lyapunov-Krasovskii functional is developed. This result is then relaxed in order to remove coupled terms and finally the closed-loop system is substituted in this matrix inequality. Finally, congruence transformations and change of variables are performed in view of linearizing the matrix inequality.

Theorem 5.1.14 *There exists a stabilizing control law of the form (5.46) for system (5.1) with* $h \in \mathscr{H}_1^\circ$ *if there exists a continuously differentiable matrix function* $\tilde{P} : U_\rho \to \mathbb{S}^n_{++}$, *matrix functions* $V_0, V_h : U_\rho \to \mathbb{R}^{m\times n}$, *constant matrices* $\tilde{Q}, \tilde{R} \in \mathbb{S}^n_{++}, Y \in \mathbb{R}^{n\times n}$, *a constant scalar* $\gamma > 0$ *and a scalar function* $\varepsilon : U_\rho \to \mathbb{R}_{++}$ *such that the LMI*

$$\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon(\rho)\mathcal{G}(\rho)^T\mathcal{G} & \mathcal{H}(\rho)^T \\ \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{5.47}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *where* $\Psi(\rho,\nu)$ *is defined by*

$$\begin{bmatrix} -(Y+Y^T) & U_{12}(\rho) & U_{13}(\rho) & E(\rho) & 0 & Y & h_{max}\tilde{R} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{R} & 0 & U_{25}(\rho) & 0 & 0 \\ \star & \star & \tilde{U}_{33} & 0 & U_{26}(\rho) & 0 & 0 \\ \star & \star & \star & -\gamma I_p & F(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_q & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$$

$$\begin{aligned} U_{12}(\rho) &= \tilde{P}(\rho) + A(\rho)Y + B(\rho)V_0(\rho) \\ U_{13}(\rho) &= A_h(\rho)Y + B(\rho)V_h(\rho) \\ U_{25}(\rho) &= Y^TC(\rho)^T + [D(\rho)V_0(\rho)]^T \\ U_{26}(\rho) &= Y^TC_h(\rho)^T + [D(\rho)V_h(\rho)]^T \\ \mathcal{G}(\rho) &= \begin{bmatrix} G(\rho)^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathcal{H} &= \begin{bmatrix} 0 & H_A(\rho)Y + H_B(\rho)V_0(\rho) & H_{A_h}(\rho)Y + B(\rho)V_h(\rho) & H_E(\rho) & 0 & 0 & 0 \end{bmatrix} \\ \tilde{U}_{22}(\rho,\nu) &= -\tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho \tilde{P}(\rho)\nu \\ \tilde{U}_{33} &= -(1-\mu)\tilde{Q} - \tilde{R} \end{aligned}$$

Moreover a suitable control gains are given by $K_0(\rho) = V_0(\rho)Y^{-1}$ *and* $K_h(\rho) = V_h(\rho)Y^{-1}$ *and the closed-loop satisfies* $||z||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

5.1.6 Hereditary State-Feedback Controller Design - approximate delay value case

This section is devoted to the design of control of the form (5.2) in which the delay $d(t)$ is different from the delay $h(t)$ of the system. In this case, we have the following control law:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \tag{5.48}$$

The approach is still similar to the others, the main difference lies in the choice of the Lyapunov-Krasovskii functional to consider. Since the closed-loop will have two delays (the system and controller one) which are coupled together by the algebraic equality $d(t) = h(t) + \varepsilon(t)$ where $|\varepsilon(t)| \leq \delta$. The fact that these delays have a relation is the main difficulty of the approach. However, with an appropriate choice of the Lyapunov-Krasovskii functional it is possible to consider this relation. This is done by using the Lyapunov-Krasovskii defined in Section 3.7 and the remaining of the method is identical as for previous ones: relax the inequalities in order to remove coupled terms then substitute the closed-loop system into the inequalities. Finally, linearize the problem through congruence transformations and changes of variables.

Theorem 5.1.15 *There exists a state-feedback control law of the form (5.48) if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}^n_{++}$*, matrix functions* $V_0, V_h : U_\rho \to \mathbb{R}^{m\times n}$*,* $\tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2 \in \mathbb{S}^n_{++}$*, a scalar* $\gamma > 0$ *and a scalar function* $\varepsilon : U_\rho \to \mathbb{R}_{++}$ *if the* *following LMIs*

$$\begin{bmatrix} U_{11}(\rho) & U_{12}(\rho) & U_{13}(\rho) & E(\rho) & 0 & \tilde{X} & h_{max}\tilde{R}_1 & \tilde{R}_2 & 0 \\ \star & U_{22}(\rho,\nu) & \tilde{R}_1 & 0 & U_{25}(\rho) & 0 & 0 & 0 & U_{29}(\rho) \\ \star & \star & U_{33} & 0 & U_{35}(\rho) & 0 & 0 & 0 & U_{39}(\rho) \\ \star & \star & \star & -\gamma I & F(\rho)^T & 0 & 0 & 0 & H_E(\rho)^T \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 & 0 \\ \star & \star & \star & \star & \star & \star & -\tilde{R}_1 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{5.49}$$

$$\begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) & \Pi_{13}(\rho) \\ \star & \Pi_{22}(\rho) & 0 \\ \star & \star & -\varepsilon(\rho)I \end{bmatrix} \prec 0 \tag{5.50}$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ *where*

$$\begin{aligned} U_{11}(\rho) &= -\tilde{X}(\rho)^H + \varepsilon(\rho)G(\rho)G(\rho)^T \\ U_{12}(\rho) &= \tilde{P}(\rho) + A(\rho)\tilde{X} + B(\rho)V_0(\rho) \\ U_{13}(\rho) &= A_h(\rho)\tilde{X} + B(\rho)V_h(\rho) \\ U_{22}(\rho,\nu) &= -\tilde{P}(\rho) + \tilde{Q}_1 + \tilde{Q}_2 + \sum_{i=1}^{N} \frac{\partial \tilde{P}}{\partial \rho_i}\nu_i - \tilde{R}_1 \\ U_{33} &= -(1-\mu)(\tilde{Q}_1 + \tilde{Q}_2) - \tilde{R}_1 \\ U_{25}(\rho) &= U_{25}(\rho)[C(\rho)\tilde{X} + D(\rho)V_0(\rho)]^T \\ U_{29}(\rho) &= [H_A(\rho)\tilde{X} + H_B(\rho)V_0(\rho)]^T \\ U_{35}(\rho) &= [C_h(\rho)X + D(\rho)V_h(\rho)]^T \\ U_{39}(\rho) &= [H_{A_h}(\rho)\tilde{X} + H_B(\rho)V_h(\rho)]^T \end{aligned}$$

$\Pi_{11}(\rho,\nu)$ *is defined by*

$$\begin{bmatrix} -\tilde{X}(\rho)^H + \varepsilon(\rho)G(\rho)G(\rho)^T & \tilde{P}(\rho) + A(\rho)\tilde{X} + B(\rho)V_0(\rho) & A_h(\rho)\tilde{X} & B(\rho)V_h(\rho) & E(\rho) \\ \star & \tilde{\Theta}_{11}(\rho,\nu) & \tilde{R}_1 & 0 & 0 \\ \star & \star & \tilde{\Psi}_{22} & (1-\mu)\tilde{R}_2/\delta & 0 \\ \star & \star & \star & \tilde{\Psi}_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix}$$

and

$$\begin{array}{rcl}
\Pi_{12}(\rho) & = & \begin{bmatrix} 0 & \tilde{X}(\rho) & h_{max}\tilde{R}_1 & \tilde{R}_2 \\ \left[C(\rho)\tilde{X} + D(\rho)V_0(\rho)\right]^T & 0 & 0 & 0 \\ \left[C_h(\rho)\tilde{X}\right]^T & 0 & 0 & 0 \\ [D(\rho)V_h(\rho)]^T & 0 & 0 & 0 \\ F(\rho)^T & 0 & 0 & 0 \end{bmatrix} \\
\Pi_{13}(\rho) & = & \begin{bmatrix} 0 \\ \left[H_A(\rho)\tilde{X} + H_B(\rho)V_0(\rho)\right]^T \\ 0 \\ [H_B(\rho)V_h(\rho)]^T \\ 0 \end{bmatrix} \\
\Pi_{22}(\rho) & = & \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 \\ \star & \star & -\tilde{R}_1 & 0 \\ \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} \end{bmatrix} \\
\tilde{\Psi}_{22} & = & -(1-\mu_h)(\tilde{Q}_1 + \tilde{R}_2/\delta) - \tilde{R}_1 \\
\tilde{\Psi}_{33} & = & -(1-\mu_d)\tilde{Q}_2 - (1-\mu)\tilde{R}_2/\delta
\end{array}$$

5.1.7 Delay-Scheduled State-Feedback Controllers

This section is devoted to the development of a new methodology to control time-delay systems with time-varying delays provided that the delay can be measured or estimated in real-time. The difference between state-feedback with memory and delay-scheduled state-feedback controllers comes from the fact that the former uses the delayed state into the control law expression while the latter uses only the instantaneous state. On the other hand, while the former uses constant gains (in the LTI case), the latter involves a matrix gain which varies in time with respect to the value of the delay, as seen in the LPV framework when gain-scheduling controllers are designed. Hence, a delay-scheduled state-feedback control law is defined by

$$u(t) = K(\hat{h})x(t) \tag{5.51}$$

Since the gain scheduling technique is a well-established method in the LPV framework through different approaches such as LPV polytopic systems, polynomial systems and 'LFT' systems , it seems necessary to develop an equivalence (at the best) between LPV systems and time-delay systems. This section will consider the following LTI time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h(t-h(t)) + Bu(t) + Ew(t) \\ z(t) & = & Cx(t) + C_h x(t-h(t)) + Du(t) + Fw(t) \end{array} \tag{5.52}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$ are respectively the system state, the control input, the exogenous inputs and the controlled outputs.

In [Briat et al., 2007a], a model transformation has been introduced in order to turn a time-delay system into an uncertain LPV system. However, this model transformations suffer from two main problems: the first one was that operator cannot consider zero delay values and the second one is the conservatism induced by the computation of the $\mathcal{L}_2$-induced norm of that operator. The model transformation presented here authorizes zero delay values and the $\mathcal{L}_2$-induced norm is tighter.

Let the operator

$$\begin{array}{rcccl}\mathcal{D}_h & : & \mathcal{L}_2 & \rightarrow & \mathcal{L}_2 \\ & & \eta(t) & \rightarrow & \dfrac{1}{\sqrt{h(t)h_{max}}}\displaystyle\int_{t-h(t)}^{t}\eta(s)ds\end{array} \tag{5.53}$$

This operator enjoys the following properties:

1. $\mathcal{D}_h$ is $\mathcal{L}_2 - \mathcal{L}_2$ stable.
2. $\mathcal{D}_h$ has an induced $\mathcal{L}_2 - \mathcal{L}_2$ norm lower than 1.

Proof: *Let us prove first that for a $\mathcal{L}_2$ input signal we get a $\mathcal{L}_2$ output signal. Assume that $\eta(t)$ is continuous and denote by $\eta_p(t)$ the signal satisfying $d\eta_p(t)/dt = \eta(t)$ then we have*

$$\mathcal{D}_h(\eta(t)) = \frac{\eta_p(t) - \eta_p(t-h(t))}{\sqrt{h(t)h_{max}}} \tag{5.54}$$

Note that as $h(t)$ is always positive then (5.54) is bounded since $\eta(t)$ is continuous and belongs to $\mathcal{L}_2$ (and hence to $\mathcal{L}_\infty$). The main problem is when the delay reaches 0. Suppose now that there exist a (possibly infinite) family of time instants $t_{i+1} > t_i \geq 0$ such that $h(t_i) = 0$. Since $\eta_p(t)$ is continuously differentiable and hence we have

$$lim_{t\to t_i}\frac{\eta_p(t) - \eta_p(t-h(t))}{\sqrt{h(t)h_{max}}} = \sqrt{\frac{h(t_i)}{h_{max}}}\eta(t_i)$$

As $\eta(t)$ is continuous and belongs to $\mathcal{L}_2$, we can state that $\eta(t_i)$ is always finite and then the output signal remains bounded even if the delay reaches zero. This proves that $\mathcal{D}_h$has a finite $\mathcal{L}_\infty$-induced norm (no singularities). Let us prove now that it has a finite induced $\mathcal{L}_2$-norm using a similar method as in Gu et al. [2003]:

$$||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} := \int_0^{+\infty}\frac{dt}{h(t)h_{max}}\int_{t-h(t)}^{t}\eta^T(\theta)d\theta\cdot\int_{t-h(t)}^{t}\eta(\theta)d\theta$$

Then using the Jensen's inequality (see [Gu et al., 2003]) we obtain

$$||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} \leq \int_0^{+\infty}\frac{dt}{h_{max}}\int_{t-h(t)}^{t}\eta^T(\theta)\eta(\theta)d\theta \tag{5.55}$$

To solve the problem we will exchange the order of integration under the assumption $\eta(\theta) = 0$ when $\theta \leq 0$. First note that the domain is contained in $t - h_M \leq \theta \leq t$, $\theta \geq 0$ and is bounded by lines $\theta = t$ and $\theta = p(t) := t - h(t)$. Since $p(\theta)$ is a non-decreasing function then the set *of segments where $\theta = p(t)$ is constant is countable. Hence for almost all θ the function $p(t)$ is increasing and the inverse $t = q(\theta) := p^{-1}(\theta) :=$ is well-defined and then we have*

$$\begin{aligned} ||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} &\leq \frac{1}{h_{max}} \int_0^{+\infty} \eta^T(\theta)\eta(\theta)d\theta \int_\theta^{q(\theta)} dt \\ &= \frac{1}{h_{max}} \int_0^{+\infty} \eta^T(\theta)\eta(\theta)(q(\theta)-\theta)d\theta \end{aligned}$$

Hence using the fact that $\theta = t - h(t)$ and that $t = q(\theta)$ then we have the equality $\theta = q(\theta) - h(q(\theta))$ and hence we have $q(\theta) - \theta = h(q(\theta))$. This leads to

$$\begin{aligned} ||\mathcal{D}_h(\eta)||^2_{\mathcal{L}_2} &\leq \frac{1}{h_{max}} \int_0^{+\infty} \eta^T(\theta)\eta(\theta)h(q(\theta))d\theta \\ &\leq ||\eta||^2_{\mathcal{L}_2} \end{aligned}$$

We have then proved that $\mathcal{D}_h$ defines a $\mathcal{L}_2 - \mathcal{L}_2$ stable operator with an $\mathcal{L}_2$-induced norm lower than 1. □

We show now how to use this operator to transform a time-delay system into a uncertain LPV system. Consider system (5.52) and note that $x_h(t) = x(t-h(t)) = \mathcal{D}_h(\dot{x}(t))$ then substituting into system (5.52) and once expressed into a LFT form we obtain then

$$\begin{array}{rcl} \dot{y}(t) &=& \bar{A}y(t) - \alpha(t)A_h w_0(t) + B_u u(t) + Ew(t) \\ z_0(t) &=& \dot{y}(t) \\ z(t) &=& \bar{C}y(t) - \alpha(t)C_h w_0(t) + D_u u(t) + Fw(t) \\ w_0(t) &=& \mathcal{D}_h(z_0(t)) \\ \bar{A} &=& A + A_h \\ \bar{C} &=& C + C_h \end{array} \tag{5.56}$$

where $\alpha(t) = \sqrt{h(t)h_M}$ and $y(t)$ is the new state of the system emphasizing that the transformed model is not always equivalent to the original one.

This system is then obviously:

- uncertain due to the presence of the "unknown" structured norm bounded LTV dynamic operator $\mathcal{D}_h$. For this part we will use results of robust stability analysis and robust synthesis.
- parameter varying (even affine in $\alpha(t)$). We will use parameter dependent Lyapunov functions to tackle this time-varying part.

It is clear that this system is not equivalent to (5.52) due to the model transformation inducing additional dynamics (see [Gu et al., 2003]). Just note that additional dynamics may be a source of conservatism in stability analysis. Nevertheless, in the stabilization problem this is less problematic since we aim to stabilize the system and hence we stabilize these additional dynamics (assuming they are stabilizable).

Before introducing the main results of this section based on this model transformation it is necessary to introduce the following sets

$$\begin{array}{rcl} H &:=& [h_{min}, h_{max}] \\ U &:=& [\mu_{min}, \mu_{max}] \\ \hat{H} &:=& [h_{min} - \delta, h_{max} + \delta] \\ \hat{U} &:=& [\mu_{min} - \nu_{min}, \mu_{max} + \nu_{max}] \end{array}$$

The set H corresponds to the set of values taken by the delay, the set U defines the set of values taken by the delay derivative. The sets $\hat{H}$ and $\hat{U}$ represent respectively the set of values taken by the measured delay and its derivative. It is worth mentioning that the measurement error belongs to $[-\delta, \delta]$ while its derivative remains within $[\nu_{min}, \nu_{max}]$.

5.1.7.1 Stability and $\mathcal{L}_2$ performances analysis

This section is devoted to the stability analysis of the transformed system using robust and LPV stability analysis tools. The robustness with respect to the operator $\mathcal{D}_h$ will be ensured using the full-block $\mathcal{S}$-procedure [Scherer, 2001] while the stability with respect to the parameter varying part will be tackled using a parameter dependent Lyapunov function. The full-block $\mathcal{S}$-procedure is used with parameter dependent D-G scalings, $D(\cdot)$ and $G(\cdot)$ being the decision variables, as shown below:

Lemma 5.1.16 *System (5.56) without control input (i.e. $u(t) = 0$) is asymptotically stable for $h \in \mathscr{H}$ and satisfies the $\mathcal{H}_\infty$-norm property $||z||_2/||w||_2 < \gamma(h, \dot{h})$ if there exist a smooth matrix function $P : H \to \mathbb{S}^n_{++}$, matrix functions $D : H \times U \to \mathbb{S}^n_{++}$, $G : H \times U \to \mathbb{K}^n$ and a function $\gamma : H \times U \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} [\bar{A}^T P(h)]^H + \frac{dP}{dh}\dot{h} & -\alpha P(h) A_h + \bar{A}^T G(h,\dot{h}) & P(h)E & \bar{C}^T & \bar{A}^T D(h,\dot{h}) \\ \star & -D(h) - [\alpha A_h^T G(h,\dot{h})]^H & G^T(h,\dot{h})E & -\alpha C_h^T & -\alpha A_h^T D(h,\dot{h}) \\ \star & \star & -\gamma(h,\dot{h}) I_p & F^T & E^T D(h,\dot{h}) \\ \star & \star & \star & -\gamma(h,\dot{h}) I_q & 0 \\ \star & \star & \star & \star & -D(h,\dot{h}) \end{bmatrix} \prec 0 \tag{5.57}$$

holds for all $h \in H$ and $\dot{h} \in U$ with $\alpha = \sqrt{h_{max} h}$.

Proof: *Let us consider system (5.56), it is possible to apply the full-block $\mathcal{S}$-procedure in order to develop an efficient stability test. Combining with $\mathcal{L}_2$ performances we obtain the following LMI*

$$\begin{aligned} &\begin{bmatrix} \dfrac{\partial P}{\partial h}\dot{h} + \bar{A}^T P(h) + P(h)\bar{A} & -\alpha P A_h & PE \\ \star & 0 & 0 \\ \star & \star & -\gamma(h,\dot{h}) I \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \bar{A}^T \\ I & -\alpha A_h^T \\ 0 & E^T \end{bmatrix} \mho(h,\dot{h}) \begin{bmatrix} 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \end{bmatrix} \\ &+\gamma^{-1}(h,\dot{h}) \begin{bmatrix} \bar{C}^T \\ -\alpha A_h^T \\ F^T \end{bmatrix} \begin{bmatrix} \bar{C}^T \\ -\alpha A_h^T \\ F^T \end{bmatrix}^T < 0 \end{aligned} \tag{5.58}$$

where $\mho(h, \dot{h})$ satisfies

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix}^T \mho(h,\dot{h}) \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix} ds > 0 \quad \text{for all } \eta \in \mathcal{L}_2 \tag{5.59}$$

The separator $\mho(h, \dot{h}) = \mho^*(h, \dot{h})$ is chosen following the following facts:

- *As $||\mathcal{D}_h||_\infty < 1$ then $\mathcal{D}_h$ may satisfy*

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix}^T \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mho_1} \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix} ds > 0 \quad \text{for all } \eta \in \mathcal{L}_2 \tag{5.60}$$

- *The uncertain operator $\mathcal{D}_h$ acts entrywise (scalar and repeated diagonally) then it satisfies:*

$$\int_0^t \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{\mho_2} \begin{bmatrix} \mathcal{D}_h(\eta) \\ I_n \end{bmatrix} ds = 0 \quad \text{for all } \eta \in \mathcal{L}_2 \tag{5.61}$$

Hence a set of separators can be parametrized as $\mho = \mho_1 \otimes D + \mho_2 \otimes G$ where $D = D^* > 0$ and arbitrary $G = G^*$. But the set of separators is limited to be hermitian and the signal are real valued then the separator becomes

$$\mho(h, \dot{h}) := \begin{bmatrix} -D(h, \dot{h}) & G^T(h, \dot{h}) \\ \star & D(h, \dot{h}) \end{bmatrix} \tag{5.62}$$

where $D : H \times U \to \mathbb{S}^n_{++}$ and $G : H \times U \to \mathbb{K}^n$. Then expand (5.58) and perform a Schur complement on quadratic term

$$-\begin{bmatrix} \bar{C}^T & \bar{A}^T D(h, \dot{h}) \\ -\alpha C_h & -\alpha A_h D(h, \dot{h}) \\ F & E^T D(h, \dot{h}) \\ & \end{bmatrix} \begin{bmatrix} -\gamma^{-1}(h, \dot{h}) I_q & 0 \\ 0 & -D^{-1}(h, \dot{h}) \end{bmatrix} (\star)^T$$

leads to inequality (5.57). □

The LMI provided by the latter theorem can be easily solved using classical LMI solvers. Moreover, if the parameter dependence is linear then a polytopic relaxation will be exact. However, if the dependence is polynomial then a more complex relaxation scheme should be adopted. For more details about these relaxations, the readers should refer to Sections 1.3.3.2, 1.3.3.3, 1.3.3.4 and 3.2.

5.1.7.2 Delay-Scheduled state-feedback design

We provide in that section the computation of a delay-scheduled state-feedback of the form (5.51) for system (5.52). In this case, the closed-loop system is then given by

$$\begin{array}{rcl} \dot{y}(t) & = & \bar{A}_{cl}(h, \delta_h) y(t) - A_h \alpha(t) w_0(t) + E w(t) \\ z(t) & = & \bar{C}_{cl}(h, \delta_h) y(t) - C_h \alpha(t) w_0(t) + F w(t) \\ z_0(t) & = & \dot{y}(t) \\ w_0(t) & = & \mathcal{D}_h(z_0(t)) \end{array} \tag{5.63}$$

with $\hat{h} = h + \delta_h$, a state feedback of the form $K(h + \delta_h)$ and closed-loop system matrices $\bar{A}_{cl}(h, \delta_h) = \bar{A} + B_u K(\hat{h})$, $\bar{C}_{cl}(h, \delta_h) = \bar{C} + D_u K(\hat{h})$.

As shown in previous sections, there exist several ways to compute this controller:

1. Use an approach involving congruence transformations and change of variable. Using this approach, it is possible to fix a desired form to the controller.

2. Elaborate a stabilizability test (independent of the controller) based on the projection lemma (see Appendix E.18). A suitable controller is then deduced either through a LMI problem or an explicit algebraic equality.

We will only provide here a solution based on a change of variable but a solution based on the projection lemma can also be employed (see Section 5.1.1 for details, differences and interests of these approaches). This approach allows us to fix the controller structure which can be independent of the delay derivative. However, the result may be conservative since it is difficult to choose adequately the controller structure. The approach using the projection lemma is interesting in the sense that it allows to compute the best $\mathcal{L}_2$ performances gain that can be reached using this approach but the controller which is computed from algebraic equations will depend on the delay-derivative.

Theorem 5.1.17 *The system (5.56) is stabilizable with a delay-scheduled state feedback $K(\hat{h}) = Y(\hat{h})X^{-1}(\hat{h})$ if there exists a smooth matrix function $X : \hat{H} \to \mathbb{S}^n_{++}$, matrix functions $Y : \hat{H} \to \mathbb{R}^{m\times n}, \tilde{D} : H \times U \times \hat{H} \times \hat{U} \to \mathbb{S}^n_{++}$, $\tilde{G} : H \times U \times \hat{H} \times \hat{U} \to \mathbb{K}^n$ and a scalar function $\gamma : H \times U \times \hat{H} \times \hat{U} \to \mathbb{R}_{++}$ such that the LMI*

$$\begin{bmatrix} U_{11}(\hat{h},\dot{\hat{h}}) & U_{12}(\hat{h}) & U_{13}(\hat{h},\dot{\hat{h}}) & \alpha A_h\tilde{D}(\xi) & E \\ \star & -\gamma(\xi)I_q & \alpha C_h\tilde{G}^T(\xi) + \bar{C}X(h) & \alpha C_h\tilde{D}(\xi) & F \\ \star & \star & -\dot{\hat{h}}\dfrac{\partial X(\hat{h})}{\partial \hat{h}} - \tilde{D}(\xi) & 0 & 0 \\ \star & \star & \star & -\tilde{D}(\xi) & 0 \\ \star & \star & \star & \star & -\gamma(\xi)I_p \end{bmatrix} \prec 0 \qquad (5.64)$$

holds for all $h \in H$, $\dot{h} \in U$, $\delta_h \in \Delta$ and $\dot{\delta}_h \in \Delta_\nu$, where $\xi = col(h, \delta_h, \dot{h}, \dot{\delta}_h)$ and

$$\begin{aligned} U_{11}(\hat{h},\dot{\hat{h}}) &= -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial \hat{h}} + [X(\hat{h})\bar{A}^T + Y^T(\hat{h})B_u^T]^H \\ U_{12}(\hat{h}) &= X(\hat{h})\bar{C}^T + Y^T(\hat{h})D_u^T \\ U_{13}(\hat{h},\dot{\hat{h}}) &= -\dot{\hat{h}}\frac{\partial X(\hat{h})}{\partial \hat{h}} + \bar{A}X(\hat{h}) + \alpha A_h\tilde{G}^T(\xi) \\ K(\hat{h}) &= Y(\hat{h})X(\hat{h})^{-1} \end{aligned}$$

Proof: *First note that the real unknown delay is $h(t)$ and the estimated one is $\hat{h}(t) = h(t) + \delta_h(t)$. X must depend on $\hat{h}(t)$ only since the controller gain is a function of X. Indeed, if X depends on $h(t)$ hence the controller would depend on $h(t)$ which is not possible since $h(t)$ is unknown. Nevertheless, other variables may depend on all the parameters (i.e. $h(t), \delta_h(t), \dot{h}(t), \dot{\delta}_h(t)$). From here let $\xi = col(h, \delta_h, \dot{h}, \dot{\delta}_h)$ for simplicity. First note that LMI*

(5.58) can be rewritten in the following form

$$(\star)^T M(h,\dot{h}) \underbrace{\left[\begin{array}{ccc} I & 0 & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & I & 0 \\ \bar{A} & -\alpha A_h & E \\ \hline 0 & 0 & I \\ \bar{C} & -\alpha C_h & F \end{array}\right]}_{\mathscr{S}} < 0 \tag{5.65}$$

where $M(h,\dot{h}) = \begin{bmatrix} \dot{h}\dfrac{dP(h)}{dh} & P(h) \\ P(h) & 0 \end{bmatrix} \oplus \mho(h,\dot{h}) \oplus [-\gamma(h,\dot{h})I_p] \oplus [\gamma^{-1}(h,\dot{h})I_q]$. *First inject the closed-loop system into (5.65). Note that* $\dim(M) = 4n+p+q$ *and* $n^-(M) = 2n+p$ *(where* $n^-(M)$ *is the number of strictly negative eigenvalues of the symmetric matrix* M, $n = \dim(x)$, $p = \dim(w)$ *and* $q = \dim(z)$*) and the latter equals the rank of the subspace* $\mathscr{S}$ *(defined in (5.65)). Then it is possible to apply the dualization lemma and we obtain*

$$(\star)^T M^{-1}(\xi) \underbrace{\left[\begin{array}{ccc} -\bar{A}_{cl}^T(\hat{h}) & -\bar{C}^T(\hat{h}) & 0 \\ I_n & 0 & I_n \\ \hline \alpha A_h^T & \alpha C_h^T & 0 \\ 0 & 0 & -I_n \\ \hline -E^T & -F^T & 0 \\ 0 & I_q & 0 \end{array}\right]}_{S^+} > 0 \tag{5.66}$$

where $M^{-1}(\xi) = \begin{bmatrix} \frac{dP(\hat{h})}{dt} & P(\hat{h}) \\ \star & 0 \end{bmatrix}^{-1} \oplus \mho^{-1}(\xi) \oplus [-\gamma^{-1}(\xi)] \oplus [\gamma(\xi)]$.

Let $X = P^{-1}$ *and then* $\dfrac{dX(\hat{h})}{dt} = -X\dfrac{dP(\hat{h})}{dt}X$, *we have* $\begin{bmatrix} \frac{dP(\hat{h})}{dt} & P(\hat{h}) \\ \star & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & X(\hat{h}) \\ \star & \frac{dX(\hat{h})}{dt} \end{bmatrix}$.

Denote also $\mho^{-1}(\xi) = \begin{bmatrix} -\tilde{D}(\xi) & \tilde{G}^T(\xi) \\ \star & \tilde{D}(\xi) \end{bmatrix}$ *with* $\tilde{D} \in \mathbb{S}^n_{++}$ *and* $\tilde{G} \in \mathbb{K}^n$. *Moreover* $\mho^{-1}(\cdot)$ *satisfies the inequality*

$$\begin{bmatrix} -I_n \\ \mathcal{D}_h^T(\cdot) \end{bmatrix}^T \begin{bmatrix} -\tilde{D}(\xi) & \tilde{G}^T(\xi) \\ \star & \tilde{D}(\xi) \end{bmatrix} \begin{bmatrix} -I_n \\ \mathcal{D}_h^T(\cdot) \end{bmatrix} < 0 \tag{5.67}$$

Then expand (5.66) and noticing that $\tilde{R}(h,\dot{h}) < 0$, *the Schur complement can be used on the quadratic term:*

$$-\begin{bmatrix} \alpha A_h\tilde{D} & E \\ \alpha C_h\tilde{D} & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{D}^{-1} & 0 \\ 0 & \gamma^{-1}(\xi)I_p \end{bmatrix} (\star)^T \tag{5.68}$$

Finally multiplying the LMI by -1 (to get a negative definite inequality) we obtain inequality (5.64) in which $Y(\hat{h}) = K(\hat{h})X(\hat{h})$ *is a linearizing change of variable.* □

We have expressed the stability and stabilizability problems as polynomially parametrized LMIs (5.57) and (5.64). Moreover the $\mathcal{L}_2$-induced norm is expressed as a positive function of the parameters and its minimization is not a well-defined problem since we cannot minimize a function. We detail in the following how to turn this problem into a tractable one.

Since the cost to be minimized needs to be unique for every parameters, the idea is here, to provide an idea on how to turn the semi-infinite number of cost (defined for each value of the parameters) into a single one. This step is performed by an integration procedure with respect to some specific measure.

Let us illustrate this on the elementary cost $\gamma(\xi)$, $\xi = \text{col}(h, \delta_h, \dot{h}, \dot{\delta}_h)$. It is possible to define several 'general' costs $\mathcal{J}(\cdot)$:

$$\mathcal{J}_\theta(\gamma) \quad := \quad \int_{H\times U\times \bar{H}\times \hat{U}} \gamma(\xi) d\theta\xi \tag{5.69}$$

where $d\theta(\xi)$ is a probability measure over $H \times U \times \bar{H} \times \hat{U}$ (i.e. $\displaystyle\int_{H\times U\times \bar{H}\times \hat{U}} d\theta\xi = 1$).

We propose here some interesting values of the measure $d\theta(\cdot,\cdot)$:

- $d\theta_1(\xi) = \mu(H \times U \times \bar{H} \times \hat{U})^{-1}$ where $\mu(\cdot)$ is the Lebesgue measure.
- $d\theta_2(\xi) = \delta(\prod_{i=1}^{f}(\xi - \xi_i))$ with $\delta(t)$ is the Dirac distribution.
- $d\theta_3(\xi) = p(\xi)$ where $p(\cdot)$ denotes for instance a probability density function.

The first one minimizes the volume below the hypersurface defined by the application $\gamma : H \times U \times \bar{H} \times \hat{U} \to \mathbb{R}_+$ with equal preference for any parameter values. The second one aims to minimize the $\mathcal{H}_\infty$-norm, specifically for certain delay, errors and their derivative values. This may be interesting for systems with discrete valued delays. The third one is dedicated when we have a stochastic model of the delay (and eventually a model for its derivative) attempts for instance to minimize in priority the $\mathcal{H}_\infty$-norm for high probable delay values.

Example 5.1.18 *We aim to stabilize the following time delay system with time-varying delay*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 11 & 23 \\ 14 & 16 \end{bmatrix} x(t) + \begin{bmatrix} 15 & 18 \\ 12 & 23 \end{bmatrix} x_h(t) + \begin{bmatrix} 12 \\ 0 \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 11 \\ 22 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \end{aligned} \tag{5.70}$$

with $h(t) \in [0, 5]$, $|\dot{h}| \le 0.2$, $\dot{\delta}_h, \delta_h \in [-0.1, 0.1]$. We choose a constant $\gamma > 0$ and constant scaling (i.e. constant $\tilde{D}, \tilde{G}$). We compute three different controllers using Theorem 5.1.17: a constant feedback, a linear state-feedback (i.e. $K(\hat{h}) = K_0 + K_1\hat{h}$) and a rational state-feedback (i.e. $K(\hat{h}) = (K_{n0} + K_{n1}\hat{h})(K_{d0} + K_{d1}\hat{h})^{-1}$). The results are summarized in table 5.1. We can see that in this example, taking a constant or linearly dependent controller does not lead to a better closed-loop $\mathcal{H}_\infty$-norm. Nevertheless, taking a rationally dependent controller leads to a better $\mathcal{H}_\infty$-norm for the closed-loop system. This happens for two main reasons:

1. *The controller have a more complex form*
2. *Through the use of a parameter dependent Lyapunov matrix, the information on the delay-derivative is embedded and then reduce the conservatism.*

controller	γ
constant	*19.4055*
linear	*19.4055*
rational	*15.5030*

Table 5.1: Minimal $\mathcal{H}_\infty$-norm of the closed-loop system

5.2 Dynamic Output Feedback Control laws

This section is devoted to the stabilization of time-delay systems by output-feedback. Two different laws will be analyzed: the first one is called 'observer based control law' which means that the controller is composed by an observer which estimate the system state and a state-feedback control law which generate the control input from the estimated state. The second type is a direct approach where the controller is a full-block and all the matrices are sought such that the closed-loop system is asymptotically stable.

The main difficulty in the synthesis of observer-based control laws is the fact that, first of all, it is not possible to exactly linearize the conditions by congruence transformations and change of variables due to a low number of degrees of freedom (two for observer based-control laws). However, the obtained controller is rather simple and then easy to implement.

In the full-block output-feedback control law framework, congruence transformations and change of variable are possible. Moreover, this approach leads to exact LMI conditions when dealing with output-feedback with memory (when the delay is exactly known). Nevertheless, such a case almost never occurs since the delay is generally not exactly known except in some vary special cases (for instance when the delay represents a variable sampling period [Fridman et al., 2004, Suplin et al., 2007]). This is the reason why memoryless controller are often preferred but are more difficult to design due to the presence of bilinear terms (non-linearizable) in the resulting conditions.

In [Sename and Briat, 2006], the problem of finding a observer-based control law for LTI time-delay systems is derived through iterative LMI conditions. The result is provided in the delay-independent framework only. This section will consider delay-dependent results only and in both memoryless and with memory types. It is also possible to elaborate delay-scheduled dynamic output feedback control laws based on the approach detailed in Section 5.1.7 but tractable conditions can only be obtained using more simple scalings than D-G scalings. Otherwise, iterative LMI conditions procedure would deal with such problems in which matrices and their inverse coexist in the same problems. This will not be treated in the current thesis.

5.2.1 Memoryless observer based control laws

This section aims at developing sufficient conditions to the existence of a memoryless observer-based control law of the form

$$\begin{array}{rcl}\dot{\xi}(t) & = & A(\rho)\xi(t) + B(\rho)u(t) + L(\rho)(y(t) - C_y(\rho)\xi(t)) \\ u(t) & = & -K(\rho)\xi(t)\end{array} \tag{5.71}$$

for LPV time-delay systems

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + D(\rho)u(t) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + F_y(\rho)w(t) \end{array} \tag{5.72}$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and $z \in \mathbb{R}^r$ are respectively the system state, the controller state, the control input, the measured output, the exogenous inputs and the controlled outputs. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ and the parameters $\rho \in U_\nu$ with $\dot{\rho} \in \text{hull}[U_\nu]$.

In such a case, the closed-loop system can be expressed by the equations

$$\begin{array}{rcl} \begin{bmatrix} \dot{e}(t) \\ \dot{x}(t) \end{bmatrix} & = & \begin{bmatrix} A(\rho) - L(\rho)C_y(\rho) & 0 \\ B(\rho)K(\rho) & A(\rho) - B(\rho)K(\rho) \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \\ & & + \begin{bmatrix} 0 & A_h(\rho) \\ 0 & A_h(\rho) \end{bmatrix} \begin{bmatrix} e(t-h(t)) \\ x(t-h(t)) \end{bmatrix} + \begin{bmatrix} E(\rho) - L(\rho)F_y(\rho) \\ E(\rho) \end{bmatrix} w(t) \end{array} \tag{5.73}$$

and we define an extended output vector $\tilde{z}(t) = \begin{bmatrix} Te(t) \\ z(t) \end{bmatrix}$ with T full row rank.

The role of the matrix T is to weight the observation error in order to reduce the impact of the disturbances on the observation error.

Before introducing the main result of the section, known methodologies will be briefly introduced here. A common methodology is to assume that the Lyapunov matrix multiplied with system matrices are block-diagonal and each block corresponds to a specific part of the augmented system (i.e. the observation error and the system-state).

It is worth mentioning that since the design matrices $K(\rho)$ and $L(\rho)$ are not multiplied in the same fashion with system matrices ($L(\rho)$ is free from the left while $K(\rho)$ is free from the right) then this suggests that congruence transformations would lead to nonlinear terms with possibility of linearization. Hence a commutation approach has been introduced (see for instance [Chen, 2007]) where a block of Lyapunov matrix is constrained such that it commutes with a system matrix in order to linearize the equations. For instance, the matrix X is constrained such that it commutes with the matrix C_y, i.e. $C_yX = \hat{X}C_y$ where $\text{rank}[C_y] = p$. In this case, the change of variable $\hat{L} = L\hat{X}$ is allowed but this considerably increases the conservatism of the approach. Actually, this approach has been introduced to deal with the static-output feedback design [Daafouz et al., 2002]. Moreover, it appears difficult when the observer gain appears in different places (e.g. $A - LC_y$ and $E - LF_y$) since in this case it is not possible to linearize the equations. In our case, this method cannot be applied since the measured output depends on the disturbances.

In the presented method no congruence transformations are applied but we use a simple approach to bound nonlinear terms. The methodology is the following: first of all a correct Lyapunov-Krasovskii functional is chosen. The LMI conditions are then relaxed in order to remove all the coupled terms and then the extended system expression is substituted into. Finally 'annoying' (nonlinear) terms are then bounded in order to get finally easily tractable LMI conditions.

Theorem 5.2.1 *There exist an observer-based control law of the form (5.71) which asymptotically stabilizes system (5.72) for all $h \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix $P : U_\rho \to \mathbb{S}_{++}^n$, matrix functions $X_0, X_c : U_\rho \to \mathbb{R}^{n\times n}$, $K : U_\rho \to \mathbb{R}^{m\times n}$, $L_o : U_\rho \to \mathbb{R}^{n\times p}$,* *constant matrices $Q, R \in \mathbb{S}^n_{++}$, scalar functions $\alpha_1, \alpha_2 : U_\rho \to \mathbb{R}_{++}$ and a constant scalar $\gamma > 0$ such that the following LMI*

$$\left[\begin{array}{cccc|c} -(X+X^T) & \Omega_2(\rho)^T & \Omega_3(\rho)^T & \Omega_5(\rho)^T & \\ \star & \Omega_4(\rho,\dot{\rho}) & \Omega_6(\rho)^T & 0 & \Omega_c^T \\ \star & \star & \Omega_8(\rho) & 0 & \\ \star & \star & \star & \Omega_{10}(\rho) & \\ \hline & & \star & & -\Omega_d \end{array}\right] \prec 0 \tag{5.74}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ where

$$\begin{array}{rcl}
\Omega_2(\rho) & = & \begin{bmatrix} P(\rho) + \begin{bmatrix} A(\rho)^T X_o(\rho) - C_y(\rho)^T L_o(\rho)^T & 0 \\ 0 & A(\rho)^T X_c \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ A_h(\rho)^T X_o(\rho) & A_h(\rho)^T X_c(\rho) \end{bmatrix} \end{bmatrix} \\
\Omega_3(\rho) & = & \begin{bmatrix} E(\rho)^T X_o(\rho) - F_y(\rho)^T L_o(\rho)^T \\ E(\rho)^T X_c(\rho) \\ 0 \\ 0 \end{bmatrix} \\
\Omega_5(\rho) & = & \begin{bmatrix} X \\ h_{max} R \end{bmatrix} \\
\Omega_6(\rho) & = & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 \\ D(\rho)K(\rho) & C(\rho) - D(\rho)K(\rho) & 0 & C_h(\rho) \end{bmatrix} \\
\Omega_8(\rho) & = & \begin{bmatrix} -\gamma(\rho) I_w & F(\rho)^T \\ \star & -\gamma(\rho) I_z \end{bmatrix} \\
\Omega_{10}(\rho) & = & \begin{bmatrix} -P(\rho) & -h_{max} R \\ \star & -R \end{bmatrix} \\
\Omega_c(\rho) & = & \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \alpha_1(\rho) X_c(\rho)^T B(\rho) & \alpha_1(\rho) X_c(\rho)^T B(\rho) & 0 & 0 \\ \hline 0 & 0 & K(\rho)^T & 0 \\ 0 & 0 & 0 & K(\rho)^T \\ \hline 0 & 0 & & \\ 0 & 0 & & \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array}\right] \\
\Omega_d(\rho) & = & \begin{bmatrix} \alpha_1(\rho) I & 0 & 0 & 0 \\ \star & \alpha_2(\rho) I & 0 & 0 \\ \star & \star & \alpha_1(\rho) I & 0 \\ \star & \star & \star & \alpha_2(\rho) I \end{bmatrix}
\end{array}$$

Moreover the observer gain $L(\rho) = X_o(\rho)^{-T} L_o(\rho)$ and the closed-loop satisfies $||\tilde{z}||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *First of all we assume that the matrix X is structured as follows:*

$$X = diag(X_o, X_c)$$

Since we are interested in a simple stabilization test, we will consider the Lyapunov-Krasovskii functional of Section 3.5.1 whose relaxation is provided in Section 3.5.2. After substitution of the extended system in the LMI of Lemma 3.5.2 we get

$$\Xi = \begin{bmatrix} -(X(\rho)+X(\rho)^T) & \Xi_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T \\ \Xi_2(\rho) & \Xi_4(\rho,\dot{\rho}) & \Xi_6(\rho)^T & 0 \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) \end{bmatrix}$$

$$\Xi_2(\rho) = \begin{bmatrix} P(\rho) + \begin{bmatrix} A(\rho)^T X_o(\rho) - C_y(\rho)^T L_o(\rho)^T & -K(\rho)^T B(\rho)^T X_c(\rho) \\ 0 & A(\rho)^T X_c - K(\rho)^T B(\rho)^T X_c(\rho) \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ A_h(\rho)^T X_o(\rho) & A_h(\rho)^T X_c(\rho) \end{bmatrix} \end{bmatrix}$$

$$\Xi_3(\rho) = \begin{bmatrix} E(\rho)^T X_o(\rho) - F_y(\rho)^T L_o(\rho)^T \\ E(\rho)^T X_c(\rho) \\ 0 \\ 0 \end{bmatrix}$$

$$\Xi_5(\rho) = \begin{bmatrix} X \\ h_{max} R \end{bmatrix}$$

$$\Xi_6(\rho) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 \\ D(\rho)K(\rho) & C(\rho) - D(\rho)K(\rho) & 0 & C_h(\rho) \end{bmatrix}$$

$$\Xi_8(\rho) = \begin{bmatrix} -\gamma(\rho) I_w & F(\rho)^T \\ \star & -\gamma(\rho) I_z \end{bmatrix}$$

$$\Xi_{10}(\rho) = \begin{bmatrix} -P(\rho) & -h_{max} R \\ \star & -R \end{bmatrix}$$

$$\Xi_4(\rho,\dot{\rho}) = \begin{bmatrix} \dfrac{\partial P(\rho)}{\partial \rho}\dot{\rho} - P(\rho) + Q - R & R \\ \star & -(1-\mu)Q - R \end{bmatrix}$$

The main difficulty comes from the bilinear term $X_c(\rho)^T B(\rho) K(\rho)$. It is worth mentioning that in this case it is not possible to find a linearizing congruence transformation. However, it is possible to use the well-known on cross-terms heavily used in time-delay systems (see Appendix F.2):

$$\begin{aligned} -2x_3^T X_c(\rho)^T B(\rho) K(\rho) x_2 &\leq \alpha_1(\rho) x_3^T X_c(\rho)^T B(\rho) B(\rho)^T X_c(\rho) x_3 + \alpha_1(\rho)^{-1} x_2^T K(\rho)^T K(\rho) x_2 \\ -2x_4^T X_c(\rho)^T B(\rho) K(\rho) x_2 &\leq \alpha_2(\rho) x_4^T X_c(\rho)^T B(\rho) B(\rho)^T X_c(\rho) x_4 + \alpha_2(\rho)^{-1} x_2^T K(\rho)^T K(\rho) x_2 \end{aligned}$$

for any real valued vectors x_2, x_3, x_4 of appropriate dimensions and real valued positive scalar functions $\alpha_1(\cdot), \alpha_2(\cdot)$. Using these inequalities it is possible to show that the following inequality implies $\Xi < 0$:

$$\Upsilon = \begin{bmatrix} -(X+X^T)+Y_1 & \Upsilon_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T \\ \Upsilon_2(\rho) & \Upsilon_4(\rho,\dot{\rho}) & \Xi_6(\rho)^T & 0 \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) \end{bmatrix} \prec 0 \tag{5.75}$$

with

$$Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & (\alpha_2(\rho)+\alpha_1(\rho))[X_c(\rho)^T B(\rho) B(\rho)^T X_c(\rho)] \end{bmatrix} \tag{5.76}$$

$$\Upsilon_2(\rho) = \begin{bmatrix} P(\rho) + \begin{bmatrix} A(\rho)^T X_o(\rho) - C_y(\rho)^T L_o(\rho)^T & 0 \\ 0 & A(\rho)^T X_c \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ A_h(\rho)^T X_o(\rho) & A_h(\rho)^T X_c(\rho) \end{bmatrix} \end{bmatrix} \tag{5.77}$$

$$\begin{aligned} \Upsilon_4(\rho,\dot{\rho}) &= \Xi_4(\rho,\dot{\rho}) + Y_2 \\ Y_2 &= \begin{bmatrix} \alpha_1(\rho)^{-1} K(\rho)^T K(\rho) & 0 \\ 0 & \alpha_2(\rho)^{-1} K(\rho)^T K(\rho) \end{bmatrix} \end{aligned}$$

Finally since

$$\begin{aligned} Y_1 &= \begin{bmatrix} 0 & 0 \\ \alpha_1(\rho) X_c(\rho)^T B(\rho) & \alpha_2(\rho) X_c(\rho)^T B(\rho) \end{bmatrix} \begin{bmatrix} \alpha_1(\rho)^{-1} I & 0 \\ 0 & \alpha_2(\rho)^{-1} I \end{bmatrix} (\star)^T \\ Y_2 &= \begin{bmatrix} K(\rho)^T & 0 \\ 0 & K(\rho)^T \end{bmatrix} \begin{bmatrix} \alpha_1(\rho)^{-1} I & 0 \\ 0 & \alpha_2(\rho)^{-1} I \end{bmatrix} (\star)^T \end{aligned}$$

where $(\star)^T$ stands for the symmetric part of the quadratic term, then Υ may be rewritten into the form

$$\begin{aligned} \Upsilon &= \begin{bmatrix} -(X+X^T) & \Upsilon_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T \\ \Upsilon_2(\rho) & \Xi_4(\rho,\dot{\rho}) & \Xi_6(\rho)^T & 0 \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) \end{bmatrix} \\ &+ \underbrace{\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \alpha_1(\rho) X_c(\rho)^T B(\rho) & \alpha_1(\rho) X_c(\rho)^T B(\rho) & 0 & 0 \\ \hline 0 & 0 & K(\rho)^T & 0 \\ 0 & 0 & 0 & K(\rho)^T \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\Upsilon_c(\rho)} \Upsilon_d(\rho)^{-1} (\star)^T \prec 0 \end{aligned}$$

with $\Upsilon_d(\rho)^{-1} = \begin{bmatrix} \alpha_1(\rho)^{-1} I & 0 & 0 & 0 \\ \star & \alpha_2(\rho)^{-1} I & 0 & 0 \\ \star & \star & \alpha_1(\rho)^{-1} I & 0 \\ \star & \star & \star & \alpha_2(\rho)^{-1} I \end{bmatrix}$.

And finally applying Schur complement we get

$$\left[\begin{array}{cccc|c} -(X+X^T) & \Upsilon_2(\rho)^T & \Xi_3(\rho)^T & \Xi_5(\rho)^T & \\ \Upsilon_2(\rho) & \Xi_4(\rho,\dot{\rho}) & \Xi_6(\rho)^T & 0 & \Upsilon_c^T \\ \Xi_3(\rho) & \Xi_6(\rho) & \Xi_8(\rho) & 0 & \\ \Xi_5(\rho) & 0 & 0 & \Xi_{10}(\rho) & \\ \hline & & \Upsilon_c & & \Upsilon_d \end{array}\right] < 0 \tag{5.78}$$

which is linear in $X_o, X_c, L_o, K, P, Q, R$ □

Remark 5.2.2 *The procedure is similar when dealing with observer-based control law with memory:*

$$\begin{array}{rcl} \dot{\xi}(t) & = & A(\rho)\xi(t) + A_h(\rho)\xi(t-h(t)) + B(\rho)u(t) \\ & & s + L(\rho)(y(t) - C_y(\rho)\xi(t) - C_{yh}(\rho)\xi(t-h(t))) \\ u(t) & = & -K(\rho)\xi(t) - K_h(\rho)\xi(t-h(t)) \end{array} \tag{5.79}$$

Indeed, in this case, the extended system would be

$$\begin{aligned} \begin{bmatrix} \dot{e}(t) \\ \dot{x}(t) \end{bmatrix} = & \begin{bmatrix} A(\rho) - L(\rho)C_y(\rho) & 0 \\ B(\rho)K(\rho) & A(\rho) - B(\rho)K(\rho) \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \\ & + \begin{bmatrix} A_h(\rho) - L(\rho)C_{yh}(\rho) & 0 \\ B(\rho)K_h(\rho) & A_h(\rho) - B(\rho)K_h(\rho) \end{bmatrix} \begin{bmatrix} e(t-h(t)) \\ x(t-h(t)) \end{bmatrix} + \begin{bmatrix} E - LF_y \\ E \end{bmatrix} w(t) \end{aligned}$$

For such a system the same procedure applies and then is not detailed here.

5.2.2 Dynamic Output Feedback with memory design - exact delay case

This section is devoted to the design of a dynamic output feedback controller with memory. The delay is assumed here to be exactly known. The advantage of such controllers resides in the existence of congruence transformations and linearizing change of variables. However, they are difficult to implement in practice due to the imprecision on the delay value knowledge. Section 2.2.2 presents methods allowing to deal a posteriori on delay uncertainty that can be used in order to give a bound on the maximal error on the delay value knowledge that can be tolerated.

The class of systems under consideration is given by:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B(\rho)u(t) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + D(\rho)u(t) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + C_{yh}(\rho)x(t-h(t)) + F_y(\rho)w(t) \end{array} \tag{5.80}$$

for which the following stabilizing controllers have to be designed

$$\begin{array}{rcl} \dot{x}_c(t) & = & A_c(\rho)x_c(t) + A_{hc}(\rho)x_c(t-h(t)) + B_c(\rho)y(t) \\ u(t) & = & C_c(\rho)x_c(t) + C_{hc}(\rho)x_c(t-h(t)) + D_c(\rho)y(t) \end{array} \tag{5.81}$$

where $x \in \mathbb{R}^n$, $x_c \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $w \in \mathbb{R}^q$ and $z \in \mathbb{R}^r$ are respectively the system state, the controller state, the control input, the measured output, the exogenous inputs and the controlled outputs. The delay $h(t)$ is assumed to belong to the set $\mathscr{H}_1^\circ$ and the parameters $\rho \in U_\nu$ with $\dot{\rho} \in \text{hull}[U_\nu]$.

The closed-loop system is given by

$$
\begin{aligned}
\dot{\bar{x}}(t) &= \underbrace{\begin{bmatrix} A + BD_cC_y & BC_c \\ B_cC_y & A_c \end{bmatrix}}_{A_{cl}} \bar{x}(t) + \underbrace{\begin{bmatrix} A_h + BD_cC_{yh} & BC_{hc} \\ B_cC_{yh} & A_{hc} \end{bmatrix}}_{A_{hcl}} \bar{x}(t-h(t)) \\
&\quad + \underbrace{\begin{bmatrix} E + BD_cF_y \\ B_cF_y \end{bmatrix}}_{E_{cl}} w(t) \\
z(t) &= \underbrace{\begin{bmatrix} C + DD_cC_y & DC_c \end{bmatrix}}_{C_{cl}} \bar{x}(t) + \underbrace{\begin{bmatrix} C_h + DD_cC_{yh} & DC_{hc} \end{bmatrix}}_{C_{hcl}} \bar{x}(t-h(t)) \\
&\quad + \underbrace{(F + DD_cF_y)}_{F_{cl}} w(t)
\end{aligned}
\tag{5.82}
$$

with $\bar{x}(t) = col(x(t), x_c(t))$ and where the dependence on the parameters has been dropped in order to improve the clarity.

The methodology to develop the main theorem is a bit different than for the other methods and is inspired from [Scherer and Wieland, 2005]. The method is based on a LMI relaxation of a Lyapunov-Krasovskii based approach. After substitution of the closed-loop system, a congruence transformation and a linearization change of variable are performed.

Theorem 5.2.3 *There exists a dynamic output feedback of the form (5.81) for system (5.80) with $h(t) \in \mathscr{H}_1^\circ$ if there exist a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}_{++}^{2n}$, constant matrices $W_1, X_1 \in \mathbb{S}_{++}^n$, $\tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^{2n}$, a scalar function $\alpha : U_\rho \to \mathbb{R}_{++}$ and a scalar $\gamma > 0$ such that the LMI*

$$
\begin{bmatrix}
-2\tilde{X} & P(\rho) + \mathcal{A}(\rho) & \mathcal{A}_h(\rho) & \mathcal{E}(\rho) & 0 & \tilde{X} & h_{max}\tilde{R} \\
\star & U_{22}(\rho,\nu) & \tilde{R} & 0 & \mathcal{C}(\rho)^T & 0 & 0 \\
\star & \star & U_{33} & 0 & \mathcal{C}_h(\rho)^T & 0 & 0 \\
\star & \star & \star & -\gamma I & \mathcal{F}(\rho)^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I & 0 & 0 \\
\star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\
\star & \star & \star & \star & \star & \star & -\tilde{R}
\end{bmatrix} \prec 0
\tag{5.83}
$$

holds for all $(\rho,\nu)\in U_\rho\times U_\nu$ *where*

$$\begin{aligned}
\tilde{X} &= \begin{bmatrix} W_1 & I \\ I & X_1 \end{bmatrix}\\
U_{22}(\rho,\dot\rho) &= U_{22}(\rho,\nu)-\tilde{P}(\rho)+\tilde{Q}-\tilde{R}+\partial_\rho\tilde{P}(\rho)\nu\\
U_{33} &= -(1-\mu)\tilde{Q}-\tilde{R}\\
\mathcal{A}(\rho) &= \begin{bmatrix} A(\rho)W_1+B(\rho)\mathcal{C}_c(\rho) & A(\rho)+B(\rho)\mathcal{D}_c(\rho)C_y(\rho) \\ \mathcal{A}_c(\rho) & X_1A(\rho)+\mathcal{B}_c(\rho)C_y(\rho) \end{bmatrix}\\
\mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho)W_1+B(\rho)\mathcal{C}_c(\rho) & A(\rho)+B(\rho)\mathcal{D}_c(\rho)C_{yh}(\rho) \\ \mathcal{A}_{hc}(\rho) & X_1A_h(\rho)+\mathcal{B}_c(\rho)C_{yh}(\rho) \end{bmatrix}\\
\mathcal{E}(\rho) &= \begin{bmatrix} E(\rho)+B(\rho)\mathcal{D}_c(\rho)F_y(\rho) \\ X_1E(\rho)+\mathcal{B}_c(\rho)F_y(\rho) \end{bmatrix}\\
\mathcal{C}(\rho) &= \begin{bmatrix} C_y(\rho)W_1+D(\rho)\mathcal{C}_c(\rho) & C_y(\rho)+D(\rho)\mathcal{D}_c(\rho)C_y(\rho) \end{bmatrix}\\
\mathcal{C}_h(\rho) &= \begin{bmatrix} C_h(\rho)W_1+D(\rho)\mathcal{C}_{yh}(\rho) & C_h(\rho)+D(\rho)\mathcal{D}_c(\rho)C_{yh}(\rho) \end{bmatrix}\\
\mathcal{F}(\rho) &= \begin{bmatrix} F(\rho)+D(\rho)\mathcal{D}_c(\rho)F_y(\rho) \end{bmatrix}
\end{aligned}$$

In this case the corresponding controller is given by

$$\begin{aligned}
\begin{bmatrix} A_c(\rho) & A_{hc}(\rho) & B_c(\rho) \\ C_c(\rho) & C_{hc}(\rho) & D_c(\rho) \end{bmatrix} &= \mathcal{M}_1(\rho)^{-1}\left(\begin{bmatrix} \mathcal{A}_c(\rho) & \mathcal{A}_{hc}(\rho) & \mathcal{B}_c(\rho) \\ \mathcal{C}_c(\rho) & \mathcal{C}_{hc}(\rho) & \mathcal{D}_c(\rho) \end{bmatrix}-\mathcal{M}_2(\rho)\right)\mathcal{M}_3(\rho)^{-1}\\
\mathcal{M}_1(\rho) &= \begin{bmatrix} X_2 & X_1B(\rho) \\ 0 & I \end{bmatrix}\\
\mathcal{M}_2(\rho) &= \begin{bmatrix} X_1A(\rho)W_1 & X_1A_h(\rho)W_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\\
\mathcal{M}_3(\rho) &= \begin{bmatrix} W_2^T & 0 & 0 \\ 0 & W_2^T & 0 \\ C_y(\rho)W_1 & C_{yh}(\rho)W_1 & I \end{bmatrix}\\
X^{-1} &= \begin{bmatrix} X_1 & X_2 \\ \star & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} W_1 & W_2 \\ \star & W_3 \end{bmatrix}
\end{aligned}$$

and the closed-loop system satisfies $||z||_{\mathcal{L}_2}\le\gamma||w||_{\mathcal{L}_2}$.

Proof: *First of all we rewrite the closed-loop system under the form*

$$\begin{aligned}
\left[\begin{array}{c|c|c} A_{cl} & A_{hcl} & E_{cl} \\ \hline C_{cl} & C_{hcl} & F_{cl} \end{array}\right] &= \Theta+\left[\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & D \end{array}\right]\Omega\left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ C_y & 0 & C_{yh} & 0 & F_y \end{array}\right]\\
\Theta &= \left[\begin{array}{cc|cc|c} A & 0 & A_h & 0 & E \\ 0 & 0 & 0 & 0 & 0 \\ \hline C & 0 & C_h & 0 & F \end{array}\right]\\
\Omega &= \begin{bmatrix} A_c & A_{hc} & B_c \\ C_c & C_{hc} & D_c \end{bmatrix}
\end{aligned}$$

For simplicity we restrict X to be a symmetric positive definite matrix such that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \qquad W := X^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} \tag{5.84}$$

By injecting the closed-loop system in LMI (3.109) of Theorem 3.5.5 we get

$$\begin{bmatrix} -2X & P(\rho) + X^T A_{cl}(\rho) & X^T A_{hcl}(\rho) & X^T E_{cl}(\rho) & 0 & X & h_{max}R \\ \star & U_{22}(\rho,\dot{\rho}) & R & 0 & C_{cl}(\rho)^T & 0 & 0 \\ \star & \star & U_{33} & 0 & C_{hcl}(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F_cl(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0$$

with

$$\begin{aligned} U_{22}(\rho,\dot{\rho}) &= -P(\rho) + Q - R + \partial_\rho P(\rho)\nu \\ U_{33} &= -(1-\mu)Q - R \end{aligned}$$

This inequality is obviously nonlinear and in order to linearize it we perform a congruence transformation with respect to the matrix $\mathrm{diag}(Z^T, Z^T, Z^T, I, I, Z^T, Z^T)$ *where*

$$Z := \begin{bmatrix} W_1 & I \\ W_2^T & 0 \end{bmatrix} \tag{5.85}$$

Then we obtain

$$\begin{bmatrix} -2Z^T XZ & V_{12}(\rho) & V_{13}(\rho) & Z^T X^T E_{cl}(\rho) & 0 & Z^T XZ & h_{max}\tilde{R} \\ \star & V_{22}(\rho,\nu) & \tilde{R} & 0 & Z^T C_{cl}(\rho)^T & 0 & 0 \\ \star & \star & V_{33} & 0 & Z^T C_{hcl}(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I & F_cl(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} \prec 0$$

with

$$\begin{aligned} V_{12}(\rho) &= \tilde{P}(\rho) + Z^T X^T A_{cl}(\rho) Z \\ V_{13}(\rho) &= V_{13}(\rho) Z^T X^T A_{hcl}(\rho) Z \\ V_{22}(\rho,\nu) &= V_{22}(\rho,\nu) - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} + \partial_\rho \tilde{P}(\rho)\nu) Z \\ V_{33} &= -[(1-\mu)\tilde{Q} + \tilde{R}] \\ \tilde{P}(\rho) &= Z^T P(\rho) Z \\ \tilde{Q} &= Z^T Q Z \\ \tilde{R} &= Z^T R Z \end{aligned}$$

Note that

$$\begin{aligned} Z^T X &= \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} \\ Z^T X Z &= \begin{bmatrix} W_1 & I \\ I & X_1 \end{bmatrix} \end{aligned}$$

and then defining

$$\mathcal{Z} = \left[\begin{array}{c|c|c} Z^T X A_{cl} Z & Z^T X A_{hcl} Z & Z^T X E_{cl} \\ \hline C_{cl} Z & C_{hcl} Z & F_{cl} \end{array}\right] \tag{5.86}$$

we get

$$\begin{aligned} \mathcal{Z} &= \left[\begin{array}{cc|cc|c} AW_1 & A & A_h W_1 & A & E \\ 0 & X_1 A & 0 & X_1 A_h & X_1 E \\ \hline C_y W_1 & C_y & C_h W_1 & C_h & F \end{array}\right] + \Theta_1 \begin{bmatrix} \mathcal{A}_c & \mathcal{A}_{hc} & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{C}_{hc} & \mathcal{D}_c \end{bmatrix} \\ \Theta_1 &= \left[\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & D \end{array}\right] \\ \Theta_2 &= \left[\begin{array}{cc|cc|c} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & C_y & 0 & C_{yh} & F_y \end{array}\right] \end{aligned} \tag{5.87}$$

$$\begin{aligned} \begin{bmatrix} \mathcal{A}_c & \mathcal{A}_{hc} & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{C}_{hc} & \mathcal{D}_c \end{bmatrix} &= \begin{bmatrix} X_1 A W_1 & X_1 A h W_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Omega_1 \begin{bmatrix} A_c & A_{hc} & B_c \\ C_c & C_{hc} & D_c \end{bmatrix} \\ \Omega_1 &= \begin{bmatrix} X_2 & X_1 B \\ 0 & I \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} W_2^T & 0 & 0 \\ 0 & W_2^T & 0 \\ C_y W_1 & C_{yh} W_1 & I \end{bmatrix} \end{aligned} \tag{5.88}$$

Finally we get

$$\mathcal{Z} = \left[\begin{array}{cc|cc|c} AW_1 + B\mathcal{C}_c & A + B\mathcal{D}_c C_y & A_h W_1 + B\mathcal{C}_c & A + B\mathcal{D}_c C_{yh} & E + B\mathcal{D}_c F_y \\ \mathcal{A}_c & X_1 A + \mathcal{B}_c C_y & \mathcal{A}_{hc} & X_1 A_h + \mathcal{B}_c C_{yh} & X_1 E + \mathcal{B}_c F_y \\ \hline C_y W_1 + D\mathcal{C}_c & C_y + D\mathcal{D}_c C_y & C_h W_1 + D\mathcal{C}_{yh} & C_h + D\mathcal{D}_c C_{yh} & F + D\mathcal{D}_c F_y \end{array}\right] \tag{5.89}$$

which shows that the equations are linearized with respect to the new variables ($\mathcal{A}_c$, $\mathcal{A}_{hc}$, $\mathcal{B}_c$, $\mathcal{C}_c$, $\mathcal{C}_{ch}$, $\mathcal{D}_c$). Finally replacing the linearized values into the inequality leads to the result. The construction of the controller is performed by the inversion of the change of variable. □

Remark 5.2.4 *The design of a memoryless controller of the form*

$$\begin{aligned} x_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{5.90}$$

is more involved since the matrix $\mathcal{Z}$ defined by

$$\begin{aligned} \mathcal{Z} &= \left[\begin{array}{cc|cc|c} AW_1 & A & A_h W_1 & A_h & E \\ 0 & X_1 A & X_1 A_h W_1 & X_1 A_h & X_1 E \\ \hline CW_1 & C & C_h W_1 & C_h & F \end{array}\right] \\ &+ \left[\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & D \end{array}\right] \begin{bmatrix} \mathcal{A}_c & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{D}_c \end{bmatrix} \left[\begin{array}{cc|cc|c} I & 0 & 0 & 0 & 0 \\ 0 & C_y & 0 & 0 & D_y \end{array}\right] \end{aligned} \tag{5.91}$$

is nonlinear due to the term $X_1 A_h W_1$. The change of variable is given by

$$\begin{bmatrix} \mathcal{A}_c & \mathcal{B}_c \\ \mathcal{C}_c & \mathcal{D}_c \end{bmatrix} = \begin{bmatrix} X_1 A W_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_2 & X_1 B \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} W_2^T & 0 \\ C_y W_1 & I \end{bmatrix} \tag{5.92}$$

Finally we would have

$$\mathcal{Z} = \left[\begin{array}{cc|cc|c} AW_1 + B\mathcal{C}_c & A + B\mathcal{D}_c C_y & A_h W_1 & A & E + B\mathcal{D}_c F_y \\ \mathcal{A}_c & X_1 A + \mathcal{B}_c C_y & X_1 A_h W_1 & X_1 A_h & X_1 E + \mathcal{B}_c F_y \\ \hline C_y W_1 + D\mathcal{C}_c & C_y + D\mathcal{D}_c C_y & C_h W_1 & C_h & F + D\mathcal{D}_c F_y \end{array}\right] \tag{5.93}$$

and the problem would be nonconvex. However it can be relaxed using the same bounding technique as for the observer based control law:

$$2x^T X_1 A_h W_1 y \leq x^T X X x + y^T W_1 A_h^T A_h W_1 y \tag{5.94}$$

5.3 Chapter Conclusion

We have developed in this chapter several control laws to stabilize LPV time-delay systems in the $\mathcal{L}_2$ performances framework. Both state-feedback and dynamic output feedback control laws have been developed in both memoryless and with-memory structures. We have emphasized the interest of the relaxations of LMI with multiple coupling in the synthesis problem in terms of computational complexity and conservativeness. Although the bilinear approach gives better results it is difficult to extend it to the case of discretized Lyapunov-Krasovskii functional due to the high number of products between system data matrices and decision variables: for a discretization of order N it would result in the introduction of N 'slack' variables and hence $2N$ bilinearities which complexifies the initialization of the iterative LMI algorithm. A new type of controllers has been introduced, the 'delay-scheduled' state-feedback controllers whose gain is smoothly scheduled by the delay value, in a similar way as for gain-scheduling strategies used in the LPV control framework.

Several dynamic output feedback controllers have been synthesized and their efficiency demonstrated through an example. It is worth mentioning that dynamic output feedback control laws are still open problems in time-delay systems framework.

Conclusion and Future Works

Summary and Main Contributions

This thesis has considered the control and observation of LPV time-delay systems using a part of the arsenal of modern control tools. Even if the problem remain open for several complex cases, the results presented in this thesis has improved many of the current results. The work has been presented in five chapters.

- In the first chapter, a state of the art on LPV systems is presented in which different types of representation coupled with their specific stability tests have been introduced.
- The second chapter, a (non-exhaustive) state of the art of time-delay systems is addressed with a particular focusing on time-domain methods, especially Lyapunov-Krasovskii functionals, small-gain, well-posedness and IQC based methods.
- The third chapter gathers parts of the theoretical contributions of this work. Two methods of relaxations for parameter dependent matrix inequalities and for matrix inequalities with particular concave nonlinearities have been developed. Known Lyapunov-Krasovskii functionals are generalized to the LPV case and relaxed using a specific approach in order to get 'easy-to-use' condition in the synthesis framework. Finally, a new Lyapunov-Krasovskii functional has been expressed in order to consider the special case of systems with two delays, in which the delays satisfy an equality, arising in the problem of stabilization of a time-delay system with a controller implementing a different delay.
- The fourth chapter used results of chapter three in order to construct observers and filters which have been shown to lead with interesting results.
- The fifth and last chapter used results of chapter 3 in order to derive different control laws: memoryless /with memory state-feedback/dynamic output feedback controllers. Moreover, a new design technique based on a LPV representation of time-delay systems has been applied to construct a new type of controller called 'delay-scheduled' controller where the controller gain depend on the delay. Using this technique, the robustness analysis with respect to delay knowledge uncertainty can be performed easily since the delay is not viewed anymore as an operator but as a scheduling parameter.

Future Works

As a perspective of the results developed in this thesis we can mention:

- The provided results only considers systems with are stable/stabilizable/detectable for zero delay (i.e. $A + A_h$ Hurwitz) and hence they may be conservative while considering systems which are not stable/stabilizable/detectable for zero delay but only from $h_{min} \neq 0$. Hence, it seems interesting and important to consider delay-range stability [He et al., 2007, Jiang and Han, 2005, Knospe and Roozbehani, 2006, 2003, Roozbehani and Knospe, 2005]. Note also that only few results exists on discretized Lyapunov-Krasovskii functionals for such systems.

- Two types of controllers have been developed in this thesis: state-feedback and dynamic output feedback control laws. It seems important to extend these results to the static-output feedback case [Li et al., 1998, Michiels et al., 2004, Peaucelle and Arzelier, 2005, Sename and Lafay, 1993, Seuret et al., 2008, Syrmos et al., 1995]. It is worth mentioning that despite of its simplicity, the static output feedback case is difficult to develop to the NP-hardness of its necessary and sufficient existence condition [Fu, 2004]. The method proposed in [Prempain and Postlethwaite, 2005] deserves attention and shall be generalized to time-delay systems and LPV systems. Moreover, delayed static-output feedback control is able to stabilize systems which are not stabilizable by instantaneous static-output feedback as noticed in [Niculescu and Abdallah, 2000]. In such a control law, the delay is an extra degree of freedom.

- Since many control systems have bounded inputs, it may be interesting to develop control laws in presence of saturations on the inputs [da Silva and Tarbouriech, 2005, Ferreres and Biannic, 2007, Henrion and Tarbouriech, 1999, Henrion et al., 2005, Wu and Lu, 2004, Wu and Soto, 2004].

- The generalization of such approach to :
  - input/output delayed systems, distributed and neutral delay systems
  - systems with delayed parameters
  - systems with parameter dependent delays

- The extension of the current work to more complex parameter-dependent Lyapunov-Krasovskii functional, e.g.

$$\begin{array}{rcl} V(x_t,\dot{x}_t) & = & V_1(x,\rho) + V_2(x_t,\rho_t) \\ V_1(x,\rho) & = & x(t)^T P(\rho) x(t) \\ V_2(x_t,\rho_t) & = & \int_{t-h(t)}^{t} x(\theta)^T Q(\rho(\theta) x(\theta) d\theta \end{array}$$

  in order to reduce the conservatism of the approach.

- The application of such control strategies on physical systems, currently, the stabilization of unstable modes in fusion plasmas (see Appendix L or [Olofsson et al., 2008]).

Chapter 6

Appendix

A Technical Results in Linear Algebra

This appendix is devoted to the introduction of some fundamentals on matrix algebra. It is supposed that matrix multiplication and inversion is known. Determinants of block matrices, notion of eigenvalues and eigenvectors, inverse of block matrices, notion of order in the set of symmetric matrices, singular value decomposition, Moore-Penrose pseudo-inverse and the resolution of specific matrix equality and inequality will be considered.

A.1 Determinant Formulae

We give here several important relations concerning the determinant. For a square matrix $A \in \mathbb{C}^{n\times n}$, its determinant is denoted $\det(A)$. If A and B are both square matrices of same dimensions, then it can be shown that

$$\det(AB) = \det(A)\det(B) = \det(BA)$$

Another well-known fact is

$$\det\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right) = \det(A)\det(D) \tag{A.1}$$

when A and D are both square. If A is square and nonsingular, then we can use the latter relations and the fact that

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

to get

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(D - CA^{-1}B)$$

which is known as the Schur (determinant) complement. Similarly, if D is nonsingular, we can show

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(D)\det(A - BD^{-1}C)$$

If $A = I$ and $D = I$ and BC is a square matrix, we arrive at the following very useful identity

$$\det(I - BC) = \det(I - CB)$$

A.2 Eigenvalues of Matrices

Definition A.1 *For a square matrix $M \in \mathbb{R}^{n\times n}$, the spectrum of M (the set of eigenvalues of M) is denoted $\lambda(A) = \text{col}_i(\lambda_i)$ each one of these zeroes the characteristic polynomial defined as*

$$\chi_M(\lambda) = \det(\lambda I - M) \tag{A.2}$$

where $\det(M)$ *is the determinant of* M.

We have the following relations:

$$\begin{array}{c} \sum_{i=1}^n \lambda_i = \text{trace}(A) \\ \prod_{i=1}^n \lambda_i = \det(A) \end{array}$$

For the special cases $n = 1, 2$ and 3 the characteristic polynomial is given by the expressions

$n = 1$**:** $\chi_M(\lambda) = \lambda - M$

$n = 2$**:** $\chi_M(\lambda) = \lambda^2 - \text{trace}(M)\lambda + \det(M)$

$n = 3$**:** $\chi_M(\lambda) = \lambda^3 - \text{trace}(M)\lambda^2 + \text{trace}[\text{Adj}(M)]\lambda + \det(M)$

where $\text{trace}(M)$ and $\text{Adj}(M)$ are respectively the trace and the adjugate matrix of M.

Let us consider now symmetric matrices, i.e. matrices such that $M = M^*$ (or $M = M^T$ if M is a real matrix). It can be shown that in this case all the eigenvalues of M are real [Bhatia, 1997]. Moreover, we have the following definition:

Definition A.2 *The eigenvectors of a symmetric square matrix M are defined to be the nonzero full column rank matrices v_i such that*

$$(A - \lambda_i)v_i = 0 \tag{A.3}$$

In this case, the matrix $M' = PMP^{-1}$ exhibits all the eigenvalues of M on the diagonal:

$$M' = \text{diag}(\lambda_i I_{m_i})$$

where the eigenvalues are repeated as many times as their order of multiplicity m_i. The matrix P is defined as $P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$. This decomposition is called **spectral decomposition**.

Remark A.3 *This suggests that for a symmetric matrix M with eigenvalues λ_j with order of multplicity m_j, there exists m_j eigenvectors u such that $(A - \lambda_j)v = 0$. It is important to emphasize that it is not the always the case for general matrices. In such a case, the matrix may be non-diagonalizable but can be reduced to a Jordan matrix. Any algebra book or course shall detail this correctly.*

The fact that every symmetric matrix can be diagonalized in an orthonormal basis is an interesting fact and makes symmetric matrices an useful tools in many fields. The interest of symmetric matrices is the ability to generalize the notion of positive and negative number to the matrix case. Indeed, since the eigenvalues of symmetric matrices are all real then it is possible to define positive and negative matrices, hence a relation of order in the set of symmetric matrices.

Definition A.4 *A symmetric matrix M is said to be positive (semi)definite if all its eigenvalues are positive (nonnegative). This is denoted by $M \succ 0 (\succeq 0)$.*

Definition A.5 *A symmetric matrix M is said to be negative (semi)definite if all its eigenvalues are negative (nonpositive). This is denoted by $M \prec 0 (\preceq 0)$.*

The notion of positivity and negativity of a symmetric matrix M is related to its associated quadratic form $x^T M x$ where x is a real vector.

Proposition A.6 *A $n \times n$ symmetric matrix M is positive (semi)definite if and only if the quadratic form $x^T M x > 0 (\geq 0$ for all $x \in \mathbb{R}^n$.*

Proof: *Sufficiency:*

Suppose all the eigenvalues of M are nonnegative. Define now the quadratic form $Q(x) = x^T M x$ and since M is nonnegative, then in virtue of the Cholesky decomposition of symmetric nonnegative matrices we have $Q(x) = x^T L^T L x$ which is equal to $||Lx||_2$ and is obviously nonnegative.

Necessity:

Suppose now that $Q(x) = x^T M x \geq 0$ for all $x \in \mathbb{R}^n$. A well-known result says that if a quadratic form is positive semidefinite then it is sum-of-squares (see Section 1.3.3.3) and writes as $Q(x) = \sum_i q_i(x)^2$. Now introduce line vectors L_i such that $q_i(x) = L_i x$ and therefore we have $q_i(x)^2 = x^T L_i^T L_i x$. Finally denoting

$$L := \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}$$

then we have $Q(x) = x^T L^T L x$ where $L^T L \succeq 0$. This concludes the proof. $\square$

Proposition A.7 *A $n \times n$ symmetric matrix M is negative (semi)definite if and only if the quadratic form $x^T M x < 0 (\leq 0$ for all $x \in \mathbb{R}^n$.*

A.3 Exponential of Matrices

Definition A.8 *The exponential of a square matrix A is given by the expression*

$$e^M = \exp(M) := \sum_{i=1}^{+\infty} \frac{M^i}{i!} \tag{A.4}$$

Theorem A.9 (Cayley-Hamilton Theorem) *Any square matrix $M \in \mathbb{R}^{n \times n}$ satisfies the equality*

$$\chi_M(M) = 0 \tag{A.5}$$

where $\chi_M(\lambda)$ is the characteristic polynomial of M.

This theorem shows that for a matrix M of dimension n, M^n can be computed as a linear combination of all other lower powers M^k, $0 \leq k < n$. For instance for $n = 2$ we have

$$M^2 = \text{trace}(M)M - \det(M) \tag{A.6}$$

It allows to compute any powers of M using a linear combination of all powers of M from 0 to $n-1$. For instance,

$$\begin{aligned} M^3 &= \text{trace}(M)M^2 - \det(M)M \\ &= \text{trace}(M)(\text{trace}(M)M - \det(M)) - \det(M)M \\ &= [\text{trace}(M)^2 - \det(M)]M - \det(M) \end{aligned} \tag{A.7}$$

One of the most important applications of this theorem is the rank condition for controllability and observability of linear systems; this will be detailed in Appendix B.6. As any power of M can be expressed in through a linear combination of powers from 0 to $n-1$ therefore any sum of power of matrices can be written is such a manner, even if the sum infinite is (but countable).

Proposition A.10 *The exponential of a matrix M, by virtue of the Cayley-Hamilton theorem, can be expressed as*

$$\exp(M) = \sum_{i=0}^{n-1} \alpha_i M^i \tag{A.8}$$

where the α_i satisfies the linear system

$$\sum_{i=0}^{n-1} \alpha_i \lambda_j^i = e^{\lambda_j} \qquad \text{for all } j = 1, \ldots, n \tag{A.9}$$

The infinite sum has been amazingly converted into a finite sum where the coefficients are determined by solving a system of linear equations.

A.4 Generalities on Block-Matrices

Let us consider the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Assuming that M is square and invertible then the inverse is given by

$$\begin{aligned} M^{-1} &= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \end{aligned} \tag{A.10}$$

The first formula is well-defined if A is invertible while the second when D is invertible. By identification of the blocks we get the well-known matrix inversion lemma:

Lemma A.11 (Matrix Inversion Lemma)

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \tag{A.11}$$

or also

$$(A - BDC)^{-1} = A^{-1} + A^{-1}B(D^{-1} - CA^{-1}B)^{-1}CA^{-1} \tag{A.12}$$

We also have the identity:

$$A^{-1}B(D - CA^{-1}B)^{-1} = (A - BD^{-1}C)^{-1}BD^{-1} \tag{A.13}$$

A.5 Kronecker operators and Matrix Tensor Sum and Product

This sections aims at providing some elementary definitions about Kronecker product and sum.

The Kronecker product is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \ddots & \vdots \\ a_{p1}B & \dots & a_{pn}B \end{bmatrix} \tag{A.14}$$

We have the following relations where α is a scalar:

$$\begin{array}{lcl} 1 \otimes A = A \otimes 1 & = & A \\ A \otimes (B + \alpha C) & = & A \otimes B + \alpha A \otimes C \\ A \otimes (B \otimes C) & = & (A \otimes B) \otimes C \\ (A \otimes B)(C \otimes C) & = & (AC) \otimes (BD) \\ (A \otimes B)^{-1} & = & A^{-1} \otimes B^{-1} \\ (A \otimes B)^T & = & A^T \otimes B^T \\ \lambda(A \otimes B) & = & \{\nu_i \mu_j \ \forall (i,j)\} \text{ where } \lambda(A) = \nu,\ \lambda(B) = \mu \\ \operatorname{trace}(A \otimes B) & = & \operatorname{trace}(A) \operatorname{trace}(B) \\ \det(A \otimes B) & = & \det(A)^n \det(B)^n,\ n = \dim(A) \\ \operatorname{rank}(A \otimes B) & = & \operatorname{rank}(A) \operatorname{rank}(B) \end{array}$$

The Kronecker sum of two matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ is defined by

$$A \oplus B = A \otimes I_m + I_n \otimes B \tag{A.15}$$

Moreover we have the following properties

$$\begin{array}{lcl} e^{A \oplus B} & = & e^A \otimes e^B \\ \lambda(A \oplus B) & = & \lambda(A) \cup \lambda(B) \end{array}$$

It is convenient to introduce the tensor product and sum $\phi_\otimes, \phi_\oplus : \mathbb{C}^{m \times m} \times \mathbb{C}^{m \times m} \to \mathbb{C}^{p \times p}$. Let us consider $P, Q \in \mathbb{C}^{m \times m}$ with $m \geq 2$ and

$$\begin{array}{lcl} p = m^2 & : & \left\{ \begin{array}{lcl} \phi_\otimes(P,Q) & = & P \otimes Q \\ \phi_\oplus(P,Q) & = & P \oplus Q \end{array} \right. \\ p = \dfrac{m(m-1)}{2} & : & \left\{ \begin{array}{lcl} \phi_\otimes(P,Q) & = & P \tilde{\otimes} Q \\ \phi_\oplus(P,Q) & = & P \tilde{\oplus} Q \end{array} \right. \end{array}$$

where $\oplus$ and $\otimes$ are Kronecker sum and product define above. On the other hand, the operators $\tilde{\oplus}$ and $\tilde{\otimes}$ are defined as follows [Qiu and Davidson, 1991]:

$$P \tilde{\otimes} Q == [c_{i,j}] \in \mathbb{C}^{p \times p}$$

where $c_{i,j} = (p_{i_1,j_1} q_{i_2,j_2} + p_{i_2,j_2} q_{i_1,j_1} - p_{i_2,j_1} q_{i_1,j_2} - p_{i_1,j_2} q_{i_2,j_1}$ where (i_1, i_2) is the i^{th} pair of sequence $(1,2), (1,3), \dots, (1,m), (2,3), \dots, (2,m), \dots, (m,m)$ and (j_1, j_2) is generated by duality. For $P \tilde{\oplus} Q$ the classical definition is extended in

$$P \tilde{\oplus} Q = P \tilde{\oplus} I_m + I_m \tilde{\oplus} Q$$

Algebraic properties of these tensor product and sum can be found in [Marcus, 1973, Qiu and Davidson, 1991].

A.6 Singular-Values Decomposition

The eigenvalue decomposition of a square matrix is the problem in finding a basis in which the matrix has an expression where the eigenvalues are all placed on the diagonal. This is called a spectral decomposition. We provide here a kind of generalization of such a procedure when the matrix M is not necessarily square: this is called the singular-value decomposition. A unitary matrix U is defined as $U^*U = I = UU^*$ where the superscript * denotes the complex conjugate transpose.

Theorem A.12 *Let $M \in \mathcal{C}^{k\times n}$ be a matrix of rank r. Then there exist unitary matrices U and V such that*

$$M = U\Sigma V^* \tag{A.16}$$

where U and V satisfy

$$MM^*U = U\Sigma\Sigma^* \qquad M^*MV = V\Sigma^*\Sigma \tag{A.17}$$

and Σ has the canonical structure

$$\Sigma = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Sigma_0 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r) \prec 0 \tag{A.18}$$

The numbers $\sigma_i > 0$, $i = 1, \ldots, r$ are called the nonzero singular values of M.

Proof: *The proof is given in Skelton et al. [1997] and for more on singular value decomposition see Horn and Johnson [1990] or many other books on linear algebra.* □

A.7 Moore-Penrose Pseudoinverse

Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix (i.e. $\det(M) \neq 0$), then there exists a matrix inverse denoted M^{-1} such that $MM^{-1} = M^{-1}M = I$. We provide here the generalization of this procedure to rectangular matrices. It has been shown that any $n \times m$ matrix M can be expressed as a singular value decomposition $M = U\Sigma V^*$.

Theorem A.13 *For every matrix $M \in \mathbb{R}^{n\times m}$, there exist a unique matrix $M^+ \in \mathbb{R}^{m\times n}$, the Moore-Penrose pseudoinverse of M, which satisfies the relation below:*

$$\begin{array}{ll} MM^+M = M & M^+MM^+ = M^+ \\ (MM^+)^* = MM^+ & (M^+M)^* = M^+M \end{array} \tag{A.19}$$

Moreover, M^+ is given by

$$M^+ := V \begin{bmatrix} \Sigma_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \tag{A.20}$$

Moreover consider the matrix $M \in \mathbb{R}^{n\times m}$ then

- if M has full row rank n then $M^+ = M^*(MM^*)^{-1}$
- if M has full column rank m then $M^+ = (M^*M)^{-1}M^*$

A.8 Solving $AX = B$

The solution X of equation $AX = B$ is trivial when A is a nonsingular matrix. We aim here at showing that there exists an explicit expression to X when A is a rectangular matrix sharing specific assumptions with B.

Theorem A.14 *Let $A \in \mathbb{R}^{n_1 \times n_2}$, $X \in \mathbb{R}^{n_2 \times n_3}$ and $B \in \mathbb{R}^{n_1 \times n_3}$. Then the following statements are equivalent:*

1. *The equation $AX = B$ has a solution X.*
2. *A, X and B satisfy $AA^+B = B$.*
3. *A, X and B satisfy $(I - AA^+)B = 0$.*

In this case all solutions are

$$X = A^+B + (I - A^+A)Z \tag{A.21}$$

where $Z \in \mathbb{R}^{n_1 \times n_3}$ is arbitrary and A^+ is the Moore-Penrose pseudoinverse of A.

A.9 Solving $BXC + (BXC)^* + Q \prec 0$

Such equation arises in control synthesis for linear finite dimensional systems and this motivates why it is presented here.

Theorem A.15 *Let matrices $B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{k \times n}$ and $Q = Q^* \in \mathbb{H}^n$ be given. Then the following statements are equivalent:*

1. *There exists a X satisfying*

$$BXC + (BXC)^* + Q < 0 \tag{A.22}$$

2. *The following two conditions hold*

$$\begin{array}{rcl} B_\perp Q B_\perp^* & \prec & 0 \quad \text{or} \quad BB^* \succ 0 \\ C_\perp^* Q C_\perp & \prec & 0 \quad \text{or} \quad C^*C \succ 0 \end{array} \tag{A.23}$$

Suppose the above statements hold. Let r_b and r_c be the ranks of B and C, respectively, and (B_ℓ, B_r) and (C_ℓ, C_r) be any full rank factors of B and C (i.e. $B = B_\ell B_r$ and $C = C_\ell C_r$). Then all matrices X in statement 1. are given by

$$X = B_r^+ K C_\ell^+ Z - B_r^+ B_r Z C_\ell C_\ell^+ \tag{A.24}$$

where Z is an arbitrary matrix and

$$\begin{array}{rcl} K & := & -R^{-1} B_\ell^* \Phi C_r^* (C_r \Phi C_r^*)^{-1} + S^{1/2} L (C_r \Phi C_r^*)^{-1/2} \\ S & := & R^{-1} - R^{-1} B_\ell^* - R^{-1} B_\ell^* [\Phi - \Phi C_r^* (C_r^\Phi C_r^*)^{-1} C_r \Phi] B_\ell R^{-1} \end{array} \tag{A.25}$$

where L is an arbitrary matrix such that $||L|| < 1$ (i.e. $\bar{\sigma}(L) < 1$) and R is an arbitrary positive definite matrix such that

$$\Phi := (B_\ell R^{-1} B_\ell^* - Q)^{-1} \succ 0 \tag{A.26}$$

The solution X is quite complicated and can be approximated by a more simple expression. If one of statements above holds, then more simple solutions are given by each of the expressions (see Iwasaki and Skelton [1995]):

$$\begin{array}{rcl} X_B & := & -\tau_B B^* \Psi_B C^T (C \Psi_B C^*)^{-1} \\ X_C & := & -\tau_C (B^* \Psi_C B)^{-1} B^* \Psi_C C^* \end{array} \tag{A.27}$$

where $\tau_B, \tau_C > 0$ are sufficiently large scalars such that

$$\begin{array}{rcl} \Psi_B & := & (\tau_B B B^* - Q)^{-1} \succ 0 \\ \Psi_C & := & (\tau_C C^* C - Q)^{-1} \succ 0 \end{array} \tag{A.28}$$

B Technical results on Dynamical Systems

This appendix is devoted to give an overview of dynamical systems. The definition of dynamical systems will be given. The notion of equilibrium points and solutions will be introduced with the concept of stability. Then we will focus on linear dynamical systems which enjoys nice and common properties. Finally, concepts that are closer to control problems, the notions of controllability and observability will be enunciated with associated structures for controllers and observers.

B.1 Finite dimensional Dynamical Systems

This section provides definitions on dynamical systems and several associated notions such as equilibrium points.

Definition B.1 *A finite dimensional dynamical system is a set of coupled ordinary differential equations (ODE):*

$$\begin{array}{rcl} \dot{x}_1(t) & = & f_1(t, x(t), w(t)) \\ \dots & & \\ \dot{x}_n(t) & = & f_n(t, x(t), w(t)) \\ x(0) & = & x_0 \end{array} \tag{B.29}$$

with $x(t) = \mathrm{col}(x_1(t), \dots, x_n(t))$.

The vector $x \in \mathbb{R}^n$ is called the *state* of the system and characterizes completely the system at each time t. The vector $w \in \mathbb{R}^m$ contains functions (inhomogeneity) affecting the evolution of the state at each time t. The term *finite dimensional* comes from the fact the state x belongs to a finite dimensional space; generally speaking an Euclidian space, here of dimension n (for instance $\mathbb{R}^n$).

The evolution of the state of a dynamical system at time t depends on the current state $x(t)$, the time t and exogenous terms $w(t)$. When the f_i's do not depend explicitly on time t the system is called *time-invariant* and if there is no $w(t)$ the system is said to be *autonomous* (the future evolution depends only on the current state). In the following, we will focus on time-invariant autonomous dynamical system.

Definition B.2 *The set of equilibrium points of system (B.29) is defined by*

$$\mathcal{E} := \{x \in \mathbb{R}^n : \ f_i(x) = 0, \ i = 1, \dots, n\} \tag{B.30}$$

Roughly speaking, once the state of the system has reached one of these equilibrium points, the state will permanently remains on until it is forced to move (for instance by action of external signals).

The solution of such a dynamical system is given by the function $\phi(t, x_0)$. This function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is called the *flow* associated to the differential equation. If x_0 is kept fixed, then the function

$$t \to \phi(t, x_0) \tag{B.31}$$

is just an alternative expression for the solution of the differential equation satisfying the initial condition x_0.

Definition B.3 *A smooth dynamical system of $\mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ where $\phi(t, x) = \phi_t(x)$ satisfies*

1. *$\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function: $\phi_0(x_0) = x_0$*
2. *The composition $\phi_t \circ \phi_s = \phi_{t+s}$ for each $t, x \in \mathbb{R}$ (Semi-group property)*

Recall that a function which is continuously differentiable means that all its partial derivatives exist and are continuous throughout its domain (here $\mathbb{R} \times \mathbb{R}^n$). The fact that the function ϕ is smooth implies that the trajectory of the state will be smooth too (and then continuously differentiable).

For a given ϕ_t, let

$$f(x) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x) \tag{B.32}$$

Then ϕ_t is just the time t map associated to the flow $\dot{x} = f(x)$.

B.2 Existence and uniqueness of solutions

It is important to characterize the solutions of a dynamical system. When such a solution exists ? How many of solutions exist ? For instance, if a system is modeled through a dynamical system, it seems important to have solutions to this model.

For simplicity, we consider in the following

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) \\ x(0) & = & x_0 \end{array} \tag{B.33}$$

where $x_0 \in \mathbb{R}^n$. We provide here a local result on existence and uniqueness of solutions:

Theorem B.4 *Consider the initial value problem (B.33) with given $x_0 \in \mathbb{R}^n$ and suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$. Then, first of all, there exists a solution of this initial value problem and secondly this is the only such solution.*

A global characterization of the existence and uniqueness of solution to the initial value problem (B.33) is given below

Theorem B.5 *For all $x_0 \in \mathbb{R}^n$ there exists an unique solution if and only if the function f is Lipschitz, i.e.*

$$\text{There exists } k > 0 \text{ for all } x, y \in \mathbb{R}^n \text{ such that } ||f(y) - f(x)|| \leq k||y - x|| \tag{B.34}$$

B.3 Continuous Dependence of Solutions

The existence and uniqueness of solutions to the initial value problem (B.33) is interesting in both the mathematical and physical senses. This result needs to be complemented with the property that the solution $x(t)$ depends continuously on the initial condition x_0. The next theorem gives a precise statement of this property:

Theorem B.6 *Consider the differential equation $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$. Suppose that $x(t)$ is a solution of this equation which is defined on the closed interval $[t_0, t_1]$ with $x(t_0) = x(0)$. Then there is a neighborhood $U \subset \mathbb{R}^n$ of x_0 and a constant k such that if $y_0 \in U$, then there is a unique solution $y(t)$ also defined on $[t_0, t_1]$ with $y(t_0) = y_0$. Moreover $y(t)$ satisfies*

$$||y(t) - x(t)|| \leq k||y_0 - x_0||e^{k(t-t_0)} \tag{B.35}$$

for all $t \in [t_0, t_1]$.

This result says that, if the solutions $x(t)$ and $y(t)$ start out close together, then they remain close together for t close to t_0. While these solutions may separate from each other, they do so no faster than exponentially. In particular, since the right-hand side of (B.35) depends on $||y_0 - x_0||$, which we may assume is small, we have:

Corollary B.7 *Let $\phi(t, x)$ be the flow of the system $\dot{x} = f(x)$ where f is $\mathcal{C}^1$. Then ϕ is a continuous function of x.*

B.4 Stability of Equilibria of Dynamical Systems

The set $\mathcal{E}$ defined in (B.30) is the set of equilibrium points of the dynamical system $\dot{x} = f(x)$). To understand the behavior of the dynamical system, it is necessary to determine how the system behaves in a neighborhood of these equilibria. This leads to the theory of stability, and more precisely to the notion of the Lyapunov stability:

Definition B.8 *Let $\phi : \mathbb{T} \times \mathbb{T} \times \mathbb{R}^n \to \mathcal{X}$ be the flow of the system $\dot{x} = f(x)$, $x(0) = x_0$ and suppose that $\mathbb{T} = \mathbb{R}$ and $\mathcal{X}$ is a normed vector space. The equilibrium point x^* is said to be*

1. ***Stable*** *(in the sense of Lyapunov) if given any $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ (not depending on t) such that*

$$||x_0 - x^*|| \leq \delta \Rightarrow ||\phi(t, x_0) - x^*|| \leq \varepsilon \text{ for all } t \geq t_0 \tag{B.36}$$

2. ***Attractive*** *if for all $t_0 \in \mathbb{T}$ there exists $\delta = \delta(t_0) > 0$ with the property that*

$$||x_0 - x^*|| \leq \delta \ \Rightarrow \ \lim_{t \to +\infty} ||\phi(t, t_0, x_0) - x^*|| = 0 \tag{B.37}$$

3. ***Exponentially Stable*** *if for all $t_0 \in \mathbb{T}$ there exists $\delta = \delta(t_0)$, $\alpha = \alpha(t_0) > 0$ and $\beta = \beta(t_0) > 0$ such that*

$$||x_0 - x^*|| \leq \delta \Rightarrow ||\phi(t, t_0, x_0) - x^*|| \leq \beta||x_0 - x^*||e^{-\alpha(t-t_0)} \text{ for all } t \geq t_0 \tag{B.38}$$

4. ***Asymptotically Stable*** *(in the sense of Lyapunov) if it is both stable (in the sense of Lyapunov) and attractive.*

5. ***Unstable*** *is it is not stable (in the sense of Lyapunov)*

6. ***Uniformly Stable*** *(in the sense of Lyapunov) if given* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *(not depending on* t_0 *such that (B.36) holds for all* $t_0 \in \mathbb{T}$.

7. ***Uniformly Attractive*** *if there exists* $\delta > 0$ *(not depending on* t_0*) such that (B.37) holds for all* $t_0\mathbb{T}$.

8. ***Uniformly Exponentially Stable*** *if there exists* $\delta > 0$ *(not depending on* t_0*) such that (B.38) holds for all* $t_0 \in \mathbb{T}$.

9. ***Uniformly Asymptotically Stable*** *(in the sense of Lyapunov) if it is both uniformly stable (in the sense of Lyapunov) and uniformly attractive.*

Note that the notion of exponential stability is the strongest one since an exponentially stable equilibrium point is also an asymptotically stable fixed point.

Definition B.9 *The region of attraction associated with a fixed point* x^* *is defined to be the set*

$$\mathcal{A}(x^*) := \left\{ x_0 \in \mathcal{X} : \phi(t, t_0, x_0) \underset{t \to +\infty}{\longrightarrow} x^* \right\} \tag{B.39}$$

If this region does not depend on t_0, it is said to be **uniform**, if it coincides with with $\mathcal{X}$ then x^* is **globally attractive**. In the same fashion, it is possible to define the region of stability, asymptotic stability and exponential stability associated with x^*. Again, these regions are said to be uniform if they do not depend on t_0. If these regions covers the whole state-space $\mathcal{X}$, then the fixed point is called globally stable, globally asymptotically stable and globally exponentially stable respectively.

Definition B.10 *We define here the notion of positive and negative invariant sets. A positive (resp. negative) invariant set* $\mathcal{X}_i^+$ *(resp.* $\mathcal{X}_i^-$*) is defined to be*

$$\mathcal{X}_i^+ := \{x_0 \in \mathcal{V} \subset \mathcal{X} : \exists t_0 \in \mathbb{T},\ \phi(t, t_0, x_0) \in \mathcal{V} \text{ for all } t \geq t_0\} \tag{B.40}$$

$$\mathcal{X}_i^- := \{x_0 \in \mathcal{V} \subset \mathcal{X} : \exists t_0 \in \mathbb{T},\ \phi(t, t_0, x_0) \in \mathcal{V} \text{ for all } t \leq t_0\} \tag{B.41}$$

A set which is both a negative and positive invariant set is called an invariant set.

These previous definitions give an idea how a dynamical system may behave around an equilibrium point. But the question is 'how can we determine which behavior the system has around a certain fixed point ?'. The following result provides a general answer to this question:

Theorem B.11 (Lyapunov Stability Theorem:) *Let* $x^* \in \mathcal{E}$ *be an equilibrium point for* $\dot{x} = f(x)$. *Let* $V : \mathcal{O} \to \mathbb{R}$ *be a differentiable function defined on an open set* $\mathcal{O}$ *containing* x^*. *Suppose further that*

1. $V(x^*) = 0$ *and* $V(x) > 0$ *if* $x \neq x^*$

2. $\dot{V} \leq 0$ *in* $\mathcal{O} - \{x^*\}$

Then x^* *is stable. Furthermore, if* V *also satisfies*

3. $\dot{V} < 0$ *in* $\mathcal{O} - \{x^*\}$*,*

then x^* *is asymptotically stable.*

A function V satisfying 1. and 2. is called a *Lyapunov function* for x^*. If 3. also holds, we call V a *strict Lyapunov function*. This theorem says that if we consider an initial condition near the equilibrium point (in $\mathcal{O}$) such that at least 1. and 2. are satisfied, then the solution stays in a neighborhood of x^* (which may be different from $\mathcal{O}$). Moreover, if 3. holds, then the solution converges to x^*.

The great advantage of such a method comes from the fact that it can be applied without solving the differential equations. It is possible, with this method, to determine both the behavior of the system in a neighborhood of an equilibrium point x^* but also find a region of attraction $\mathcal{A}(x^*)$ if the the fixed point x^* is, at least, attractive. Nevertheless, there does not exist any systematic approach to build such functions. Sometimes it is easy to find functions representing the energy of the system but in many cases they are not trivial functions.

B.5 Linear Dynamical Systems

We detail here the very special case of linear dynamical systems which can be easily represented over $t \geq t_0$ as

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ x(t_0) & = & x_0 \end{array} \tag{B.42}$$

where $A \in \mathbb{R}^{n\times n}$. A linear dynamical system can be linear due to the framework used to model system such as first order approximation of heat transfer, or low frequency electronic filter or also simple mechanical systems (mass and spring). They can also be obtained from nonlinear systems (models) using a linearization procedure explained hereafter.

Let us consider the nonlinear dynamical system described by $\dot{x} = f(x)$ and let $x^* \in \mathcal{X}$ be one of its equilibrium points. Then we have

$$f(x) = f(x^*) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^*)[x - x^*] + \dots \tag{B.43}$$

Then a first order approximation of $f(x)$ leads to a linear dynamical system of the form (B.42) where

$$A := \left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \dots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}v & \dots & \frac{\partial f_n}{\partial x_n}(x^*) \end{bmatrix} \tag{B.44}$$

Proposition B.12 *Let* $\phi(t, t_0, x_0)$ *be the flow of a linear dynamical system, then the flow is linear in* $\mathcal{X}_0$*.*

Proposition B.13 *Let* $\phi : \mathbb{T} \times \mathbb{T} \times \mathcal{X}$ *be a flow of a linear dynamical system with* $\mathbb{T} = \mathbb{R}$ *and suppose that* x^* *is a fixed point then*

1. x^* *is attractive if and only if* x^* *is globally attractive.*

2. *x^* is asymptotically stable if and only if x^* is globally asymptotically stable.*

3. *x^* is exponentially stable if and only if x^* is globally exponentially stable.*

If ϕ is time-invariant then

1. *x^* is stable if and only if x^* uniformly stable.*

2. *x^* is asymptotically stable if and only if x^* uniformly asymptotically stable.*

3. *x^* is exponentially stable if and only if x^* uniformly exponentially stable.*

Proof: *The proof can be found in Scherer and Wieland [2005].* □

This result is very interesting in the sense that every property for a fixed point is global for linear dynamical systems. This justifies the fact that a linear system with a stable equilibrium point is called a stable system.

It is possible to define a Lyapunov function for such systems, this is provided in the following proposition:

Proposition B.14 *Consider system (B.42). The following statements are equivalent:*

1. *The origin is an asymptotically stable equilibrium.*

2. *The origin is a globally asymptotically stable equilibrium.*

3. *All eigenvalues $\lambda(A)$ lie in the complex open left-half plane (i.e. have strictly negative real part).*

4. *The quadratic function $V(x) = x^T P x$ with $P = P^T > 0$ is a Lyapunov function for (B.42).*

5. *The following Linear Matrix Inequalities are satisfied*

$$A^T P + PA \prec 0 \qquad P = P^T \succ 0 \tag{B.45}$$

The last statement involves two Linear Matrix Inequalities and at first sight, it may seem difficult to deal with such a condition. Actually, it is more simple to consider LMIs than a direct computation of the eigenvalues since first of all when considering time-varying systems the set of eigenvalues is infinite and hence hard to compute. Moreover, LMIs are convex problems and hence can be efficiently solved using convex optimization programs (See Appendix D). Finally, it is possible to construct more complex LMIs capturing more information on the linear dynamical system (e.g. the rate of convergence which are also called Lyapunov Exponents). More information about this in Appendix E.

B.6 Controllability and Observability of Dynamical Systems

In this section we will consider dynamical systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t), u(t)) \\ y(t) & = & h(x(t), u(t)) \\ x(t_0) & = & x_0 \end{array} \tag{B.46}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$ and $y \in \mathcal{Y} \subset \mathbb{R}^p$ are respectively the state, the control input and the controlled output.

Definition B.15 *System (B.46) is said to be controllable from time s if there exists $t > s$ such that any state $x(s)$ can be transferred to any state $x(t)$ through an appropriate choice of the input $u(\cdot)$ over $[s,t]$.*

Definition B.16 *System (B.46) is said to be reachable at time t if there exists $t > s$ such that any state $x(s)$ can be transferred to any state $x(t)$ through an appropriate choice of the input $u(\cdot)$ over $[s,t]$.*

Definition B.17 *System (B.46) is said to be observable at time s if there exists $t > s$ such that the state $x(s)$ can be determined from the output $y(\cdot)$ over $[s,t]$.*

Definition B.18 *System (B.47) is said to be reconstructable from time t if there exists $t > s$ such that the state $x(s)$ can be determined from the output $y(\cdot)$ over $[s,t]$.*

Proposition B.19 *In the special linear case, the reachability and controllability imply each others, are global (for all $x \in \mathcal{X}$) and uniform (for all $t, s \in \mathbb{T}$) properties. This motivates the denomination of completely controllable systems. The observability and reconstructability satisfy the same properties.*

In other words, for linear systems, controllability means the existence of a control input u which transfers any state value to any arbitrary state value. Observability means that for any trajectory of the output y, it is possible to reconstruct the state value over the trajectory.

Let us consider system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t)) \\ x(t_0) & = & x_0 \end{array} \tag{B.47}$$

Definition B.20 *System (B.47) is completely controllable if one of the following equivalent statements holds:*

1. *For all $t_0, t_1 \in \mathbb{T}$, $x_0, x_1 \in \mathcal{X}$ there exist a control law u over $[t_0, t_1]$ transferring $x(t_0) = x_0$ to $x(t_1) = x_1$.*
2. $\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$
3. *For all $\alpha > 0$, there exists $P = P^T \succ 0$ and Y such that the LMIs hold*

$$AP + PA^T + BY + Y^T B^T \prec -2\alpha P \qquad P = P^T \succ 0 \tag{B.48}$$

Statement 2. is very interesting because, it translate an a priori infinite dimensional problem (we are looking for the existence of a function u) into a finite dimensional problem which can be verified easily. If a system is completely controllable then each one of its states can be placed where sought and independently from each others. This means that if the system is unstable then, by a suitable choice of the control input u, the unstable states can be placed where desired.

Statement 3. is a LMI condition to the fact that the control law $u(t) = Kx(t)$ allows to place all the eigenvalues of the matrix $A + BK$ below than $-\alpha$ for all $\alpha > 0$.

In some cases, some of the states cannot be set anywhere. In this case, all we can do is expecting that these states do not depend on unstable modes (their dynamical behavior is stable), this brings to the notion of stabilizability:

Definition B.21 *System (B.47) is stabilizable if one of the following equivalent statements holds:*

1. $\operatorname{rank}\begin{bmatrix} sI-A & B \end{bmatrix} = n$ *for all* $s \in \mathbb{C}^+$
2. *There is no loss of rank when evaluating the rank condition*

$$\operatorname{rank}\begin{bmatrix} sI-A & B \end{bmatrix}$$

   at each nonnegative eigenvalue of A.

3. *There exists* $P = P^T \succ 0$ *such that the LMIs hold*

$$\mathcal{N}_{U^T}^T(AP + PA^T)\mathcal{N}_{U^T} \prec 0 \qquad P = P^T \succ 0 \tag{B.49}$$

4. *There exists* $P = P^T \succ 0$ *and* Y *such that the LMIs hold*

$$AP + PA^T + BY + Y^TB^T \prec 0 \qquad P = P^T \succ 0 \tag{B.50}$$

Statement 1 is difficult to verify in all the case since it is a semi-infinite rank constraint. Statement 2, says that it suffices to verify a rank property only at a finite number of value for s. Said differently, if all the uncontrollable modes are asymptotically stable then the system is stabilizable. Statements 3 and 4, provides LMI conditions for the stabilizability of the linear system. Moreover, if statement 4 holds then a stabilizing state-feedback control law is given by $u(t) = YP^{-1}x(t)$.

Definition B.22 *System (B.47) is completely observable if one of the following equivalent statements holds:*

1. *For all* $t_0, t_1 \in \mathbb{T}$, $x_0, x_1 \in \mathcal{X}$ *the state can be reconstructed from the knowledge of the measured output* $y(t)$ *over* $[t_0, t_1]$.
2. $\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$
3. *For all* $\alpha > 0$, *there exists* $P = P^T \succ 0$ *and* Y *such that the LMIs hold*

$$PA + A^TP - YC - C^TY^T \prec -2\alpha P \qquad P = P^T \succ 0 \tag{B.51}$$

Similarly as for the controllability, statement 2 and 3 provide a finite dimensional test for complete controllability (global and unform). Statement 4 is a LMI condition to the existence of an observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \tag{B.52}$$

and ensures that for all $\alpha > 0$ the modes of the matrix $A - LC$ are all smaller than $-\alpha$. Moreover a suitable observer gain is given by $L = P^{-1}Y$.

Definition B.23 *System (B.47) is detectable if one of the following equivalent statements holds:*

1. $\text{rank}\begin{bmatrix} sI-A \\ C \end{bmatrix} = n$ *for all* $s \in \mathbb{C}^+$

2. *There is no loss of rank while evaluating the rank condition* $\text{rank}\begin{bmatrix} sI-A \\ C \end{bmatrix}$ *at each nonnegative eigenvalue of* A.

3. *There exists* $P = P^T \succ 0$ *such that the LMIs hold*

$$\mathcal{N}_C^T(A^TP + PA)\mathcal{N}C \prec 0 \qquad P = P^T \succ 0 \tag{B.53}$$

4. *There exists* $P = P^T \succ 0$ *and* Y *such that the LMIs hold*

$$PA + A^TP - YC - C^TY^T \prec 0 \qquad P = P^T \succ 0 \tag{B.54}$$

As for stabilizability, it suffices to check that there is no loss of rank for all non asymptotically stable modes only. Said differently, if all the non observable modes are asymptotically stable then the system is detectable.

B.7 Control and Observation of Dynamical Systems

We present several processes, with different objectives, that can be synthesized in view of controlling or estimating variables.

Control of dynamical systems

There are different way to control (and thus stabilize) a dynamical system. All of these are based on a feedback loop which allows for the stabilization with simple controllers.

State-Feedback

$$u(t) = f(x(t)), \qquad u(t) = Kx(t)$$

Static Output-Feedback

$$u(t) = f(y(t)), \qquad u(t) = Ky(t)$$

Dynamic Output-Feedback

$$\begin{array}{rcl} \dot{x}_c(t) & = & f_c(x_c(t), y(t)) \\ u(t) & = & h_c(x_c(t), y(t)) \end{array}$$

where x_c is the controller state.

Each controller can be classified in one of these classes, for instance consider a rotating mechanical device where the angle of the rotation is measured and state is composed by the angular velocity and the angle itself:

- A PD controller is a state-feedback

- A proportional controller is a static output feedback
- A PID controller is a dynamic output feedback

The state-feedback is the easiest to synthesize since it is based on a full-information on the system. Most of the systems can be stabilized by this way. Nevertheless, it is assumed that all the states are known and may be difficult from a practical point of view, even using an observer (for instance in large systems or due to unobservable states).

Static output feedback controllers are simple in their forms, they are just gains acting on the measured output, but despite of their simplicity it is very difficult to synthesize them in some framework. First of all, for a given measurement, many systems cannot be stabilized by static output-feedback. Moreover, in the general case, the pole placement using such controllers is a NP-Hard problem Blondel and Tsitsiklis [1997], Fu [2004] even in the linear case. In the robust-control framework using Linear Matrix Inequalities, the synthesis of static-output feedback controllers is still an open-problem.

Dynamic output feedback controllers are the most spread controllers used in industries, it only relies on measured variables and its dynamic nature allows for improving robustness margins and performances. There exists a lot of ways to synthesize such controllers. In the robust control framework using LMIs, if the order of the controller equals the dimension of the system, then the synthesis is simple and can be efficiently solved, in a reasonable time, by computers (the synthesis problem admits a necessary and sufficient condition in terms of LMI). Nevertheless, if the order of the controller is strictly less than the system, it becomes a NP-Hard problem Scherer and Wieland [2005]; the static output feedback falls into this class (controller of order 0).

Observation of dynamical systems

The role of the observer is to estimate a state (or a part of the state) from the knowledge of the measured output and parts of the system inputs (such as the control input). The second role of the observer is to filter the data in order to remove noises and/or to make the estimate unperturbed by the external disturbances which do not affect this state. For sake of brevity, only observers for linear dynamical systems are presented.

The most common structure of observers are called Luenberger's observers corresponding to system (B.47) and are of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t) - Du(t)) \tag{B.55}$$

where $\hat{x}$ is the state of the observer and estimate the state of the system, u, y are respectively the control input and the measured output of the system. In this the value of L is sought such that the estimation error $x(t) - \hat{x}(t)$ is asymptotically stable (i.e. $x(t) - \hat{x}(t) \to 0$ and $t \to +\infty$).

A more general form is given by

$$\begin{array}{rcl} \dot{z}(t) & = & Mz(t) + Ny(t) + Su(t) \\ \hat{x}(t) & = & z(t) + Hy(t) \end{array} \tag{B.56}$$

where $z(t)$ is the state of the observer. Note here that the state of the observer $z(t)$ is different from the estimate $\hat{x}(t)$ whenever $y(t) \neq 0$. Moreover the number of matrices to determine is greater than in the previous and these additional degrees of freedom add flexibility compared to the 'one-gain' observer [Darouach et al., 1994].

There exist many classes of observer, let me mention the interesting class of *reduced-order observer* for which only part of the state is estimated and the *unknown-input observers* whose role is to estimate the state in presence of unknown inputs. They allow to decouple the estimation error and the unknown-inputs [Koenig and Marx, 2004, Koenig et al., 2004]. Observers can also be used in order to estimate unknown inputs as shown in [Cherrier et al., 2006] where inputs of a chaotic cryptography system are estimated.

All the controllers and observers presented above can be directly extended to the case of nonlinear systems, but their structure is not general due to the high diversity of nonlinear dynamical systems. In this case, controllers and observers need to be adapted to the considered system and remains an important challenge of nonlinear systems theory.

C $\mathcal{L}_q$ and $\mathcal{H}_q$ Spaces

This appendix is devoted to the introduction of very important signals and systems spaces.

Let us consider here linear systems of the form

$$\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+Ew(t)\\ z(t) & = & Cx(t)+Fw(t)\end{array}\tag{C.57}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^p$ and $z \in \mathcal{Z} \subset \mathbb{R}^q$ are respectively the state, the inputs and the outputs of the system.

C.1 Norms for Signals

The functions of time $x(t), w(t)$ and $z(t)$ are generally referred as *signals* since these functions, whatever they represent (temperature, speed, position...), are considered in an abstract space where the physical signification is not useful anymore. This is the reason why spaces of signals must be considered and these spaces are called $\mathcal{L}_q^n$ defined hereunder:

$$\mathcal{L}_q := \left\{ u \in \mathcal{F}([0,+\infty),\mathbb{R}^n) : \left(\int_0^{+\infty} ||u(t)||_q^q dt\right)^{1/q} < \infty \right\}\tag{C.58}$$

Only signals with support $[0, +\infty]$ are considered here by simplicity but is possible to define such sets for signals evolving over the more general support $[t_0, t_1]$.

It is possible to associate a norm to each one of this signals set and is denoted by $||\cdot||_{\mathcal{L}_q}$ and called $\mathcal{L}_q$ norm (with a slight abuse). Recall that a norm satisfies all the following properties:

1. $||u||_{\mathcal{L}_q} \geq 0$
2. $||u||_{\mathcal{L}_q} = 0 \Leftrightarrow u(t) = 0$ for all $t \geq 0$
3. $||\alpha u|||_{\mathcal{L}_q} = |\alpha| \cdot ||u||_{\mathcal{L}_q}$
4. $||u+v||_{\mathcal{L}_q} \leq ||u||_{\mathcal{L}_q} + ||v||_{\mathcal{L}_q}$ where α is a constant

The $\mathcal{L}_q$ norm is then defined as

$$||u|||_{\mathcal{L}_q} := \left(\int_0^{+\infty} ||u(t)||_q^q dt\right)^{1/q}\tag{C.59}$$

and hence a signal $u(t)$ belongs to the space $\mathcal{L}_q^n$ if and only if its $\mathcal{L}_q$-norm is bounded. There are three main norms for signals

$\mathcal{L}_1$-norm The 1-norm of a signal $u(t)$ is the integral of its absolute value

$$||u||_{\mathcal{L}_1} := \int_0^{+\infty} ||u(t)||_1 dt \tag{C.60}$$

In some papers and books, the $\mathcal{L}_1$ norm is treated as an electrical consumption. However, the $\mathcal{L}_1$ norm is rarely considered in the literature.

$\mathcal{L}_2$-norm The 2-norm of a signal $u(t)$ is

$$||u||_{\mathcal{L}_2} := \left(\int_0^{+\infty} ||u(t)||_2^2 dt\right)^{1/2} \tag{C.61}$$

From a physical point of view, the $\mathcal{L}_2$ norm represents the energy of a signal.

$\mathcal{L}_\infty$-norm The $\mathcal{L}_\infty$ norm of a signal is the larger upper bound of the absolute value

$$||u||_{\mathcal{L}_\infty} := \max_i \sup_{t\in[0,+\infty)} |u_i(t)| \tag{C.62}$$

The $\mathcal{L}_\infty$ norm of a signal is the maximum value that the signals under some input values. This norm is useful when the amplitude of signals needs to be constrained.

Remark C.1 *If the second property of norms (i.e. there exists $u(t) \neq 0$ such that $||u||_{\mathcal{L}_q} = 0$) is not satisfied, the term semi-norm is used instead of norm. For instance the power of a signal is a semi-norm and is referred in the literature as the power semi-norm:*

$$||y||_P := \lim_{T\to+\infty} \frac{1}{T}\int_0^T y(t)^* y(t) dt \tag{C.63}$$

These spaces enjoy a non trivial inclusion relationship described by the following Venn diagram depicted in Figure 6.1 inspired from Doyle et al. [1990].

For instance, let $\mathcal{Z} = \mathbb{R}^2$ and $\mathcal{W} = \mathbb{R}$, no behavior is captured by these definitions. For instance, if the system is $\mathcal{L}_2 - \mathcal{L}_2$ stable (for all $\mathcal{L}_2$ input, we have $\mathcal{L}_2$ outputs), then it is possible to consider the sets $\mathcal{L}_2(\mathbb{R}^+, \mathbb{R})$ and $\mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^2)$. Note that we cannot say $\mathcal{W} = \mathcal{L}_2(\mathbb{R}^+, \mathbb{R})$ since $\mathcal{W}$ is an Euclidian space of dimension 2 while $\mathcal{L}_2(\mathbb{R}^+, \mathbb{R})$ is a functional space which is infinite dimensional.

C.2 Norms for Systems

While physical magnitudes can be viewed as signals only, the relation between these signals and how they evolve in time (dynamical behavior) is called *system*. A system may be viewed as a physical process but also as a (linear) operator mapping functional space to another. For instance, system (C.57) maps the Euclidian space $\mathcal{W}$ to $\mathcal{Z}$. Note that these spaces are not functional spaces but Euclidian space containing values taken by input and output signals. However, by considering these Euclidian spaces, only few information is considered and It is possible (and more interesting) to capture greater information on the operator by considering functional spaces instead. This is the role of $\mathcal{L}_q$ spaces: rather than considering functional spaces where no constraints apply on elements, $\mathcal{L}_q$ spaces consider elements with a specific (desired) behavior, allowing to tackle more information on the system and its related signals.

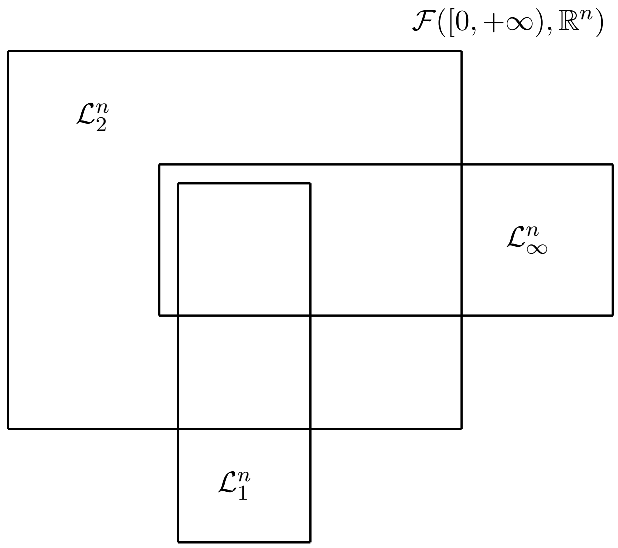

Figure 6.1: Inclusion of Signal Sets

By considering these norms, it seems interesting to develop a similar framework for systems and this brings us to the notion of norms for systems denoted $\mathcal{H}_q$. The letter $\mathcal{H}$ stands for *Hardy space* and is defined for functions holomorphic over $\mathbb{D}$ as

$$\mathcal{H}_q := \left\{ f \in \mathcal{F}(\mathbb{D},\mathbb{C}) : f \text{ holomorphic over } \mathbb{D} \text{ and } \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} [f(re^{\imath\theta})]^p d\theta \right)^{1/p} < \infty \right\} \tag{C.64}$$

For $0 < p < q < \infty$, it can be shown that $\mathcal{H}_q$ is a subset of $\mathcal{H}_p$. Variations of the latter definition exists for other domains than the unit open disc $\mathbb{D}$, in our case the domain which has to be considered in the right-half plane $\mathbb{C}^+$.

Definition C.2 **$\mathcal{H}_2$-norm** *The $\mathcal{H}_2$-norm of a signal $u(t)$ is*

$$||H||_{\mathcal{H}_2} := \left(\text{trace} \, \frac{1}{2\pi} \int_0^{+\infty} H(j\omega)H(j\omega)^* d\omega \right)^{1/2} \tag{C.65}$$

The $\mathcal{H}_2$ norm can be viewed as the maximal energy gain from the inputs to the outputs.

$\mathcal{H}_\infty$-norm *The $\mathcal{H}_\infty$-norm of a signal is the least upper bound of its absolute value*

$$||H||_{\mathcal{H}_\infty} := \max \bar{\sigma}(H(j\omega)) \tag{C.66}$$

The $\mathcal{H}_\infty$ norm can be viewed as the maximal energy gain from the inputs to the outputs.

An important property of the $\mathcal{H}_\infty$-norm is the submultiplicative property:

$$||M_1 M_2||_{\mathcal{H}_\infty} \leq ||M_1||_{\mathcal{H}_\infty} ||M_2||_{\mathcal{H}_\infty} \tag{C.67}$$

which has deep consequences in robustness analysis and robust control synthesis. Note that some system norms do not satisfy such a property, for instance the $\mathcal{H}_2$ does not.

It is worth noting that the $\mathcal{H}_2$ norm is the same as in Definition C.2. By the way, both definitions coincide in the SISO case but the *induced-norm* version (norm of a system induced by the norm of the input and output spaces) holds in the MIMO case and therefore defines a generalization of Definition C.2. This is the reason why this extended version is called *Generalized* $\mathcal{H}_2$ *norm* or $\mathcal{L}_2 - \mathcal{L}_\infty$ *induced norm*.

To conclude this section, the great interest of induced norms are their domain of validity. Indeed, the system norms are generally expressed in terms of functions over the frequency domain (restricting the validity of the definitions over LTI systems only) and the signal norms over the time domain (quite general framework). Hence, this duality allows for the computation of norms of systems which are neither time-invariant nor linear by considering the quotient of norms of the output signals space over norms of the input signal spaces. This opens the doors to gain analysis of time-varying, parameter varying, nonlinear and distributed systems. Therefore, the energy gain of time-varying system will then be referred as its $\mathcal{L}_2$-gain. The correspondence between signal and system norms is summarized in Table 6.1.

	$\|\|w\|\|_{\mathcal{L}_2}$	$\|\|w\|\|_{\mathcal{L}_\infty}$
$\|\|z\|\|_{\mathcal{L}_2}$	$\|\|H\|\|_{\mathcal{H}_\infty}$	∞
$\|\|z\|\|_{\mathcal{L}_\infty}$	$\|\|H\|\|_{\mathcal{H}_2}$	$\|\|H\|\|_{\mathcal{H}_1}$

Table 6.1: Correspondence between norms of signals and systems

D Linear Matrix Inequalities

This appendix aims at providing a brief overview of Linear Matrix Inequalities (LMIs). A brief history is given, then some preliminary definitions and methods to solve them are introduced.

D.1 Story

Historically, the first LMI appeared in the pioneering work of Lyapunov (actually its Ph.D thesis in 1890) which was on the 'General Stability of Motion' and where what is called 'the Lyapunov's theory' is defined with its the fundamental tools. In this work, the stability of a linear time-invariant dynamical systems $\dot{x} = Ax$ is equivalent to the feasibility of the Linear Matrix Inequality:

$$A^T P + PA \prec 0 \qquad P = P^T \succ 0 \tag{D.68}$$

In his work Lyapunov wrote down all the foundation of modern control theory and therefore remains, in this field, one of the major works since it has been developed. Since then, many results have been grafted over it. Indeed, in 1940, Lu're and Postnikov et al. applied Lyapunov' s theory to control problems involving nonlinearityin the actuator. This has lead to Lu're systems which are defined as

$$\dot{x} = Ax + B\phi(x) \tag{D.69}$$

where $\phi(\cdot)$ is a nonlinear function of x. Although the stability criteria ware not in a LMI form (in reality they were polynomially frequency dependent inequalities), they actually were equivalent to a LMI formulation. The bridge, which was unknown at this time, between frequency dependent inequalities and LMI has been emphasized in an important result derived by Yakubovich, Popov, Kalman, Anderson... and is called the Positive Real Lemma (some precision on it and its link to passivity are introduced in Appendix E.4). This positive real lemma, reduces the solution of a LMI into simple graphical criterion in the complex plane (which is linked to Popov, circle and Tsypkin criteria). In 1962, Kalman derived one of the most important work of this century: the 'Kalman-Yakubovitch-Popov' Lemma which bridges completely graphical tests in the complex plane and a family of LMIs (see Appendix E.3) and allows by now to switch easily from frequency domain to time-domain criteria.

In 1970, Willems focused on solving algebraic equations such as Lyapunov's or Ricatti's equations (ARE), rather than LMIs. Indeed, the solvability was not well established at this time and the numerical algebra was developed to solve algebraic equations rather than LMIs. To understand the power of LMIs, it has been necessary to develop complex mathematical tools and algorithms to solve them.

In 1919, the 'Ellipsoid Algorithm' of Khachiyan was the first algorithm to exhibit a polynomial complexity (polynomial bound on worst-case iteration count) for Linear Programming. Linear Programming problems are optimization problems where the optimization cost and constraints are all affine in the unknown variables. In 1984, Karmarkar introduced 'Interior Point' methods for LP which has lead to lower complexity and better efficiency than ellipsoidal methods.

The particularity of LMIs is that, although the cost and the constraints are affine on the unknown variables, the inequalities are not componentwise but represent the location of the eigenvalues of the matrix inequality. Therefore, the problem is obviously non linear since the location of the eigenvalues of a symmetric matrix depend on the sign of its principal minors. By computing the principal minors of a LMI, it appears that we obtain a set of polynomial scalars inequality and therefore is nonlinear. However, although the optimization problem is non linear, it can be shown that the optimization problem is a convex problem, one of the most studied field in optimization (see Boyd and Vandenbergue [2004]) and hence now LMI benefits of a huge arsenal of solid tools.

In 1988, Nesterov, Nemirovskii and Alizadeh (see Nesterov and Nemirovskii [1994]) extend IP methods for Semidefinite Programming (SDP) which is the class of problems where LMIs belong. Since then, IP methods have been heavily developed and is now the most powerful tools to solve numerically LMIs.

Finally, in 1994, the research effort on application of LMI to control culminated in Boyd et al. [1994] where many other authors brought important contributions, for instance Apkarian, Bernussou, Gahinet, Geromel, Peres...

Since then many solvers for SDP have been developed for instance SeDuMi Sturm [2001, 1999], DSDP, SDPT3... Since all this solvers have been developed for the mathematical framework of SDP and since the representation of LMI in the field of automatic control is based on a matrix representation, softwares called 'parsers' have been developed as interface between these notations, for instance SeDuMi Interface and the best one: Yalmip Löfberg [2004].

D.2 Definitions

A Linear Matrix Inequality (LMI) is an inequality of the form

$$\mathcal{L}(x) := \mathcal{L}_0 + \sum_{i=1}^{m} \mathcal{L}_i x_i \succ 0 \tag{D.70}$$

where $x \in \mathbb{R}^m$ is the variable and the symmetric matrix $\mathcal{L}_i \in \mathbb{S}^n$, $i = 1, \ldots, n$ are given data. The inequality symbol $\prec$ means that $\mathcal{L}(x)$ is positive definite (i.e. $y^T \mathcal{L}(x) y > 0$ for all $y \in \mathbb{R}^n - \{0\}$. This inequality is equivalent to m polynomial inequalities corresponding to the leading minor of $\mathcal{L}(x)$.

The LMI (D.70) is a convex constraint on x: the subset $\{x \in \mathbb{R}^m : \mathcal{L}(x) \succ 0\}$ is convex.

Multiple LMI $\mathcal{L}^{(1)}(x) > 0, \ldots, \mathcal{L}^{(q)}(x) \prec 0$ can be expressed as a single LMI diag$(\mathcal{L}^{(1)}(x) > 0, \ldots, \mathcal{L}^{(q)}(x)) \succ 0$. This shows that the intersection of LMI constraints is also a LMI. This can be connected with the property that the intersection of convex sets is also a convex set.

Notation (D.70) is the 'mathematical' notation while the following is the notation used by in the field of automatic control and system theory

$$A^T P + PA \prec 0 \qquad P = P^T \succ 0 \tag{D.71}$$

where the matrix $P = P^T \succ 0$ is the variable and $A \in \mathbb{R}^{n \times n}$ a given data. It is not possible to give a general formulation of LMIs where matrices are variable since there is a large variety of different forms. Nevertheless, any LMI in 'matrix variable' form can be written into the mathematical form (but the converse is not necessarily true). To write this, just decompose $P = P^T \succ 0$ over a basis of symmetric matrices of dimension n denoted by P_i. Hence $P := P(x) = \sum_{i=1}^m P_i x_i$ with $m = \dfrac{n(n+1)}{2}$. Finally by identification we get $\mathcal{L}_0$ and $\mathcal{L}_i = -A^T P_i - P_i A$.

Definition D.1 *A LMI $\mathcal{M}(x) \preceq 0$ is feasible if said to be feasible if and only if there exists x such that $\mathcal{M}(x) \preceq 0$. It is said to be strictly feasible if and only if there exists x such that $\mathcal{M}(x) \prec 0$.*

D.3 How to solve them ?

Several approaches allowing the determination of the solution of LMIs are presented here.

Algebraic Methods

In order to solve simple LMI, algebraic methods can be used using linear algebra. This is possible when dealing with only few decision matrices. For instance, let us consider the well-known Lyapunov stability LMI condition for linear time-invariant systems:

$$A^T P + PA \prec 0 \tag{D.72}$$

Assume there exists $P = P^T \succ 0$ such that the LMI is satisfied, i.e. the system $\dot{x}(t) = Ax(t)$ is asymptotically stable (all the eigenvalues of A lie in the left half complex plane).

Let $P_0 = \int_0^{+\infty} e^{A^T t} Q e^{At} dt$ with some matrix $Q = Q^T \succ 0$ and inject the expression of P_0 into the latter LMI, we get

$$\begin{aligned} \int_0^{+\infty} A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A dt &= \int_0^{+\infty} \frac{d}{dt}\left[e^{A^T t} Q e^{At}\right] dt \\ &= \lim_{t \to +\infty} e^{A^T t} Q e^{At} - Q \end{aligned}$$

Since the system is asymptotically stable then $\lim_{t \to +\infty} e^{A^T t} Q e^{At} = 0$ an then a parametrization of the solutions of the LMI (D.72) is given by P_0.

It is clear that this method can become very complicated while dealing with LMIs of high dimensions and with a large number of decision matrices.

Algorithms

At this time, Interior Points algorithms are mainly used. Simple algorithms are presented in Boyd et al. [1994] while the complete theory of IP algorithms with barrier function, in an unified framework, is detailed in the (very difficult to understand in detail for nonspecialists) book Nesterov and Nemirovskii [1994]. The idea of barrier function is briefly explained here:

Consider the optimization problem:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^T x \\ \text{s.t.} & F_i(x) \succeq 0, \ i = 1, \ldots, p \end{array} \tag{D.73}$$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $F_i(x)$ are respectively the cost vector, the decision variable and the LMIs constraints. It is important to note that every LMI optimization problem can be rewritten in the latter form.

The idea of interior point algorithm with barrier function, is to turn a constrained optimization problem into an unconstrained one. By introducing the set

$$\mathcal{X}_f := \{x \in \mathbb{R}^n : \ F_i(x) \succeq 0, \ i = 1, \ldots, p\} \tag{D.74}$$

the optimization problem (D.73) is equivalent to

$$\min_{x\in\mathcal{X}_f} \quad c^T x \tag{D.75}$$

The key idea to define implicitly the set $\mathcal{X}_f$ (since it is difficult and time consuming to define it explicitly) is to define a function which is small in the interior of $\mathcal{X}_f$ and tends to infinity for each sequence of points converging to the boundary of $\mathcal{X}_f$. This function is called a *barrier function*. It also important, for mathematical purpose, that this barrier function be analytic (differentiable), convex and self-concordant. Indeed, if the barrier function is convex then the optimization problem will be convex and hence the theory of convex optimization applies. The differentiability of the barrier function (actually it must be C^3) allows for the computation of gradient and hessian in the iterative optimization procedure. Finally, the self-concordance of a barrier function is a property, which has been introduced specifically in the framework of SDP optimization, which guarantees nice convergence properties of the Newton algorithm used to solve these unconstrained optimization problems. This notion has been introduced in the book Nesterov and Nemirovskii [1994] and the definition is given below:

Let $F(x)$ by function which is convex and analytic. It is said to be self-concordant with parameter a if

$$|D^3F(x)[h, h, h]| \leq 2a^{-1/2}(D^2F(x)[h, h])^{3/2} \tag{D.76}$$

in a metric defined by the hessian itself and

$$|DF(x)[h]| \leq b(D^2F(x)[h, h])^{3/2} \tag{D.77}$$

where $D^kF(x)[h_1, \ldots, h_k]$ is the k^{th} differential of F taken at x along the collection of direction $[h_1, \ldots, h_k]$. The first inequality defines the Lipschitz continuity of the Hessian of the barrier with respect to the local Euclidian metric defined by the Hessian itself. The second inequality defines the Lipschitz continuity of the barrier itself with respect to the same local Euclidian structure. The signification of term self-concordant is not easy to see. The first idea could be that the absolute value of the third derivative is bounded by a function of the second one. This establishes a link between them and shows that the third order term in the Taylor expansion can always be bounded by the second order term. Another idea is that the third order derivative can be approximated by an expression involving the the Hessian.

A good barrier function for SDP is the logarithmic barrier

$$f(x) = -\log\det F(x) = \log\det F(x)^{-1} \tag{D.78}$$

This function is analytic, convex and self-concordant on $\{x: \ F(x) \succ 0\}$.

Finally the constrained optimization problem (D.73) (and equivalently (D.75)) is converted into the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} \qquad c^Tx + \log\det F^{-1}(x) \tag{D.79}$$

Finally, the Newton algorithm is used to find the solution of the optimization problem (D.79). It can be shown that the optimum of (D.79) coincides with the optimum of (D.75) and therefore no modification of the problem is done when adding the self-concordant barrier function to the cost.

The Newton algorithm aims to find zeros of functions, say $f(x)$ and the iteration procedure is

$$x_{k+1} = x_k - [\nabla^2 f(x)]^{-1}\nabla f(x) \tag{D.80}$$

where $\nabla^2 f(x)$ and $\nabla f(x)$ are respectively the Hessian and the gradient of f evaluated at x. Despite of its apparent simplicity, this iteration procedure converges quadratically provided that the initial condition x_0 belongs satisfies

$$\frac{L}{2m^2}||f'(x_0)||_2 < 1 \tag{D.81}$$

where L is the Lipschitz continuity constant of the Hessian and m is defined as $h^T f''(x)h \geq m||h||_2^2$. It can be shown that in the case of unconstrained optimization with self-concordant barrier functions, the Newton procedure can find very efficiently the global optimum of optimization problems (D.73)-(D.79).

In Nesterov and Nemirovskii [1994], it is shown that for every allowable x_i (i.e. $x_i \in \mathcal{X}_f$) the next value x_{i+1} remains in $\mathcal{X}_f$ (is allowable too) and $f(x_{i+1}) \leq f(x_i)$. Then for a good initialization of the iterative procedure, it suffices to find a point in $\mathcal{X}_f$. For this purpose, most solvers implement an initialization procedure resulting in the determination of an initial feasible point from which the optimum of the optimization problem can be easily computed.

E Technical Results in Robust Analysis, Control and LMIs

This appendix aims at providing a catalog of important definitions and theorems extensively used in the literature.

Let us consider multivariable finite dimensional linear time-invariant systems of the form:

$$Z(s) = H(s)W(s) \tag{E.82}$$

where s stands for the Laplace variable, $H(s)$ the transfer function of the system and $W(s)$, $Z(s)$ are respectively the input and the output.

Assume that (E.82) admits the following minimal realization Σ_l:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{array} \tag{E.83}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^p$ and $z \in \mathcal{Z} \subset \mathbb{R}^q$ are respectively the state, the inputs and the outputs.

E.1 Dissipative Systems and Supply Rates

The dissipativity is a theory devoted to study the stability of a non-autonomous systems of any kind. The main principle of the theory is really simple: let us consider the general system Σ governed by the equations

$$\begin{array}{rcl} \dot{x}(t) & = & f(x, w) \\ z(t) & = & h(x, w) \end{array} \tag{E.84}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^p$ and $z \in \mathcal{Z} \subset \mathbb{R}^q$ are respectively the state, the inputs and the outputs.

Let $s(w, z)$ be a mapping from $\mathcal{W} \times \mathcal{Z} \to \mathbb{R}$. It is assumed that for any $t_0, t_1 \in \mathbb{R}$ and for all input-output pairs (w, z) satisfying (E.84), the function $s(w, z)$ is absolutely integrable (i.e. $\int_{t_0}^{t_1} |s(w(t), z(t))| dt < \infty$). This mapping is referred as the **supply function** and its meaning will be detailed just after the following definition:

Definition E.1 *The system (E.84) with supply function s is said to be dissipative if there exists a function $V : \mathcal{X} \to \mathbb{R}$ such that*

$$V(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t)) dt \geq V(x(t_1)) \tag{E.85}$$

for all $t_0 \leq t_1$ and all signals (w, x, z) which satisfies (E.84). The pair (Σ, s) is said to be conservative if the equality holds for all $t_0 \leq t_1$ and all signals (w, x, z) which satisfies (E.84).

The supply-rate s should be interpreted as the supply delivered to the system. This means that $s(w, z)$ represents the rate at which supply circulates into the system if the pair (w, z) is generated. Hence, when the integral $\int_0^T s(w(t), z(t)) dt$ is positive then the work is done on the system while the work is done by the system when the integral is negative. The function V is called the storage function and generalizes the notion of an energy for a dissipative system.

Thanks to this interpretation, inequality (E.85) says that for any interval $[t_0, t_1]$, the change of internal storage $V(x(t_1)) - V(x(t_0))$ will never exceed the amount of supply that flows into the system. This means that part of what is supplied is stored while the remaining part is dissipated.

For more details on dissipativity and dissipative systems, please refer to Scherer and Weiland [2004].

E.2 Linear Dissipative Systems and Quadratic Supply Rates

We detail here the special case of linear system governed by expressions (E.83). Suppose that $x^* = 0$ is the point of neutral storage and consider quadratic supply functions $s : \mathcal{W} \times \mathcal{Z} \to \mathcal{R}$ defined by

$$s(w(t), z(t)) = \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \tag{E.86}$$

We provide here the essential result about dissipativity of linear systems with quadratic supply rate.

Theorem E.2 *Suppose that system Σ_l defined by (E.83) is controllable and let the supply function be defined by(E.86). Then the following statements are equivalent:*

1. *(Σ_l, s) is dissipative*
2. *(Σ_l, s) admits a quadratic storage function $V(x) = x^T P x$ with $P = P^T$*
3. *There exists $P = P^T$ such that*

$$F(P) := \begin{bmatrix} A^T P + PA & PB \\ \star & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \preceq 0 \tag{E.87}$$

4. *For all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$, the tranfer function $H(s) = C(sI - A)^{-1}E + F$ satisfies*

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ H(j\omega) \end{bmatrix} \succeq 0 \tag{E.88}$$

The proof can be found in Scherer and Wieland [2005].

E.3 Kalman-Yakubovich-Popov Lemma

The Kalman-Yakubovitch-Popov lemma shows that, amongst others, the frequency condition given by Popov is equivalent to the existence of a Lyapunov function. This is a very important result in linear system theory. Some important considerations are provided in Rantzer [1996], Scherer and Wieland [2005], Willems [1971], Yakubovitch and references therein.

Lemma E.3 *For any triple of matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \in \mathbb{S}^{n+m}$, the following statements are equivalent:*

1. *There exists a symmetric matrix $P = P^T$ such that*

$$M + \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \prec 0 \tag{E.89}$$

2. *$M_{22} \prec 0$ and for all $\omega \in \mathbb{R}$ and complex vectors $\mathrm{col}(x, w) \neq 0$*

$$\begin{bmatrix} A - j\omega I & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \quad \text{implies} \quad \begin{bmatrix} x \\ w \end{bmatrix}^* M \begin{bmatrix} x \\ w \end{bmatrix} < 0 \tag{E.90}$$

If (A, E) is controllable, the corresponding equivalence also holds for non-strict inequalities.
Finally, if

$$M = -\begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \tag{E.91}$$

then statement 2 is equivalent to the condition that for all $\omega \in \mathbb{R}$, *with* $\det(j\omega I - A) \neq 0$ *we have*

$$\begin{bmatrix} I \\ C(j\omega I - A)^{-1}B + D \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ C(j\omega I - A)^{-1}B + D \end{bmatrix} \succ 0 \tag{E.92}$$

The KYP-lemma establishes a relation between the frequency domain analysis and a linear matrix inequality expressed in the time-domain. This proves that geometric considerations is the complex plane (such as Popov' criterion, circle criterion...) have time-domain counterparts which can be expressed through linear matrix inequalities, or equivalently algebraic Ricatti inequalities. Another interesting fact is that, it turns a semi-infinite matrix inequality (due to the frequency variable $\omega \in [0, +\infty)$) into a finite dimensional matrix inequality involving a finite dimensional variable $P = P^T \succ 0$. Another examples follow.

E.4 Positive real lemma

The positive real lemma is highly related to the passivity of a system and has played a crucial role in questions related to the stability of control systems and synthesis of passive electrical networks.

An LMI formulation to passivity can be derived using the dissipativity framework by considering the supply function $s(w, z) = z^T w + w^T z$. This leads to:

Lemma E.4 *System (E.83) is passive (or positive real) if and only if there exists a matrix* $P \in \mathbb{S}^n_{++}$ *such that*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ \star & -(D + D^T) \end{bmatrix} \prec 0 \tag{E.93}$$

Then for all $\omega \in \mathbb{R}$ *with* $\det(j\omega I - A) \neq 0$ *one has* $H(j\omega)^* + H(j\omega) \succeq 0$.

Moreover, $V(x) = x^T P x$ *defines a quadratic storage function if and only if* P *satisfies LMI (E.93).*

Proof: The proof is an application of the Kalman-Yakubovicth-Popov lemma with quadratic supply function $s(w, z) = w^T z + z^T w$. □

E.5 $\mathcal{H}_2$ Performances

The $\mathcal{H}_2$ norm of a system measures the output energy in the impulse responses of the system.

Lemma E.5 *Suppose system (E.83) with* $F = 0$ *is asymptotically stable. Then* $||H||_{\mathcal{H}_2} < \nu$ *if and only if there exists* $P = P^T \succ 0$, Z *and* $\nu > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -I \end{bmatrix} \prec 0 \qquad \begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} \succ 0 \qquad \text{trace}(Z) < \nu^2 \tag{E.94}$$

Proof: See Scherer and Wieland [2005]. □

E.6 Generalized $\mathcal{H}_2$ performances

The generalized $\mathcal{H}_2$ performance is defined as the $\mathcal{L}_2 - \mathcal{L}_\infty$ induced norm. The system is then defined as an operator from the set of signals of bounded energy to set of signals with finite amplitude (energy to peak norm). In the scalar case, $\mathcal{L}_2 - \mathcal{L}_\infty$ induced norm coincides with the $\mathcal{H}_2$ norm which is the reason why it is called generalized $\mathcal{H}_2$ norm.

Lemma E.6 *Suppose system (E.83) with $F = 0$ is asymptotically stable. Then $||H||_{\mathcal{L}_2,\mathcal{L}_\infty} < \nu$ if and only if there exists $P = P^T \succ 0$ and $\nu > 0$ such that*

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & -\nu I \end{bmatrix} \prec 0 \qquad \begin{bmatrix} P & C^T \\ C & \nu I \end{bmatrix} \succ 0 \qquad \text{trace}(Z) < \nu^2 \tag{E.95}$$

Proof: *See Scherer and Wieland [2005].* □

E.7 Bounded-Real Lemma - $\mathcal{H}_\infty$ Performances

The bounded real lemma is a well known lemma allowing for the computation of the $\mathcal{H}_\infty$ norm of a linear system. It can be obtained in the dissipativity framework while considering the supply function $s(w, z) = \gamma w^T w - \gamma^{-1} z^T z$.

Lemma E.7 *System (E.83) is asymptotically stable if and only if there exists $P \in \mathbb{S}^n_{++}$ and $\gamma > 0$ such that*

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ \star & -\gamma I & D^T \\ \star & \star & -\gamma I \end{bmatrix} \prec 0 \tag{E.96}$$

Then for all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$ one has $H(j\omega)^* H(j\omega) \preceq \gamma^2 I$. Moreover, $V(x) = x^T P x$ defines a quadratic storage function if and only if P satisfies LMI (E.96).

Proof: *The proof is an application of the Kalman-Yakubovicth-Popov lemma with quadratic supply function $s(w, z) = \gamma w^T w - \gamma^{-1} z^T z$.* □

This result is extremely important and has brought to lots of works in systems and control theory. As a first interpretation, it implies that we have the following relation

$$||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2} \tag{E.97}$$

and thus γ is the energy gain of the system. This means that for an input of unit energy, the energy of the output is less than γ. Moreover, in the time-invariant case, it is possible to show that the $\mathcal{L}_2$ induced norm coincides with the $\mathcal{H}_\infty$ norm of the system.

The bounded real lemma is a useful tool in robust control since the $\mathcal{H}_\infty$ norm is sub-multiplicative which means that for two asymptotically stable transfer functions $M_1(s)$ and $M_2(s)$, the following relation holds:

$$||M_1(s)M_2(s)||_\infty \leq ||M_1(s)||_\infty \cdot ||M_2(s)||_\infty$$

This can be modified to have the useful implication for $\alpha, \beta > 0$:

$$||M_1(s)||_\infty < \beta/\alpha \text{ and } ||M_2(s)||_\infty < \alpha \Rightarrow ||M_1(s)M_2(s)||_\infty < \beta$$

which is the basis of small-gain theorem (see Appendix E.10).

E.8 $\mathcal{L}_\infty - \mathcal{L}_\infty$ Performances

The $\mathcal{L}_\infty$ induced norm is also called peak to peak norm since it considers the system as an operator from the set of signals with finite amplitude to another set of signals of finite amplitude.

Lemma E.8 *Consider system (E.83) then if there exists $P = P^T > 0$ and scalars $\alpha, \beta > 0$ such that*

$$\begin{bmatrix} A^TP + PA + \alpha P & PB \\ B^TP & -\beta I \end{bmatrix} \preceq 0 \qquad \begin{bmatrix} \alpha P & 0 & C^T \\ 0 & (\alpha - \beta)I & D^T \\ C & D & \alpha I \end{bmatrix} \succ 0 \tag{E.98}$$

then the peak-to-peak norm of the system is lower than α: $||H||_{\mathcal{L}_\infty - \mathcal{L}_\infty} < \alpha$.

Proof: *See Scherer and Wieland [2005].* □

E.9 $\mathcal{S}$-procedure

The $\mathcal{S}$-procedure allows to deal with implications in the LMI framework. Indeed, we aim to express the following problem

$$\text{for all } \xi \in \mathbb{R}^n \text{ such that } \xi^T M_i \xi \leq 0,\ i = 1, \ldots, N \Rightarrow \xi^T M_0 \xi < 0 \tag{E.99}$$

as an LMI problem.

It is obvious that if there exists scalars $\tau_1, \ldots, \tau_N \geq 0$ such that

$$M_0 - \sum_{i=1}^{N} \tau_i M_i \prec 0 \tag{E.100}$$

then (E.99) holds. The converse is not true in general unless $N = 1$ for real valued problems or $N = 2$ for complex valued problems.

Despite of its conservatism, it is a very useful tool in robust analysis and control theory and plays a crucial role in the full-block $\mathcal{S}$-procedure (in some sense) Scherer [2001], IQC framework Rantzer and Megretski [1997], Lu're systems...

E.10 Small Gain Theorem

Let us consider the interconnection depicted on figure 6.2

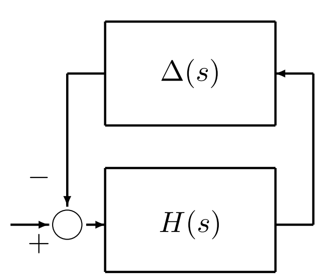

Figure 6.2: Interconnection of systems

It is clear that the closed-loop system $H_{cl}(s) = \dfrac{H(s)}{1+\Delta(s)H(s)}$ and hence the $\mathcal{H}_\infty$ norm of the closed-loop system can be bounded as

$$||H_{cl}||_{\mathcal{H}_\infty} \leq \frac{||H||_{\mathcal{H}_\infty}}{1-|\Delta H||_{\mathcal{H}_\infty}} \tag{E.101}$$

It is well-known that the closed-loop system is stable if and only if $||\Delta H||_{\mathcal{H}_\infty} < 1$. It is not difficult to verify that ΔH satisfies this property by using standard LMI arguments (see the computation of $\mathcal{H}_\infty$-norm in Appendix E.7). Let us assume, for some reasons, the computation of the $\mathcal{H}_\infty$-norm of the product ΔH cannot be performed.

Since the $\mathcal{H}_\infty$-norm is submultiplicative then we have the inequality

$$||H_{cl}||_{\mathcal{H}_\infty} \leq \frac{||H||_{\mathcal{H}_\infty}}{1-||H||_{\mathcal{H}_\infty} \cdot ||\Delta||_{\mathcal{H}_\infty}} \tag{E.102}$$

and hence the closed-loop is asymptotically stable if $||H||_{\mathcal{H}_\infty} \cdot ||\Delta||_{\mathcal{H}_\infty} < 1$ Therefore if it possible to compute (estimate) the $\mathcal{H}_\infty$ norm of $\Delta(s)$ then the closed-loop system is asymptotically stable if

$$||H||_{\mathcal{H}_\infty} \leq \frac{1}{||\Delta||_{\mathcal{H}_\infty}} \tag{E.103}$$

Assuming that $H(s)$ admits realization (A, B, C, D), it is easy to determine the stability sufficient condition is given by the *Small-Gain Thorem*:

Theorem E.9 *The closed-loop system is stable if there exist $P = P^T \succ 0$ such that the LMI holds*

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ \star & -I & D^T \\ \star & \star & -I \end{bmatrix} \prec 0 \tag{E.104}$$

Due to the use of the submultiplicative property of the $\mathcal{H}_\infty$ norm, stability conditions are sufficient only and are in many cases, very conservative.

E.11 Scalings and Scaled-Small Gain theorem

In order to reduce the conservatism of the small-gain theorem which takes into account norms only, some *scalings* are introduced in the loop. These scalings do not modify the interconnection but allows for a reduction of conservatism. Let us consider an uncertain square matrix Δ containing, for simplicity, unknown real valued parameters ρ_i and full-blocks gathered on the diagonal:

$$\begin{array}{rcl} \Delta &=& \text{diag}(\Delta_s, \Delta_f) \\ \Delta_s &:=& \text{diag}_i(\rho_i I_{s_i}) \\ \Delta_f &:=& \text{diag}_i(F_i) \end{array} \tag{E.105}$$

where s_i is the number of occurrence of scalar parameter ρ_i and F_i are full-blocks.

The idea is to capture the structure of the uncertain matrix Δ by a matrix commutation property

$$L\Delta = \Delta L$$

which can also be defined by an identity relation

$$\Delta = L^{-1}\Delta L$$

The set of scalings corresponding to the uncertain structure Δ is defined by

$$\mathcal{S}(\Delta) := \{L \in \mathbb{S}_{++} : \ L\Delta = \Delta L\} \tag{E.106}$$

This set enjoys the following properties:

1. $I \in \mathcal{S}(\Delta)$ and therefore the small-gain is a particular case (more conservative) of this approach.
2. $L \in \mathcal{S}(\Delta) \Longrightarrow L^T \in \mathcal{S}(\Delta)$
3. $L \in \mathcal{S}(\Delta) \Longrightarrow L^{-1} \in \mathcal{S}(\Delta)$
4. $L_1 \in \mathcal{S}(\Delta), L_2 \in \mathcal{S}(\Delta) \Longrightarrow L_1L_2\Delta = \Delta L_1L_2$ not that the matrix L_1L_2 is not necessarily symmetric.
5. $\mathcal{S}(\Delta)$ is a convex subset of $\mathbb{R}^k$ where k is the dimension of $Delta$.

The structure of $L \in S(\Delta)$ can be expressed easily by

$$\begin{array}{rcl} L &=& \text{diag}(L_s, L_f) \\ L_s &=& \text{diag}_i(L_i^s), \ L_i^s \in \mathbb{S}_{++}^{s_i} \\ L_f &=& \text{diag}_i(l_i I_{n_i}) \end{array} \tag{E.107}$$

where n_i is the size of square full-block F_i.

Using this scaling it is possible modify the small-gain theorem into another refined version called *Scaled-Small Gain Theorem*

Theorem E.10 *The closed-loop system is stable if there exist $P = P^T \succ 0$ and $L \in \mathcal{S}(D)$ such that the following LMI holds*

$$\begin{bmatrix} A^TP + PA & PB & C^TL \\ \star & -L & D^TL \\ \star & \star & -L \end{bmatrix} \prec 0 \tag{E.108}$$

Despite of the conservatism reduction, this result is still conservative since it stills considers the norm of the operator and it would be more interesting to capture a more complex (complete) set of uncertainty. Actually the scaled-small gain can be obtained in the dissipativity framework by considering a supply-function:

$$s(w(t), z(t)) = w(t)^T Lw(t) - z(t)^T Lz(t)$$

for some L which satisfies $L - \Delta^T L\Delta \succeq 0$.

E.12 Full-Block $\mathcal{S}$-procedure

The Full-Block $\mathcal{S}$-procedure unifies all the frameworks of scalings into a single one, where small-gain and scaled-small gain results are particular cases only.

The full-block $\mathcal{S}$-procedure considers a full-block supply-function of the form

$$s(w(t), z(t)) = \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \tag{E.109}$$

such that

$$\int_0^{+\infty} s(w(t), z(t))dt \geq 0$$

Hence we have the following theorem:

Theorem E.11 *The closed-loop system is stable if there exist* $P = P^T \succ 0$, $Q = Q^T \prec 0$ *and* $R = R^T \succeq 0$ *and* S *such that the LMIs*

$$\begin{bmatrix} A^TP + PA & PB \\ \star & 0 \end{bmatrix} + \begin{bmatrix} 0 & C^T \\ I & D^T \end{bmatrix} \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0 \tag{E.110}$$

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \preceq 0 \tag{E.111}$$

hold.

E.13 Dualization Lemma

The dualization lemma (which has been discovered simultaneously but separately in Scherer [1999] and Iwasaki and Hara [1998]) allows to turn an LMI into another equivalent one provided that some strong assumptions are satisfied:

Lemma E.12 *Let* $M \in \mathbb{S}^n$ *nonsingular and* $S \in \mathbb{R}^{n\times p}$ *with* $\text{rank}(B) = p$ *such that* $n^-(M) = rank(B) = p$ *then the following statements are equivalent:*

1. *The LMI* $S^TMS \prec 0$ *holds*
2. *The LMI* $S^{\perp T}M^{-1}S^{\perp} \prec 0$ *holds where* $S^{\perp}$ *is a basis of the orthogonal complement of* $Im(S)$ *(i.e.* $S^TS^{\perp} = 0$*).*

At first sight, this result may seem superfluous, but actually it is very useful in the robust/LPV control context. Indeed, when using multipliers to study systems expressed through LFR, it has the property of decoupling data matrices from multipliers and Lyapunov matrix, making the problem convex (see for instance Scherer [1999], Wu [2003]).

Nevertheless, the *rank* constraint provides a very strong condition and such lemma is difficult to apply to other type of systems. For instance, by considering time-delay systems and the Lyapunov-Krasovskii theorems, the rank condition is not satisfied due to a high number of Lyapunov matrices.

E.14 Bounding Lemma

The bounding lemma de Souza and Li [1999], Xie et al. [1992] is used to remove uncertainties from matrix inequalities in the robust analysis/control framework. It deals with both real and complex parameter uncertainties. We provide here the real version of the result:

Lemma E.13 *Let* $\Psi \in \mathbb{S}^n$ *a symmetric matrix and* $P \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{p\times n}$ *and* $\Delta(t) \in \mathbf{\Delta}$ *be an uncertain matrix (possibly time-varying) with*

$$\mathbf{\Delta} := \{\Delta(t) \in \mathbb{R}^{m\times p} : p \leq m,\ \Delta^T\Delta \leq R,\ R > 0\}$$

then the following statements are equivalent:

1. *The LMI*
$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^T P \prec 0 \tag{E.112}$$
holds for all $\Delta(t) \in \mathbf{\Delta}$

2. *There exists a scalar* $\varepsilon > 0$ *such that the LMI*
$$\Psi + \varepsilon P^T RP + \varepsilon^{-1}Q^T Q \prec 0$$
holds.

Proof: It seems interesting to provide the proof of this result. It is actually an old (and interesting result) and then the proof is not simple to find since, generally, provided references are not original one. The original paper which has provided for the first time this result is [good question, I am still looking for it]. Moreover, most of the technical results involved in the proof can be found in Khargonekar et al. [2001], Petersen [1987]. Without loss of generality, let us consider for simplicity here that $R = I$.

Sufficiency:

Assume that matrices Ψ*,* P *and* Q *in contain decision matrices such that (E.112) is a LMI, and let us denote all of them in a compact form* D_M*. Suppose that exists* $D_M = D_M^s$ *and* $\varepsilon > 0$ *such that* $\Psi + \varepsilon P^T P + \varepsilon^{-1}Q^T Q \prec 0$ *holds.*

We immediately need the following well-known fact

Proposition E.14 *For any matrices* X *and* Y *with appropriate dimensions, we have* $X^TY + Y^TX \preceq \beta X^TX + \beta^{-1}Y^TY, \quad \text{for any } \beta > 0.$ *The latter inequality can be viewed as a consequence of the inequality* $(\beta^{-1/2}X - \beta Y)^T(\beta^{-1/2}X - \beta^{1/2}Y) \succeq 0$.

Whatever the inertia of the matrix inequality $\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP$*, there always a scalar* $\varepsilon > 0$ *such that*

$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP \quad \preceq \quad \Psi + \varepsilon P^TP + \varepsilon^{-1}Q^TQ \qquad \text{for some} \varepsilon > 0 \tag{E.113}$$

Hence by assumption, the left-hand side is negative definite. The sufficiency is shown.

Necessity:

Before showing the necessity we need the following results whose proofs can be found in Petersen [1987].

Lemma E.15 *Given any* $x \in \mathbb{R}^n$ *we have*

$$\max_{\Delta(t)\in\mathbf{\Delta}} \left\{(x^TM_1M_2\Delta(t)M_3x)^2\right\} = x^TM_1M_2M_2^TM_1xx^TM_3^TM_3x$$

where $M_1 = M_1^T$.

Lemma E.16 *Let* X, Y *and* Z *be given* $r \times r$ *matrices such that* $X \succeq 0$*,* $Y \prec 0$ *and* $Z \succeq 0$*. Furthermore, assume that*

$$(\xi^TY\xi)^2 - 4(\xi^TX\xi\xi^TZ\xi) > 0$$

for all $\xi \in \mathbb{R}^r$ *with* $\xi \neq 0$*. Then there exists a constant* $\lambda > 0$ *such that*

$$\lambda^2X + \lambda Y + Z \prec 0$$

The proof of sufficiency follows the same lines as the proof of theorem 2.3 of Petersen [1987] and is recalled here.

Assume that there exists $D_M = D_M^n$ such that

$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP \prec 0$$

holds and that the LMI is satisfied for the nominal system (i.e. $Delta(t) = 0$) and therefore $\Psi\mathbb{S}^n_{--}$. Thus we have

$$\begin{array}{lcl} \Psi & \prec & -P^T\Delta(t)Q - Q^T\Delta(t)^TP \\ x^T\Psi x & < & -2x^TP^T\Delta(t)Qx, \text{ for all} x \in \mathbb{R}^n \\ x^T\Psi x & < & -2\max_{\Delta(t)\in\mathbf{\Delta}}\{x^TP^T\Delta(t)Qx\} \\ (x^T\Psi x)^2 & > & 4\max_{\Delta(t)\in\mathbf{\Delta}}\{(x^TP^T\Delta(t)Qx)^2\} \end{array} \tag{E.114}$$

By application of lemma E.15 with $M_1 = I$, $M_2 = P^T$ and $M_3 = Q$, we get

$$\begin{array}{ll} (x^T\Psi x)^2 & > 4x^TP^TPxx^TQ^TQx \\ (x^T\Psi x)^2 - 4x^TP^TPxx^TQ^TQx > 0 & \end{array} \tag{E.115}$$

Note that $P^TP \succeq 0$, $Q^TQ \succeq$ and $\Psi \prec 0$ hence lemma E.16 applies with $Y = \Psi$, $X = P^TP$ and $Z = Q^TQ$. Therefore there exists $\lambda > 0$ such that

$$\lambda^2P^TP + \lambda\Psi + Q^TQ \prec 0 \tag{E.116}$$

Finally, multiplying the latter inequality by λ^{-1} and letting $\varepsilon = \lambda^{-1}$ we get inequality

$$\Psi + \varepsilon P^TP + \varepsilon^{-1}Q^TQ \prec 0 \tag{E.117}$$

This concludes the proof of sufficiency. □

There also exist a 'dual' version of the previous lemma where the uncertainty satisfies $\Delta(t)\Delta(t)^T < R$ in the case $m \leq p$. In this case we obtain

Lemma E.17 *Let $\Psi \in \mathbb{S}^n$ a symmetric matrix and $P \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{p\times n}$ and $\Delta(t) \in \mathbf{\Delta'}$ be an uncertain matrix (possibly time-varying) with*

$$\mathbf{\Delta'} := \{\Delta(t) \in \mathbb{R}^{m\times p} : m \leq p,\ \Delta\Delta^T \leq R,\ R > 0\}$$

then the following statements are equivalent:

1. *The LMI*
$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^TP \prec 0$$
holds for all $\Delta(t) \in \mathbf{\Delta'}$

2. *There exists a scalar $\varepsilon > 0$ such that the LMI*
$$\Psi + \varepsilon P^TP + \varepsilon^{-1}Q^TRQ \prec 0$$
holds.

The bounding lemma can neither be used to deal with rational uncertainties nor dynamical operators (such as dynamical systems or infinite dimensional operators…). This is the main drawback of the bounding lemma but, on the other hand, it provides simple and easy to use results in many cases and this motivates its utilization in many works. The bounding-lemma provides the same result as the scaled-small gain for one single full uncertainty block. We aim to show that with this framework it is possible to retrieve and small-gain like and the full-block multiplier results.

Equivalence with scaled-small gain

In the scaled-small gain result, the uncertainty are assumed to satisfy the commutative relation

$$L\Delta = L\Delta,\ L = L^T \succ 0 \tag{E.118}$$

and therefore we have $\Delta = L^{-1}\Delta L$. Finally, we get the following result:

Lemma E.18 *Let $\Psi \in \mathbb{S}^n$ a symmetric matrix and $P \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{p\times n}$ and $\Delta(t) \in \mathbf{\Delta}$ be an uncertain matrix (possibly time-varying) with*

$$\mathbf{\Delta}_1 := \{\Delta(t) \in \mathbb{R}^{m\times p} : p \leq m,\ \Delta^T\Delta \leq I\}$$

then the following statements are equivalent:

1. *The LMI*
$$\Psi + P^T\Delta(t)Q + Q^T\Delta(t)^T P \prec 0$$
holds for all $\Delta(t) \in \mathbf{\Delta}$

2. *The LMI*
$$\Psi + P^T L^{-1}\Delta(t)LQ + Q^T L^T\Delta(t)^T L^{-T}P \prec 0$$
holds for all $\Delta(t) \in \mathbf{\Delta}_1$ and some $L \in \mathcal{S}(\Delta)$.

3. *There exists a scalar $\tilde{L} \in \mathcal{S}(\Delta)$ such that the LMI*
$$\begin{bmatrix} \Psi + P^T\tilde{L}P & Q^T\tilde{L} \\ \star & -\tilde{L} \end{bmatrix} \prec 0 \tag{E.119}$$
holds.

Proof: *The equivalence between the first and second statement is done by replacing Δ by $L^{-1}\Delta L$. The third statement is obtained by similar argument than for obtaining statement two of lemma E.17. Then a change of variable $\tilde{L} \leftarrow \varepsilon L$ and a Schur's complement leads to LMI (E.119).* □

To see clearly the equivalence with the scaled-small gain, let us consider system

$$\dot{x} = (A + B\Delta C)x \tag{E.120}$$

which can be rewritten as an interconnection depicted in Figure 6.2 where $H(s) = C(sI - A)^{-1}B$.

The robust stability of the system is ensured if there exists $P = P^T \succ 0$ such that the LMI

$$(A + B\Delta C)^T P + P(A + B\Delta C) \prec 0$$

holds. This LMI can be rewritten in the form

$$\Psi + \mathcal{P}^T \Delta(t) \mathcal{Q} + \mathcal{Q}^T \Delta(t)^T \mathcal{P} \prec 0$$

where $\Psi = A^T P + PA$, $\mathcal{P}^T = PB$ and $\mathcal{Q} = C$. Apply lemma E.18, we obtain

$$\begin{bmatrix} \Psi + P^T \tilde{L} P & Q^T \tilde{L} \\ \star & -\tilde{L} \end{bmatrix} \prec 0 \tag{E.121}$$

which is identical to

$$\begin{bmatrix} A^T P + PA + PB\tilde{L}B^T P & C^T \\ \star & -\tilde{L} \end{bmatrix} \prec 0 \tag{E.122}$$

A Schur complement on the latter inequality and letting $\tilde{L}' = \tilde{L}$ (see properties of the set $\mathcal{S}(\Delta)$) leads

$$\begin{bmatrix} A^T P + PA & PB & C^T \tilde{L}' \\ \star & -\tilde{L}' & 0 \\ \star & \star & -\tilde{L}' \end{bmatrix} \prec 0 \tag{E.123}$$

which is exactly the scaled bounded real lemma.

Equivalence with full-block $\mathcal{S}$-procedure

Let us consider the set of uncertainties $\boldsymbol{\Delta}_q$ defined by

$$\boldsymbol{\Delta}_q := \left\{ \Delta \in \mathbb{R}^{m \times p} : p \leq m, \begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} U & V \\ \star & W \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \preceq 0 \right\} \tag{E.124}$$

Now consider equation (E.112) and rewrite it into

$$\Psi + \begin{bmatrix} P^T & Q^T \end{bmatrix} \begin{bmatrix} 0 & \Delta(t) \\ \Delta(t)^T & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \prec 0 \tag{E.125}$$

We need to transform the quadratic inequality defining the set $\boldsymbol{\Delta}_q$. Note that in virtue of the dualization lemma (Scherer [1999] or Appendix E.13) we have

$$\begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} xU & V \\ \star & W \end{bmatrix}^{-1} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} \prec 0$$

Let $\begin{bmatrix} U & V \\ \star & W \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{U} & \tilde{V} \\ \star & \tilde{W} \end{bmatrix}$ and expand the latter inequality yields

$$\begin{aligned} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix}^T \begin{bmatrix} \tilde{U} & \tilde{V} \\ \star & \tilde{W} \end{bmatrix} \begin{bmatrix} -I \\ \Delta^T \end{bmatrix} &= \tilde{U} - \tilde{V}\Delta^T - \Delta\tilde{V}^T + \Delta\tilde{W}\Delta^T \prec 0 \\ &= (\Delta - \tilde{V}\tilde{W}^{-1})\tilde{W}(\Delta^T - \tilde{W}^{-1}\tilde{V}^T) + \tilde{U} - \tilde{V}\tilde{W}\tilde{V}^T \prec 0 \end{aligned}$$

Since $\tilde{W} \succ 0$ and $\tilde{U} \prec 0$ then $\tilde{U} - \tilde{V}\tilde{W}\tilde{V}^T \prec 0$. Let $U' = \tilde{U} - \tilde{V}\tilde{W}\tilde{V}^T$, $V' = \tilde{V}\tilde{W}^{-1}$ and $W' = \tilde{W}^{-1}$ hence the latter inequality is equivalent to

$$(\Delta - V')(W')^{-1}(\Delta^T - V'^T) + U' \prec 0$$

A Schur complement yields

$$\begin{bmatrix} U' & \Delta - V' \\ \star & -W' \end{bmatrix} \prec 0$$

and finally we have

$$\begin{bmatrix} 0 & \Delta \\ \star & 0 \end{bmatrix} \prec \begin{bmatrix} -U' & V' \\ \star & W' \end{bmatrix}$$

Now inject the bound on matrix $\begin{bmatrix} 0 & \Delta \\ \star & 0 \end{bmatrix}$ into inequality (E.125) leads to

$$\Psi + \begin{bmatrix} P^T & Q^T \end{bmatrix} \begin{bmatrix} -U' & V' \\ \star & W' \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \prec 0 \tag{E.126}$$

Despite of the apparent difference with results obtained from the full-block $\mathcal{S}$-procedure , they are actually identical. We have used a linearization procedure which has turned the quadratic definition of the uncertainty set into a linear definition. This linear definition has been used to bound the uncertainty into the LMI. A similar result has been provided in Scherer [1996].

E.15 Schur's complement

The Schur complement Boyd et al. [1994] allows to exhibit convex linear matrix inequalities from nonlinear matrix inequality apparently nonconvex.

Lemma E.19 *The following statements are equivalent:*

1. $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \prec 0$
2. $M_{11} \prec 0$ *and* $M_{22} - M_{12}^T M_{11}^{-1} M_{12} \prec 0$
3. $M_{22} \prec 0$ *and* $M_{11} - M_{12} M_{22}^{-1} M_{12}^T \prec 0$

It is difficult to see that statements 2 and 3 provides convex inequalities. But in virtue of this lemma, they can be cast as a convex linear constraints in M_{ij} which is useful in the LMI framework.

It is important to say that when a matrix is positive definite then all its Schur complement must be positive definite. The following example shows a trap of the Schur complement.

Example E.20 *Let us for instance consider the following LMI:*

$$\begin{bmatrix} -E^T P E - Q & A^T P & Q & \\ \star & -P & 0 & 0 \\ \star & \star & -Q & \end{bmatrix} \prec 0 \tag{E.127}$$

where $P = P^T \succ 0$, $Q = Q^T \succ 0$ and E, A are square. Is this LMI satisfied ? First of all, the diagonal terms must be negative definite: this is the case, due to the assumptions on the matrices. However, by performing the Schur complement with respect to the right-lower block we obtain the two underlying inequalities:

$$\begin{array}{rcl} -Q & \prec & 0 \\ \begin{bmatrix} -E^TPE & A^TP \\ \star & -P \end{bmatrix} & \prec & 0 \end{array} \tag{E.128}$$

While the first equality is satisfied, the second may be not satisfied if E is not of full rank since in this case the term E^TPE would have zero eigenvalues.

This example shows that Schur complements should be used with care.

There also exists a non-strict version of the Schur complement Boyd et al. [1994].

Lemma E.21 *The following statements are equivalent:*

1. $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \succeq 0$
2. *The following relations hold*

$$R \succeq 0, \qquad M_{11} - M_{12}M_{22}^+M_{12}^T \succeq 0, \qquad S(I - M_{22}M_{22}^+) = 0$$

where M_{22}^+ is the Moore-Penrose pseudoinverse of M_{22}.

E.16 Finsler's Lemma

The Finsler's lemma Skelton et al. [1997] is a very useful tool in robust control and is defined below:

Lemma E.22 *The following statements are equivalent:*

1. $x^TMx < 0$ *for all* $x \in \mathcal{X} := \{x \in \mathbb{R}^n : Bx = 0\}$
2. $B_\perp^T M B_\perp \prec 0$
3. *There exists a scalar $\tau \in \mathbb{R}$ such that $M - \tau B^TB \prec 0$ and if such τ exists, it must satisfy*

$$\tau > \tau_{min} := \lambda_{max}([D^T(M - MB_\perp(B_\perp^TMB_\perp)^{-1}B_\perp M)D]$$

where $D := (B_rB_l^T)^{-1/2}B_l^+$ with (B_r, B_l) is any full rank factor of B (i.e. $B = B_lB_r$).

4. *There exists an* ***unconstrained*** *matrix N such that*

$$M + N^TB + B^TN \prec 0$$

5. *There exists a matrix $W \in \mathbb{S}_+^{n+m}$ and a scalar $\tau > 0$ such that*

$$\begin{bmatrix} M & B^T \\ B & -\tau I_m \end{bmatrix} \prec W \qquad \text{rank}(W) = m$$

The last statement has been recently added in Kim and Moon [2006] to deal with reduced-order output feedback and constrained controllers Kim and Moon [2006], Kim et al. [2007].

There also exists a matrix version of statement 3 Skelton et al. [1997] where the scalar τ is replaced by a matrix, say $X \in \mathbb{S}$, such that $M + B^T X B \prec 0$.

If the matrix N is constrained then the equivalence is lost and statement 4 implies the others only.

E.17 Generalization of Finsler's lemma

A generalization of the Finsler's lemma has been provided in [Iwasaki, 1998, Scherer, 1997] and is recalled here. Indeed, the Finsler's lemma is generally applicable when the matrix B is known and hence the basis of the null-space can be easily computed. The generalization allows for the use of unknown matrices.

Lemma E.23 *Let matrices $M = M^T$, B and a compact subset of real matrices $\mathcal{K}$ be given. The following statements are equivalent:*

1. *for each $K \in \mathcal{K}$*
$$x^T M x < 0, \qquad \forall\ x \neq 0 \text{ s.t. } KFx = 0$$
2. *there exists $Z = Z^T$ such that*
$$\begin{array}{lcl} M + F^T Z F & \prec & 0 \\ \mathrm{Ker}[K]^T Z \mathrm{Ker}[K] & \succeq & 0 \qquad \forall\ K \in \mathcal{K} \end{array} \tag{E.129}$$

Proof: *Suppose 1) holds. Choose $K \in \mathcal{K}$ arbitrarily then in virtue of the Finsler's lemma (Appendix E.16) there exists a real scalar τ such that*

$$M + \tau F^T K^T K F \prec 0$$

Since $\mathcal{K}$ is compact then τ can be chosen independently of K. Hence we have

$$M \prec -F^T S F \qquad \forall\ S \in \{\tau K^T K :\ K \in \mathcal{K}\}$$

It has been shown in [Iwasaki, 1998] that the latter inequality is equivalent to the existence of a symmetric matrix Z such that

$$M + F^T Z F \prec 0 \qquad \text{and } -Z \preceq \tau K^T K, \quad \forall\ K \in \mathcal{K}$$

Then performing a congruence transformation on the second inequality with respect to Ker$[H]$ *yields*

$$\mathrm{Ker}[K]^T Z \mathrm{Ker}[K] \succeq 0 \qquad \forall\ K \in \mathcal{K}$$

Suppose now 2) holds. Set $x \neq 0$ and $K \in \mathcal{K}$ such that $KFx = 0$. Then it is possible to find η such that $Fx = \mathrm{Ker}[K]\eta$ and hence we have

$$\begin{array}{lcl} x^T(M + F^T Z F)x & < & 0 \\ x^T M x & < & -x^T F^T Z F x < -\eta^T \mathrm{Ker}[K]^T Z \mathrm{Ker}[K]\eta \leq 0 \end{array}$$

and we get 1). $\square$

E.18 Projection Lemma

The projection lemma is used to remove a decision matrix and gives a necessary and sufficient condition to the existence of such a matrix. Generally, the controller matrix is removed to obtain LMIs instead of a BMI (see for instance Apkarian and Gahinet [1995], Scherer [1999]).

Lemma E.24 *Let $\Psi \in \mathbb{S}^n$ and P, Q matrices of appropriate dimensions, then the following statements are equivalent:*

1. *There exists an* ***unconstrained*** *matrix Ω such that*
$$\Psi + P^T \Omega Q + Q^T \Omega^T P \prec 0$$
2. *The two following underlying LMIs hold*
$$\begin{array}{ll} P_\perp^T \Psi P & \prec 0 \\ Q_\perp^T \Psi Q & \prec 0 \end{array}$$
3. *There exists two scalars $\tau \in \mathbb{R}$ such that*
$$\begin{array}{lll} \Psi - \tau P^T P & \prec & 0 \\ \Psi - \tau Q^T Q & \prec & 0 \end{array}$$

The proof can be found in Gahinet and Apkarian. The assumption that Ω is unconstrained plays a central role in the proof and in the equivalence between the two statements. This means that when dealing with constrained controllers such as 0 blocks or bounded coefficients, equivalence is lost and statement 2 may admit a solution while statement 1 does not (but this is not always the case). For instance, in some papers, the authors remove uncertain or symmetric terms invoking the projection lemma, but this is wrong since, for the first case, the projection lemma provides an existence condition of the removed matrix and we do not care of finding a uncertainty for which the condition is satisfied... we want to ensure that the LMI is satisfied for all uncertain terms belonging in a known defined set; for the second case, the matrix is symmetric and hence constrained which does not fall into the projection lemma conditions of application.

E.19 Completion Lemma

This theorem shows that it is possible to construct a matrix and its inverse from only block of each only. It has consequence in the construction of Lyapunov matrices in the dynamic output feedback synthesis problem (see Packard et al. [1991]).

Theorem E.25 *Let $X \in \mathbb{S}^n_{++}$ and $Y \in \mathbb{S}^n_{++}$. There exist $X_2 \in \mathbb{R}^{n \times r}$, $X_3 \in \mathbb{R}^{r \times r}$, $Y_2 \in \mathbb{R}^{n \times r}$ and $Y_3 \in \mathbb{R}^{r \times r}$ such that*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} \succ 0 \qquad \text{and} \qquad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \succeq 0 \qquad \text{and} \qquad \text{rank} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r \tag{E.130}$$

E.20 Application of the Projection Lemma

This appendix shows an application of the Projection Lemma in the context of the synthesis of a parameter dependent dynamic output feedback. The synthesis is performed using the scaled-small gain theorem.

Let us consider the following LPV system in 'LFT' form

$$\begin{array}{rcl} \dot{x} & = & Ax(t) + B_0 w(t) + B_1 u(t) \\ z(t) & = & C_0 x(t) + D_{00} w(t) + D_{01} u(t) \\ y(t) & = & C_1 x(t) + D_{10} w(t) \end{array} \tag{E.131}$$

where x, u, w, z and y are respectively the system state, the control input, the parameters input, the parameters output and the measured output.

We seek a controller of the form:

$$\begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \\ u(t) \end{bmatrix} = \Omega \begin{bmatrix} x_c(t) \\ w_c(t) \\ y(t) \end{bmatrix} \tag{E.132}$$

where x_c, w_c and z_c are respectively the controller state, the parameter input and the parameter output.

The parameters input and output are defined by

$$\begin{bmatrix} w(t) \\ w_c(t) \end{bmatrix} = \mathrm{diag}(\Theta(\rho), \Theta(\rho)) \begin{bmatrix} z(t) \\ z_c(t) \end{bmatrix}$$

From this description, the system is scheduled by the parameters through the signals w and z while the controller is scheduled through the signals w_c and z_c. We introduce the scaling L is defined such that

$$L\,\mathrm{diag}(\Theta(\rho), \Theta(\rho)) = \mathrm{diag}(\Theta(\rho), \Theta(\rho))L$$

It is possible to rewrite the system as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \hline z \\ z_c \\ \hline x_c \\ w_c \\ y \end{bmatrix} = \left[\begin{array}{cc|cc|ccc} A & 0 & B_0 & 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline C_0 & 0 & D_{00} & 0 & 0 & 0 & D_{01} \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ C_1 & 0 & D_{10} & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ x_c \\ \hline w \\ w_c \\ \hline \dot{x}_c \\ z_c \\ u \end{bmatrix} \tag{E.133}$$

The closed-loop system is given by

$$\begin{bmatrix} \dot{\bar{x}}(t) \\ \bar{z}(t) \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B}_1 \Omega \bar{C}_1 & \bar{B}_0 + \bar{B}_1 \Omega \bar{D}_{10} \\ \bar{C}_0 + \bar{D}_{01} \Omega \bar{C}_1 & \bar{D}_{00} + \bar{D}_{01} \Omega \bar{D}_{10} \end{bmatrix} \tag{E.134}$$

where

$$\begin{array}{rclrclrcl} \bar{A} & = & \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} & \bar{B}_0 & = & \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} & \bar{B}_1 & = & \begin{bmatrix} 0 & 0 & B_1 \\ I & 0 & 0 \end{bmatrix} \\ \bar{C}_0 & = & \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} & \bar{D}_{00} & = & \begin{bmatrix} D_{00} & 0 \\ 0 & 0 \end{bmatrix} & \bar{D}_{01} & = & \begin{bmatrix} 0 & 0 & D_{01} \\ 0 & I & 0 \end{bmatrix} \\ \bar{C}_1 & = & \begin{bmatrix} 0 & I \\ 0 & 0 \\ C_1 & 0 \end{bmatrix} & \bar{D}_{10} & = & \begin{bmatrix} 0 & 0 \\ 0 & I \\ D_{10} & 0 \end{bmatrix} & & & \end{array} \tag{E.135}$$

The stability of the closed-loop system is ensured, in virtue of the scaled-small gain theorem if the following nonlinear matrix inequality is satisfied

$$\begin{bmatrix} (\bar{A}+\bar{B}_1\Omega\bar{C}_1)^TP+P(\bar{A}+\bar{B}_1\Omega\bar{C}_1) & P(\bar{B}_0+\bar{B}_1\Omega\bar{D}_{10}) & (\bar{C}_0+\bar{D}_{01}\Omega\bar{C}_1)^T \\ \star & -L & (\bar{D}_{00}+\bar{D}_{01}\Omega\bar{D}_{10})^T \\ \star & \star & -L^{-1} \end{bmatrix} \prec 0 \tag{E.136}$$

which can be rewritten into

$$\begin{bmatrix} \bar{A}^TP+P\bar{A} & P\bar{B}_0 & \bar{C}_0^T \\ \star & -L & \bar{D}_{00}^T \\ \star & \star & -L^{-1} \end{bmatrix} + \begin{bmatrix} P\bar{B}_1 \\ 0 \\ \bar{D}_{01} \end{bmatrix} \Omega \begin{bmatrix} \bar{C}_1 & \bar{D}_{00} & 0 \end{bmatrix} + (\star)^T \prec 0 \tag{E.137}$$

Let

$$\begin{aligned} P &= \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{22} \end{bmatrix} & X = P^{-1} &= \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \\ L &= \begin{bmatrix} L_{11} & L_{12} \\ \star & L_{22} \end{bmatrix} & J = L^{-1} &= \begin{bmatrix} J_{11} & J_{12} \\ \star & J_{22} \end{bmatrix} \end{aligned}$$

A basis of the null space of $\begin{bmatrix} \bar{C}_1 & \bar{D}_{10} & 0 \end{bmatrix}$ is given by

$$\mathrm{Ker}\left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ C_1 & 0 & D_{10} & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]$$

with $C_1N_1 + D_{10}N_2 = 0$ and a basis of the null space of $\begin{bmatrix} P\bar{B}_1 \\ 0 \\ \bar{D}_{01} \end{bmatrix}^T$ is given by

$$\mathrm{Ker}\left[\begin{array}{ccc} P\begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & P\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & D_{01} \\ 0 & I & 0 \end{array}\right]^T = \mathrm{diag}(X, I, I)\left[\begin{array}{ccc} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

where $B_1^TM_1 + D_{01}^TM_2 = 0$. Hence, in virtue of the projection lemma we get the two underlying matrix inequalities:

$$
\begin{array}{c}
\left[\begin{array}{ccc} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T
\left[\begin{array}{ccc} \bar{A}^T P + P\bar{A} & P\bar{B}_0 & \bar{C}_0^T \\ \star & -L & \bar{D}_{00}^T \\ \star & \star & -L^{-1} \end{array}\right]
\left[\begin{array}{ccc} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0 \\
\left[\begin{array}{ccc} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]^T
\left[\begin{array}{ccc} X\bar{A}^T + \bar{A}X & \bar{B}_0 & X\bar{C}_0^T \\ \star & -L & \bar{D}_{00}^T \\ \star & \star & -L^{-1} \end{array}\right]
\left[\begin{array}{ccc} M_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \prec 0
\end{array}
\tag{E.138}
$$

Removing lines and columns corresponding to zero lines and columns to null-spaces leads to

$$
\begin{array}{c}
\left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T
\left[\begin{array}{c|c|cc} A^T P_{11} + P_{11}A & P\bar{B}_0 & \bar{C}_0^T & 0 \\ \hline \star & -L_{11} & D_{00}^T & 0 \\ \hline \star & \star & -J_{11} & -J_{12} \\ \star & \star & \star & -J_{22} \end{array}\right]
\left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0 \\
\left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \end{array}\right]^T
\left[\begin{array}{c|cc|c} X_{11}A^T + X_{11}A & \bar{B}_0 & 0 & X_{11}\bar{C}_0^T \\ \hline \star & -L_{11} & -L_{12} & \bar{D}_{00}^T \\ \star & \star & -L_{22} & 0 \\ \hline \star & \star & \star & -J_{11} \end{array}\right]
\left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ \hline M_2 & 0 & 0 \end{array}\right] \prec 0
\end{array}
\tag{E.139}
$$

Reorganize columns and rows yields

$$
\begin{array}{c}
\left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T
\left[\begin{array}{c|c|cc} A^T P_{11} + P_{11}A & P\bar{B}_0 & \bar{C}_0^T & 0 \\ \hline \star & -L_{11} & D_{00}^T & 0 \\ \hline \star & \star & -J_{11} & -J_{12} \\ \star & \star & \star & -J_{22} \end{array}\right]
\left[\begin{array}{ccc} N_1 & 0 & 0 \\ \hline N_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0 \\
\left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline M_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right]^T
\left[\begin{array}{c|c|cc} X_{11}A^T + X_{11}A & X_{11}\bar{C}_0^T & \bar{B}_0 & 0 \\ \hline \star & -J_{11} & D_{00} & 0 \\ \hline \star & \star & -L_{11} & -L_{12} \\ \star & \star & \star & -L_{22} \end{array}\right]
\left[\begin{array}{ccc} M_1 & 0 & 0 \\ \hline M_2 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \prec 0
\end{array}
\tag{E.140}
$$

Finally applying Schur's complement (see Appendix E.15), we get

$$
\begin{array}{c}
\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^T \left(\begin{bmatrix} A^T P_{11} + P_{11}A & P\bar{B}_0 \\ \star & -L_{11} \end{bmatrix} + \begin{bmatrix} \bar{C}_0^T & 0 \\ D_{00}^T & 0 \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ \star & L_{22} \end{bmatrix} (\star)^T \right) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \prec 0 \\
\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^T \left(\begin{bmatrix} X_{11}A^T + X_{11}A & X_{11}\bar{C}_0^T \\ \star & -J_{11} \end{bmatrix} + \begin{bmatrix} \bar{B}_0 & 0 \\ D_{00} & 0 \end{bmatrix} \begin{bmatrix} J_{11} & J_{12} \\ \star & J_{22} \end{bmatrix} (\star)^T \right) \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \prec 0
\end{array}
\tag{E.141}
$$

and equivalently

$$\begin{array}{l}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^T \left(\begin{bmatrix} A^T P_{11} + P_{11} A & P\bar{B}_0 \\ \star & -L_{11} \end{bmatrix} + \begin{bmatrix} \bar{C}_0^T \\ D_{00}^T \end{bmatrix} L_{11} \begin{bmatrix} \bar{C}_0^T \\ D_{00}^T \end{bmatrix}^T \right) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \prec 0 \\ \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^T \left(\begin{bmatrix} X_{11} A^T + X_{11} A & X_{11}\bar{C}_0^T \\ \star & -J_{11} \end{bmatrix} + \begin{bmatrix} \bar{B}_0 \\ D_{00} \end{bmatrix} J_{11} \begin{bmatrix} \bar{B}_0 \\ D_{00} \end{bmatrix}^T \right) \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \prec 0 \end{array} \tag{E.142}$$

The above matrix inequalities are LMIs. Indeed, by considering only one block of each matrix and their inverse, the condition is LMI. Moreover, the whole matrices P, X, L, J can be constructed uniquely from these blocks using singular value decomposition (see Appendix A.6) and completion lemma (see Appendix E.19) as shown below.

First of all, it is possible to construct P_12 and X_{12} from P_{11} and X_{11} using the singular value decomposition. Indeed, we have

$$P_{11}X_{11} + P_{12}X_{12}^T = I$$

and perform a singular value decomposition on $I - P_{11}X_{11} = U^T \Sigma V$, by identification we get

$$P_{12} = U^T \Sigma^{1/2} \text{ and } X_{12} = V^T \Sigma^{1/2}$$

Finally, P is the solution of the algebraic equation

$$P \begin{bmatrix} X_{11} & I \\ X_{12}^T & 0 \end{bmatrix} = \begin{bmatrix} I & P_{11} \\ 0 & P_{12}^T \end{bmatrix}$$

In an identical way, the other matrices can be computed.

E.21 Matrix Elimination Results

There exist a lot of result allowing to reduce the number of variables into a LMI, the projection lemma (see Appendix E.18 is one of them. Some additional results are provided here.

Lemma E.26 *There exists a matrix X such that*

$$\begin{bmatrix} P & Q & X \\ Q^T & R & V \\ X^T & V^T & S \end{bmatrix} \succ 0 \tag{E.143}$$

if and only if

$$\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \succ 0 \qquad \begin{bmatrix} R & V \\ V^T & S \end{bmatrix} \succ 0 \tag{E.144}$$

Proof: *The proof is a straightforward application of the projection lemma (Appendix E.18.*
□

Lemma E.27 *There exists a matrix X such that*

$$\begin{bmatrix} P & Q + XE & X \\ (Q + XE)^T & R & V \\ X & V^T & S \end{bmatrix} \succ 0 \tag{E.145}$$

if and only if

$$\begin{bmatrix} P & Q \\ Q^T & R - VE - E^T V^T + E^T SE \end{bmatrix} \succ 0 \qquad \begin{bmatrix} R & V \\ V^T & S \end{bmatrix} \succ 0 \tag{E.146}$$

Proof: *The proof is also an application of the projection lemma (Appendix E.18).* □

Lemma E.28 *There exists a symmetric matrix X such that*

$$\begin{bmatrix} P_1 - LXL^T & Q_1 \\ Q_1^T & R_1 \end{bmatrix} \succ 0 \qquad \begin{bmatrix} P_1 + X & Q_2 \\ Q_2^T & R_2 \end{bmatrix} \succ 0 \tag{E.147}$$

if and only if

$$\begin{bmatrix} P_1 + LP_2L^T & Q_1 & LQ_2 \\ Q_1^T & R_1 & 0 \\ Q_2^T L^T & 0 & R_2 \end{bmatrix} \succ 0 \tag{E.148}$$

Proof: *The proof is again an application of the projection lemma (Appendix E.18).* □

E.22 Parseval's Theorem

The Parseval's theorem allows to bridge the energy of a signal in the time-domain to an expression into the frequency domain. This equality is heavily used in IQC analysis [Rantzer and Megretski, 1997] where it is used to connect time-domain properties and frequency domain properties of signals.

Theorem E.29 *Let $x(t)$ be a $\mathcal{L}_2$ signal and define its Fourier transform as $X(\omega)$ where $\omega = 2\pi f$ and f is the frequency then the following equality holds*

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega \tag{E.149}$$

Proof:

$$\begin{aligned} \int_{-\infty}^{+\infty} |x(t)|^2 dt &= \int_{-\infty}^{+\infty} x(t)^* x(t) dt \\ &= \int_{-\infty}^{+\infty} \left[\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega')^* e^{-j\omega' t} d\omega'\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega\right)\right] dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[X(\omega')\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(X(\omega)\left(\int_{-\infty}^{+\infty} e^{j(\omega-\omega')} dt\right)\right) d\omega\right] d\omega' \end{aligned} \tag{E.150}$$

Note that $\int_{-\infty}^{+\infty} e^{j(\omega-\omega')} dt = 2\pi\delta(\omega-\omega')$ *by the definition of the Dirac pulse δ and the Fourier transform. This leads to*

$$\begin{aligned} \int_{-\infty}^{+\infty} |x(t)|^2 dt &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[X(\omega')\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(X(\omega)\cdot 2\pi\delta(\omega-\omega')\right) d\omega\right] d\omega' \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[X(\omega')\int_{-\infty}^{+\infty}\left(X(\omega)\delta(\omega-\omega')\right) d\omega\right] d\omega' \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)^* X(\omega) d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega \end{aligned} \tag{E.151}$$

□

We have the following corollary where a symmetric matrix is inserted in the energy expression:

Theorem E.30 *Let $x(t)$ be a $\mathcal{L}_2$ signal and define its Fourier transform as $X(\omega)$ where $\omega = 2\pi f$ and f is the frequency then the following equality holds*

$$\int_{-\infty}^{+\infty} x(t)^* M x(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)^* \hat{M} X(\omega) d\omega \tag{E.152}$$

Proof: *The proof follows the same lines as for the standard version of the Parseval's theorem.* □

It is possible to consider a more complete form for the Parseval's theorem which consider a frequency weighting through the use of the frequency dependent matrix $\hat{M}(j\omega)$:

Theorem E.31 *Let $x(t)$ be a $\mathcal{L}_2$ signal and define its Fourier transform as $X(\omega)$ where $\omega = 2\pi f$ and f is the frequency then the following equality holds*

$$\int_{-\infty}^{+\infty} \sigma(x_f(t), x(t))^* dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega)^* \hat{M}(j\omega) X(\omega) d\omega \tag{E.153}$$

where $\sigma(x(t), x_f(t))$ is a quadratic form and $\dot{x}_f(t) = A_f x_f(t) + B_f x(t)$.

F Technical Results in Time-Delay Systems

In this Appendix, we will give the reader further results used in the time-delay stability analysis framework.

We will consider, in the following, the time-delay system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h(x(t-h(t))) + Ew(t) \\ z(t) & = & Cx(t) + C_h x(t-h(t)) + Fw(t) \end{array} \tag{F.154}$$

F.1 Jensen's Inequality

This inequality comes from statistics and probability but is very useful in robust and time-delay system stability analysis (see Gu et al. [2003]).

Definition F.1 *Let ϕ be a convex function and $f(x)$ is integrable over $[a(t), b(t)]$, $a(t) < b(t)$ for some parameter $t \in U$. Then the following inequality holds*

$$\phi\left(\int_{a(t)}^{b(t)} f(x)dx\right) \leq |b(t) - a(t)| \int_{a(t)}^{b(t)} \phi(f(x))dx \tag{F.155}$$

The Jensen's inequality is often used in the $\mathcal{H}_\infty$ norm analytical computation of integral operators in time-delay systems framework. It is also used in approaches based on Lyapunov-Krasovskii functionals as a efficient bounding technique. A example of application is given below:

$$\left(\int_{t-h}^{t} \dot{x}(\theta)d\theta\right)^T P \left(\int_{t-h}^{t} \dot{x}(\theta)d\theta\right) \leq h \int_{t-h}^{t} \dot{x}(\theta)^T P \dot{x}(\theta)d\theta \tag{F.156}$$

with $P = P^T \succ 0$. The convex function is $\phi(z) = z^T Pz$ since $P = P^T \succ 0$, $f(t) = \dot{x}(t)$ and $b(t) - a(t) = h$.

F.2 Bounding of cross-terms

The use of model-transformations for stability analysis and control synthesis of time-delay systems may lead to annoying terms named 'cross terms', generally involving products between signals and integrals. A common cross-term is

$$-2x(t)^T A^T P A_h \int_{t-h}^{t} \dot{x}(s)ds \tag{F.157}$$

and appears, for instance, while using the Euler model-transformation with a quadratic Lyapunov-Razumikhin function of the form $V(x(t)) = x(t)^T Px(t)$.

Proposition F.2 *For any $Z = Z^T \succ 0$ we have*

$$\begin{aligned} \pm 2x(t)^T A^T P A_h \int_{t-h}^{t} \dot{x}(s)ds &\leq x(t)^T A^T Z A x(t) + \left(\int_{t-h}^{t} \dot{x}(s)ds\right)^T A_h^T P Z^{-1} P A_h \left(\int_{t-h}^{t} \dot{x}(s)ds\right) \\ &\leq x(t)^T A^T Z A x(t) + h \int_{t-h}^{t} \dot{x}(s)^T A_h^T P Z^{-1} P A_h \dot{x}(s)ds \\ &\leq hx(t)^T A^T Z A x(t) + \int_{t-h}^{t} \dot{x}(s)^T A_h^T P Z^{-1} P A_h \dot{x}(s)ds \end{aligned} \tag{F.158}$$

Proof: *The idea is to use completion of the squares, the first line is obtained by writing*

$$\begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h \int_{t-h}^{t} \dot{x}(s)ds \end{bmatrix}^T \begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h \int_{t-h}^{t} \dot{x}(s)ds \end{bmatrix} \geq 0$$

for some $Z = Z^T \succ 0$. Expand the latter expression leads to first inequality. Then apply Jensen's inequality onto the quadratic integral term leads to second inequality. Finally, the last inequality is obtained by completion of the squares too but in another fashion:

$$\int_{t-h}^{t} \begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h\dot{x}(s)ds \end{bmatrix}^T \begin{bmatrix} Z^{-1/2}Ax(t) \\ \pm Z^{-1/2}A_h\dot{x}(s) \end{bmatrix} \geq 0$$

Expanding the latter quadratic form leads to the last inequality. □ The latter bounding technique is relatively inaccurate since the cross terms may admits negative values even though the right-hand side term is always positive. This drove Park to introduce a new bound (in Park [1999], Park et al. [1998] and is generally referred to as Park's bound). The idea is to use a more complete completion by the squares and is given below in his own terminology:

Lemma F.3 *Assume that $a(\alpha) \in \mathbb{R}^{n_x}$ and $b(\alpha) \in \mathbb{R}^{n_y}$ are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$ and any matrix $M \in \mathbb{R}^{n_y \times n_y}$, the following holds*

$$-2\int_\Omega b(\alpha)^T a(\alpha) d\alpha \leq \int_\Omega \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XM \\ M^T X & (M^T X + I)X^{-1}(XM + I) \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} \tag{F.159}$$

This model transformation has led to an great improvement of results in this time (see comparison with contemporary results in Park [1999], Park et al. [1998]). The obtained result is presented in Section 2.2.1.9.

Inspired from the latter bound, another one has been employed in Moon et al. [2001] and is sometimes referred as Moon's inequality.

Lemma F.4 *Assume that $a(\cdot) \in \mathbb{R}^{n_a}$, $b(\cdot) \in \mathbb{R}^{n_b}$ and $\mathcal{N}(\cdot) \in \mathbb{R}^{n_a \times n_a}$ are defined on the interval Ω. Then, for any matrices $X \in \mathbb{R}^{n_a \times n_a}$, $Y \in \mathbb{R}^{n_a \times n_b}$ and $Z \in \mathbb{R}^{n_b \times n_b}$, the following holds*

$$-2\int_\Omega a(\alpha)^T \mathcal{N} b(\alpha) d\alpha \leq \int_\Omega \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y - \mathcal{N} \\ \star & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} \tag{F.160}$$

where

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0$$

Proof: *See Moon et al. [2001].* $\square$

Although this result is less accurate than the Park's bound, its more simple form allows for easy design techniques than by using Park's inequality.

F.3 Power Series

We will introduce here the notion of power series (or entire series) which will be necessary to define the Padé approximation of a continuous function.

A power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - c)^n$$

where a_n represents the coefficient of the n^{th}, c is a constant and x varies around c.

Since, the series may converge for some value of x and diverge for others, it is intrusting to determine its domain of convergence. From the expression above, it is clear that it converges for $x = c$ and hence we are then interested in finding a radius defining a ball centered at c in which the series converges. The radius $r \in [0, +\infty]$ is then sdetermined by the relation

$$r^{-1} := \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

if this limit exists. And in this case the series converges absolutely for $|x - c| < r$ and converges uniformly on every compact subset of $\{x : |x - c| < r\}$.

One of the most interesting property of power series is to approximate functions, at least in a compact domain. This is actually a generalization of Taylor series. In order to determine a power series approximating a function, say $g(x)$, it is interesting to define this function as a solution of a differential equation. As an example, let us consider $g(x) = e^x$ and thus we have the following differential equation

$$g'(x) - g(x) = 0 \tag{F.161}$$

Now defining $h(x) = \sum_{n=0}^{+\infty} a_n x^n$ and substituting into the differential equation leads

$$\begin{aligned} h'(x) - h(x) &= 0 \\ &= \sum_{i=0}^{+\infty} n a_n x^{n-1} - \sum_{i=0}^{+\infty} a_n x^n \\ &= \sum_{i=0}^{+\infty} \left[(n+1)a_{n+1} - a_n\right] x^n \end{aligned}$$

Since the value of the series is identically zero then we must have

$$(n+1)a_{n+1} - a_n = 0$$

for all $n \in \mathbb{N}$. Moreover, the power series converges for $x = 0$ and its value is a_0. Since $h(0) = g(0) = 1$ then we have $a_0 = 1$. Finally we find $a_{n+1} = \frac{1}{n+1} a_n$ and therefore

$$a_n = \frac{1}{n!}$$

The radius of convergence is then given by

$$\begin{aligned} r^{-1} &= \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \to +\infty} \left| \frac{1}{n+1} \right| \\ &= 0 \end{aligned}$$

This means that $r = +\infty$ and hence the power series converges for all $x \in \mathbb{R}$. This means that

$$e^x = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n$$

for all $x \in \mathbb{R}$. The equality also holds when x is a complex number.

For series in which negative and fractional powers (e.g. x^{-n} and $x^{1/2}$) are allowed see respectively Laurent series and Puiseux expansion.

F.4 Padé Approximants

This appendix introduced the Padé approximation of a continuous function. This approximation is of great interest in the framework of time-delay systems [Zhang et al., 1999].

System (F.154) with constant time-delay h can be rewritten as an interconnection of two subsystems:

$$\begin{array}{rcl} \dot{x} & = & Ax + A_h w_0 + Ew \\ z_0 & = & x \\ z & = & Cx + C_h w_0 + Fw \\ w_0 & = & e^{-sh} z_0 \end{array} \tag{F.162}$$

In order to analyze stability of the interconnection it may be interesting to approximate the operator e^{-sh} by a proper (stable) transfer function. A power series cannot be used since the transfer function would be not proper. The Padé approximants play here an important role by approximating a function by a rational function with arbitrary degree for the denominator and numerator.

Let us consider a function $f(x)$ which is sought to be approximated by a rational function $R_{m,n}(x)$ defined as

$$R_{m,n}(x) := \frac{P_m(x)}{Q_n(x)} = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{i=0}^{n} b_i x^i} \tag{F.163}$$

where polynomials $P_m(x)$ and $Q_n(x)$ are of degree m and n respectively. These polynomials can be found using a relation linking the truncated power series of $f(x)$ and polynomials $P_m(x)$ and $Q_n(x)$. The truncated power series $Z_m(x)$ of $f(x)$ of degree m is given by

$$Z_m(x) := \sum_{i=0}^{m} c_i x^i \tag{F.164}$$

In this case we look for a_i and b_i such that

$$\sum_{i=0}^{m} c_i x^i = \frac{P_m(x)}{Q_n(x)} \tag{F.165}$$

or equivalently

$$Q_n(x) \sum_{i=0}^{m} c_i x^i = P_m(x) \tag{F.166}$$

This results into an homogenous system of $n+m+1$ equations with $n+m+2$ unknowns and so admits infinitely many solutions. However, it can be shown that the generated rational functions $R_{m,n}(x)$ are all the same (the obtained polynomials are not prime at a constant factor). Table 6.2 summarizes few of Padé approximants for the exponential function e^z with complex argument z:

m \ n	1	2	3
0	$\dfrac{1}{1-z}$	$\dfrac{1}{1-z+\frac{1}{2}z^2}$	$\dfrac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$
1	$\dfrac{1+\frac{1}{2}z}{1-\frac{1}{2}z^2}$	$\dfrac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$	$\dfrac{1+\frac{1}{4}z}{1-\frac{3}{4}z+\frac{1}{4}z^2-\frac{1}{24}z^3}$
2	$\dfrac{1+\frac{2}{3}z+\frac{1}{6}z^3}{1-\frac{1}{3}z}$	$\dfrac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$	$\dfrac{1+\frac{2}{5}z+\frac{1}{20}z^2}{1-\frac{3}{5}z+\frac{3}{20}z^2-\frac{1}{60}z^3}$
3	$\dfrac{1+\frac{3}{4}z+\frac{1}{4}z^2+\frac{1}{24}z^3}{1-\frac{1}{4}z}$	$\dfrac{1+\frac{3}{5}z+\frac{3}{20}z^2+\frac{1}{60}z^3}{1-\frac{2}{5}z+\frac{1}{20}z^2}$	$\dfrac{1+\frac{1}{2}z+\frac{1}{10}z^2+\frac{1}{120}z^3}{1-\frac{1}{2}z+\frac{1}{10}z^2-\frac{1}{120}z^3}$

Table 6.2: First Padé's approximants of the function e^s

The column $n = 0$ has been omitted since it coincides with the truncation of power series.

A particularity of Padé approximants of the exponential is the regularity of the numerator and the denominator when $m = n$. Indeed, denote $N_m(z)$ the numerator of $R_{m,m}(z)$ and then we have

$$R_m(z) := R_{m,m}(z) = \frac{N_m(z)}{N_m(-z)} \tag{F.167}$$

It is proved in [Zhang et al., 1999] that the proper transfer function is asymptotically stable, that is the polynomial $N_m(-z)$ has all its roots in the complex left-half plane.

F.5 Maximum Modulus Principle

The maximum modulus principle is a interesting result very useful in complex analysis which is necessary to study the bounded of some function norms.

Theorem F.5 *Let f be a holomorphic function on come connected open subset $D \subset \mathbb{C}$ and taking complex values. If z_0 is a point such that*

$$f(z_0) \geq f(z) \tag{F.168}$$

for all z in any neighborhood of z_0, then the function f is constant on D.

This can viewed otherwise, if f is a holomorphic function f over a connected open subset D, then its modulus cannot $|f|$ exhibit a true local maximum on D. Hence the maximum modulus is attained on the boundary of ∂D. This has strong consequences in system theory, as illustrated in the following example:

Example F.6 *This example shows how the maximum modulus principle can be used in order to prove the stability of a system. Let us consider for simplicity a SISO system* $H(s) = N(s)/D(s)$ *where* $N(s)$ *and* $D(s)$ *are arbitrary. The system is proper if the degree of* $N(s)$ *is lower than the degree of* $D(s)$ *and it is asymptotically stable if all the zeros of* $D(s)$ *have negative real part (*$H(s)$ *has all its poles with negative real part). Hence this means that the norm of* $H(s)$ *denoted by* $||H(s)||$ *is bounded for all* $\mathbb{C}^+$*. By the maximum modulus principle the maximum cannot be reached in the interior of* $\mathcal{C}^+$ *hence it suffices to consider the boundary* $\partial\mathbb{C}^+$ *only to check the boundedness of* $||H(s)||$ *over* $\mathbb{C}^+$ *only. Noting that* $\partial\mathbb{C}^+ = \mathbb{C}^0 \cup +\infty$ *(the boundary of* $\mathcal{C}^+$ *is constituted of the imaginary axis* $\mathbb{C}^0$ *and a point at infinity) then this means that if* $||H(s)||$ *is bounded over* $\partial\mathbb{C}^+$ *we have*

- $||H(j\omega)|| < +\infty$ *for all* $\omega \in \mathbb{R}$ *and hence* $H(s)$ *has no poles on the imaginary axis.*
- $||H(+\infty|| < +\infty$ *then the transfer function* $H(s)$ *is proper.*

This implies that the stability of a system can be checked only by verifying the boundedness of the transfer function over the boundary of $\mathbb{C}^+$*. This can be easily generalized to MIMO systems by considering the maximum singular value as the norm. This is the definition of the* $\mathcal{H}_\infty$*-norm and this justifies that the following equality for a strictly proper MIMO transfer function:*

$$\sup_{s\in\mathbb{C}^+} \sigma\bar{H}(s)| = \sup_{\omega\in\mathbb{R}} \sigma\bar{H(}j\omega) \tag{F.169}$$

For more information about the maximum modulus principle please refer to [Levinson and Redheffer, 1970].

F.6 Argument principle

The winding number of a closed-curve C in the plane around a given point z_0 is number representing the total number of times that curve travels counterclockwise around the point z_0. The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise. This notion has given rise to the celebrated Nyquist criterion and is also a first towards the elaboration of the Rouché's theorem presented in Appendix F.7.

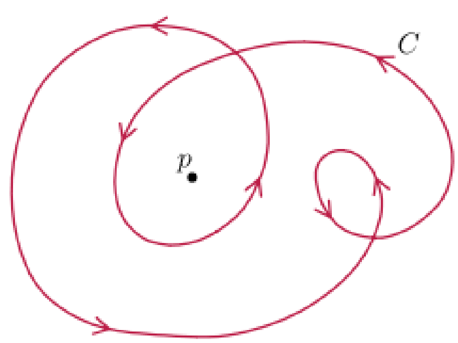

Figure 6.3: This curve has winding number two around the point p

This has important results in many applications and notably in the Nyquist criterion:

Theorem F.7 *A closed-loop continuous time system is asymptotically stable if and only if the open-loop transfer function $H_{ol}(s)$ travels N times around the critical point -1 counter-clockwise when s sweeps the imaginary axis and where N is the number of unstable poles of $H_{ol}(s)$.*

More generally we have the following theorem which is also called the argument principle:

Theorem F.8 *Let $f(z)$ be a function and C be a closed contour on $\mathbb{C}$ such that no poles and zeros are on C and C may contain any poles and zeros ($f(z)$ is meromorphic inside C), then the following formula holds:*

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi j(N - P) \tag{F.170}$$

denote respectively the number of zeros and poles of $f(z)$ inside the contour C, with each zero and pole counted as many times as its multiplicity and order respectively.

More generally, suppose that C is a curve, oriented counter-clockwise, which is contractible to a point inside an open set D in the complex plane. For each point $z \in D$, let $n(C, z)$ be the winding number of C around the point z. Then

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi j \left(\sum_a n(C, a) - \sum_b c(C, b) \right) \tag{F.171}$$

where the first summation is over all zeros a of f counted with their multiplicities, and the second summation is over the poles b of f.

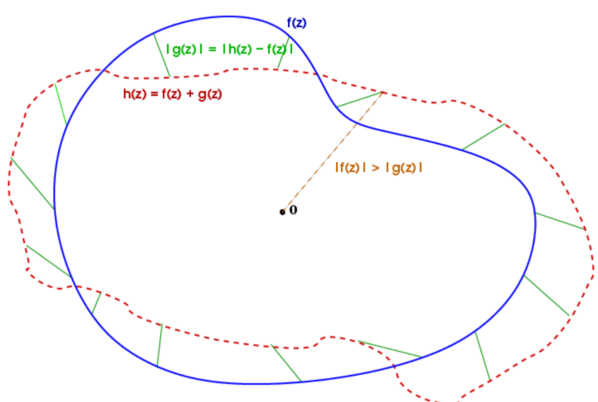

Figure 6.4: Illustration of the meaning of the Rouché's theorem

This makes a relation between the maximal principle and the winding number of a function of a complex variable. For more information about this please refer to [Levinson and Redheffer, 1970].

F.7 Proof of Rouché's theorem

This theorem is important in complex analysis and has important consequence in the stability analysis of time-delay systems. It can be used in order to get information on the number of zeroes of a function over a compact set without computing them but only by knowing the number of zeroes of another given function.

The theorem is recalled below for readability:

Theorem F.9 *Given two functions f and g analytic (holomorphic) inside inside and on a contour C. If $|g(z)| < |f(z)|$ for all z on C, then f and $f+g$ have the same number of roots inside C.*

Let us define the function h such that $h = f + g$. It is holomorphic since it is the sum of two holomorphic functions. From the argument principle (see appendix F.6), we have

$$N_h - P_h = I_h(C,0) = \frac{1}{2\pi j}\oint_C \frac{h'(z)}{h(z)}dz \tag{F.172}$$

where N_h is the number of zeroes of h inside C, P_h is the number of poles, and $I_h(C,0)$ is the winding number of $h(C)$ about 0. Since h is analytic inside and on C, it follows that $P_h = 0$ and then

$$N_h = I_h(C,0) = \frac{1}{2\pi j}\oint_C \frac{h'(z)}{h(z)}dz \tag{F.173}$$

One has that $\dfrac{h'}{h} = \mathcal{D}[\log(h(z))]$, where $\mathcal{D}$ denotes the complex derivative. Keeping in mind that $h = f + g$, we find

$$\begin{aligned}
N_h &= \frac{1}{2\pi j}\oint_C \frac{h'(z)}{h(z)} \\
&= \frac{1}{2\pi j}\oint_C \mathcal{D}[\log(h(z))]dz \\
&= \frac{1}{2\pi j}\oint_C \mathcal{D}[\log(f(z)+g(z))] \\
&= \frac{1}{2\pi j}\oint_C \mathcal{D}\left[\log\left(f(z)\left(1+\frac{g(z)}{f(z)}\right)\right)\right]dz \\
&= \frac{1}{2\pi j}\oint_C \frac{f'(z)}{f(z)} + \frac{1}{2\pi j}\oint_C \frac{\mathcal{D}(1+g(z)/f(z))}{1+g(z)/f(z)}dz \\
&= I_f(C,0) + I_{1+g(z)/f(z)}(C,0)
\end{aligned}$$

The winding number of $1 + g/f$ over C is zero. This is because we supposed that $|g(z)| < |f(z)|$, so g/f is constrained to a circle of radius 1, and adding 1 to g/f shifts it away from zero, and thus $1 + g/f$ is constrained to a circle of radius 1 about 1, and C under $1 + g/f$ cannot wind around 0. Finally we get

$$N_h = I_f(C,0)$$

which equals to N_f or the number of zeros of f. This concludes the proof.□

Example F.10 *An example of application is the determination of the number of roots of a* 3^{th} *order polynomial, say* $z^3 + z^2 - 1$*, contained in the disk* $|z| < 2$*. The idea is to remove the higher order term to use it as a bound on the rest of the polynomial. Indeed, define* $f(z) = z^3$ *and* $g(z) = z^2 - 1$*, the contour is defined by* $|z| = 2$*. Hence for all* z *on this contour we have* $|g(z)| \leq 5$ *and* $|f(z)| = 8$ *showing that we have* $|g(z)| < |f(z)|$ *for any* z *such that* $|z| = 2$*. This shows that* z^3 *and* $z^3 + z^2 - 1$ *have the same in the disc* $|z| < 2$*, which is 3.*

For more information about the Rouché's theorem, please refer to [Levinson and Redheffer, 1970].

G Frequency-Domain Stability Analysis of Time-Delay Systems

The frequency domain analysis is chronologically the first to have been deployed but is limited to the analysis of constant delays Gu et al. [2003], Niculescu [2001]. On the other hand, it is possible to turn a system with time-varying delay into an uncertain system with constant delay in order to apply robust stability analysis tools and frequency domain methods. This is not detailed here but the readers should refer, for instance, to Gu et al. [2003], Michiels and Niculescu [2007], Michiels et al. [2005] and references therein. This part does not aim at providing a complete overview of frequency domain methods but some facts on well-known and simple methods only.

G.1 Zeros of quasipolynomials

The essence of frequency domain analysis is to find where the poles of the studied system are located in the complex plane. For a finite dimensional system, it is a well-known fact that if at least one of the poles has nonnegative real part, the system is not asymptotically stable (or unstable). Fortunately, the same fact holds for linear time-delay systems [K.Hale and Lunel, 1991] and thus frequency domain methods can be exploited for stability analysis. At the difference of finite dimensional systems, the number of poles of a time-delay systems may be different from n and may take a finite number as well as an infinite (but countable) number of characteristic roots. To see this, let us consider the time-delay system

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + Ew(t) \tag{G.174}$$

where x, w and h are respectively the system state, the inputs and the delay. The Laplace transform of such a system yields:

$$sX(s) = AX(s) + A_h X(s)e^{-sh} + EW(s)$$

and finally

$$(sI - A - A_h e^{-sh})X(s) = EW(s) \tag{G.175}$$

Hence the system is asymptotically stable if the characteristic quasipolynomial in $s \in \mathbb{C}$

$$\det(sI - A - A_h e^{-sh}) \tag{G.176}$$

has all its zeroes in the open left half plane. The name 'quasipolynomial' comes from the mixing of powers and exponential terms making the search for zeroes of (G.176) a rather difficult task since it is not a polynomial.

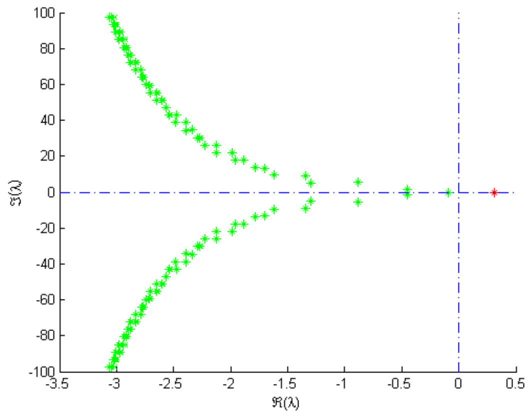

Figure 6.5: Zeros of linearized system (G.178)

Example G.1 *Let us consider the following model of two coupled neurons with time-delayed connections [Engelborghs et al., 2001]*

$$\begin{array}{rcl} \dot{x}_1(t) &=& -\kappa x_1(t) + \beta \tanh(x_1(t-\tau_s)) + \delta \tanh(x_2(t-\tau_2)) \\ \dot{x}_2(t) &=& -\kappa x_2(t) + \beta \tanh(x_2(t-\tau_s)) + \gamma \tanh(x_1(t-\tau_1)) \end{array} \quad \text{(G.177)}$$

where $\kappa = 0.5$*,* $\beta = -1$*,* $\delta = 1$*,* $\gamma = 2.34$*,* $\tau_1 = \tau_2 = 0.2$ *and* $\tau_s = 1.5$*. The linearized equations around equilibrium* $(0, 0)$ *are given by*

$$\dot{x}(t) = \begin{bmatrix} -\kappa & 0 \\ -\kappa & 0 \end{bmatrix} x(t) + \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} x(t-\tau_s) + \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x(t-\tau_1) + \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix} x(t-\tau_2) \quad \text{(G.178)}$$

The zeroes of the characteristic equation corresponding to linear system (G.178) are computed. Their distribution over the complex plane is depicted on Figure 6.5. Since there is one real zero with positive real part, $(0, 0)$ *is an unstable equilibrium.*

We summarize here few results on the location of zeros of quasipolynomial, which are mostly borrowed from Gu et al. [2003], Michiels and Niculescu [2007].

Let us consider a general quasipolynomial of the form

$$\begin{array}{rcl} f(s) &:=& \displaystyle\sum_{k=0}^{n} \sum_{i=0}^{m} (a_{ki} + j b_{ki}) s^{n-k} e^{(\alpha_i + j\beta_i)s} \\ &=& \displaystyle\sum_{i=0}^{m} p_i(s) e^{(\alpha_i + j\beta_i)s} \\ &=& \displaystyle\sum_{k=0}^{n} \psi_k(s) s^{n-k} \end{array} \quad \text{(G.179)}$$

where a_{ki}, b_{ki}, α_i and β_i are real numbers and

$$\begin{array}{rcll} p_i(s) &=& \sum_{k=0}^{n} (a_{ki} + j b_{ki}) s^{n-k}, & i = 0, 1, \ldots, m, \\ \psi_k(s) &=& \sum_{i=0}^{n} (a_{ki} + j b_{ki}) e^{(\alpha_i + j\beta_i)s}, & k = 0, 1, \ldots, n. \end{array}$$

Under assumptions

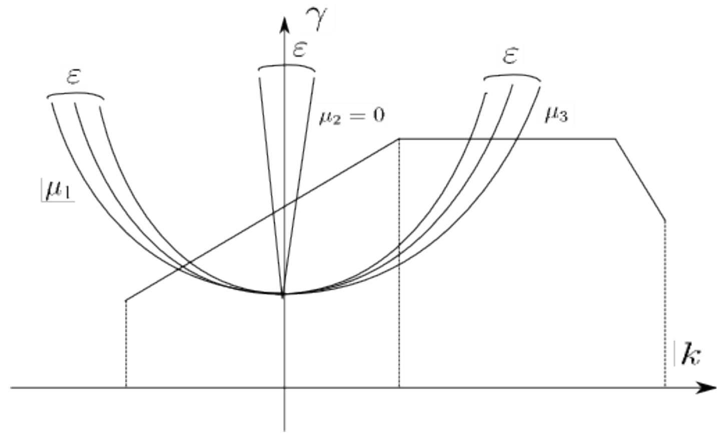

Figure 6.6: Potential diagram

- exponential coefficients $\alpha_i + j\beta_i$, $i = 0, 1, \ldots, m$ are distinct complex numbers;
- polynomials $p_i(s)$, $i = 0, 1, \ldots, m$ are not trivial

it is possible to show that $f(s)$ may have a finite number of zeros only in the case $m = 0$ (i.e. system with one delay). In the following we will assume that $m > 0$. The following proposition has been shown in [Gu et al., 2003]:

Proposition G.2 *If at least two of the exponential coefficients $\alpha_i + j\beta_i$, $i = 0, 1, \ldots, m$, have distinct imaginary parts, then $f(s)$ has zeros with arbitrarily large positive real parts.*

This means that a stable quasipolynomial has only real exponent coefficients and then $f(s)$ must reduce to

$$f_s(s) = \sum_{k=0}^{n} \sum_{i=0}^{m} (a_{ki} + jb_{ki}) s^{n-k} e^{\alpha_i s} \tag{G.180}$$

with $\alpha_0 < \alpha_1 < \ldots < \alpha_n$. The quasipolynomial $f_s(s)$ can also be written as

$$f_s(s) = \sum_{l=0}^{n} c_l s^{k_l} e^{\gamma_l s}$$

where all complex coefficients c_l are supposed to be nonzero, and additionally we assume that no couple (k_l, γ_l) are identical (i.e. $(k_l, \gamma_l) \neq (k_q, \gamma_q)$ for all $l, q = 1, \ldots, N$, $l \neq q$).

Now we will briefly explain how the potential diagram may give a result on the locations of the zeros of quasipolynomial $f_s(s)$. Let us plot on the complex plane all points characterized by $z_l = \gamma_l + jk_l$. Then construct the upper part of the envelope of these points as shown in Figure 6.6.

The upper part is known as the *potential diagram* of $f_s(s)$ and it consists of a finite number of segments. Let M be the number of these segments and we associate at each one, a logarithmic curve

$$\Lambda_\kappa := \{s = x + jy : \ x = \mu_\kappa \ln(y), \ y \in (-1, +\infty)\}, \qquad \kappa = 1, 2, \ldots, M \tag{G.181}$$

Here μ_κ is a number such that the vector $(\mu_\kappa, 1)$ is along the direction of the outer normal to the corresponding segment. If $\mu_\kappa > 0$, then Λ_κ lies in the right-half complex plane, while the curve corresponding to μ_κ belongs to the left-half complex plane. For $\mu_\kappa = 0$, the corresponding Λ_κ coincides with the imaginary axis.

For sufficiently small $\varepsilon > 0$ the logarithmic ε-sectors

$$\Lambda_\kappa(\varepsilon) := \{s = x + jy : \; x \in [(\mu_\kappa - \varepsilon/2)\ln(y), (\mu_\kappa + \varepsilon/2)\ln(y)], \; y \in (-1, +\infty)\}, \quad \kappa = 1, 2, \ldots, M \tag{G.182}$$

have no common points except $z_c = j$.

The following theorem describing distribution of zeros of quasipolynomial (G.180) can be found with proof in [Gu et al., 2003].

Theorem G.3 *For every $\varepsilon > 0$ there exists a constant $R(\varepsilon)$ depending on ε, such that all zeros of $f_s(s)$ in the upper half complex plane with magnitudes greater than $R(\varepsilon)$ lie in the union of logarithmic ε-sectors $\Lambda_\kappa(\varepsilon)$, $\kappa = 1, 2, \ldots, M$.*

The zeros of $f_s(s)$ belonging to the lower half plane lie into the union of logarithmic ε-sectors obtained by the mirror image of Λ_κ, $\kappa = 1, 2, \ldots, M$, with respect to the real axis.

Remark G.4 *Applying the principle of argument (see Appendix F.6) it may be shown that $f(s)$ has infinite (countable) number of zeros in every logarithmic ε-sector $\Lambda_\kappa(\varepsilon)$.*

It follows that $f_s(s)$ has zeros with arbitrarily large positive real parts when at least one of the values μ_κ is positive. Using the exponential diagram of $f_s(s)$, one can conclude that such positive μ_κ exists only if the outer normal of one of the segments forming the diagram points toward the right-half complex plane. In order to guarantee the absence of zeros of $f_s(s)$ with arbitrarily large positive real parts, one has to assume that one of the term in (G.180), say $c_0 s^{k_0} e^{\gamma_0 s}$, satisfies the two conditions

- $k_0 \geq k_j$, $j = 1, 2, \ldots, N$;
- $\gamma_0 \geq \gamma_j$, $j = 1, 2, \ldots, N$

When such a term exist it is called *principal term* of quasipolynomial (G.180).

We end this section on the following corollary:

Corollary G.5 *Quasipolynomial (G.180) may have all zeros in the open left half complex plane if it has a principal term.*

Some enlightenments have been provided on the location and the number of zeros of a quasipolynomial of the form (G.180). Time-delay systems have all the properties to be stable (i.e. presence of a principal term and argument of the exponentials purely imaginary).

Due to the infinite number of zeros of characteristic quasipolynomial of time-delay systems and the high computational complexity of their computation, it seems more convenient to try to determine the stability of a time-delay system without any explicit computation of zeros, this is the goal of the recent work of Michiels and Niculescu [2007], Sipahi and Olgac [2006]. The next sections aim at providing simple stability tests.

G.2 Classical simple stability test: 2-D stability test

We briefly explain here a simple stability test allowing to compute the delay margin of a system with commensurate delays. Recall that, for systems involving several delays h_i, $i = 1, \ldots, N$, it is said to they are commensurate if there exists $k_i \in \mathbb{Q}_+^{N-1}$ such that $h_{i+1} = k_i h_i$.

The approach is based on the stability of two-dimensional polynomials which are very often encountered in signal and image processing [Bose, 1982]. The main idea, here, is to turn the quasipolynomial of a linear time-delay system into a bivariate polynomial which may analyzed as a characteristic polynomial of a 2-D system.

Let us consider a system with single delay having characteristic quasipolynomial $a(s, z)$ where $z = e^{-sh}$ and h is the delay. Introduce the following bilinear transformation

$$s = \frac{1+\kappa}{1-\kappa}$$

which maps s from the open right half plane $\mathbb{C}^+$ to κ in the open unit disc $\mathbb{D}$. Construct the 2-D polynomial

$$b(\kappa, z) := (1-\kappa)^n a\left(\frac{1+\kappa}{1-\kappa}, z\right) \tag{G.183}$$

where $a(s, z) = \det(sI - A - A_h z)$ is the characteristic quasipolynomial of the system with $z = e^{-sh}$. It is evident that $a(s, z) = 0$ for some $(s, z) \in \partial\mathbb{C}^+ \times \partial\mathbb{D}$ if and only if $b(\kappa, z) = 0$ for some $(\kappa, z) \in \partial\mathbb{D} \times \partial\mathbb{D}$. In addition, for $h > 0$, the quasipolynomial $a(s, e^{-sh})$ has no root in $\mathbb{C}^+$ if and only if $b(\kappa, z)$ is stable; here by stability we mean that all its roots lie outside the closed region $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$. Hence, under the assumption that the system is stable at $h = 0$, to verify the delay-independent stability of the system, it suffices to check the stability of the 2-D polynomial $b(\kappa, z)$. On the other hand, the delay-dependent stability is determined by computing the roots of $b(\kappa, z)$. To do this, introduce the conjugate polynomial

$$\bar{a}(s, z) := z^n a(-s, z^{-1}) \tag{G.184}$$

By the conjugate symmetry of $a(s, z)$, it follows that $(s, z) \in \partial\mathbb{C}^+ \times \partial\mathbb{D}$ is a root of $a(s, z)$ if and only if it is also a root of $\bar{a}(s, z)$. Thus in order to find the roots of $a(s, z)$ on $\partial\mathbb{C}^+ \times \partial\mathbb{D}$, it suffices to solve the system of polynomial equations:

$$\begin{aligned} a(s, z) &= 0 \\ \bar{a}(s, z) &= 0 \end{aligned} \tag{G.185}$$

When no solution exists, and when the system is stable in the delay free case, then the system is delay independent stable. Otherwise, when the system of equations does admit a common solution, it is possible to eliminate one variable, resulting in a polynomial in one variable. From the computation of the roots of this latter polynomial, it is possible to give the delay margin of the system (see [Gu et al., 2003] for more details and examples).

Example G.6 *Consider the time-delay system (2.14) with constant time-delay. In this case we have*

$$\begin{aligned} a(s, z) &= s + z = 0 \\ \bar{a}(s, z) &= za(-s, z^{-1}) = -sz + 1 = 0 \end{aligned}$$

From the first equation we get $z = -s$ and substituting the expression of z in the second equality we get $s^2 + 1 = 0$. The solutions of the latter equation are given by $s = \pm i$ and hence the set of solutions (s, z) is given by

$$(s, z) \in \{(i, -i), (-i, i)\}$$

Finally, since $z = e^{-sh}$ *then we have*

$$\begin{aligned} i &= e^{ih} \\ -i &= e^{-ih} \end{aligned}$$

Both identities lead to $h = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$. *The delay-margin is given by the smallest positive value of* h *and is then* $h < \pi/2$. *The result of [Kharitonov and Niculescu, 2003] is retrieved.*

Other approaches are provided in Gu et al. [2003], Niculescu [2001] and references therein.The main drawback of 2-D stability tests (and other simple tests provided in Gu et al. [2003]) comes from the fact that important symbolic computation or calculation 'by hand' are needed, which is highly undesirable. Hence the application of these methods for high order systems and/or with many delays becomes very difficult, making the tests inefficient and less applicable.

G.3 Frequency Sweeping Tests

Frequency sweeping tests are based on the analysis of the (generalized) eigenvalues of the system when the Laplace variable s sweeps the imaginary axis. If at least one of the eigenvalues crosses the imaginary axis, then this implies that the system will be unstable. According to sweeping tests, the system would either delay-dependent or delay-independent stable. Both simple tests are provided below.

A sufficient condition to delay-independent stability taken from [Gu et al., 2003] is given below

Theorem G.7 *System (G.174) is delay-independent stable of delay if*

1. *A is stable*
2. *$A + A_h$ is stable*
3. $\bar{\rho}[(j\omega I - A)^{-1}A_h] < 1, \quad \text{for all } \omega > 0$

where $\bar{\rho}(\cdot)$ *denotes the spectral radius (i.e.* $\max_i |\lambda_i(\cdot)|$*).*

Statements 1 and 2 implies that the system is stable for $h = +\infty$ and $h = 0$. The last statement ensures that for any $\omega > 0$ and $h \in (0, +\infty)$, the system admits no eigenvalues on the imaginary axis. Therefore, the system is stable independent of the delay. System (2.10) of Example 2.2.2 satisfies these conditions and thus is delay-independent stable.

It is also possible to provide a necessary and sufficient condition to delay-independent stability [Gu et al., 2003]:

Lemma G.8 *System (G.174) is delay-independent stable if and only if*

1. *A is Hurwitz*
2. $\bar{\rho}(A^{-1}A_h) < 1$, *or* $\bar{\rho}(A^{-1}A_h) = 1$ *with* $\det(A + A_h) \neq 0$
3. $\bar{\rho}(j\omega I - A)^{-1}A_h < 1$, *for all* $\omega > 0$.

An extension of Theorem G.7 allows to compute the delay margin when the system is not delay-independent stable, provided that it is stable for $h = 0$ [Gu et al., 2003].

Theorem G.9 *Suppose that system (G.174) is stable at $h = 0$. Let us define $q := \text{rank}[A_h]$ and*

$$\bar{h}_i := \begin{cases} \min_{1\le k\le n} \theta_k^i/\omega_k^i & \text{if } \lambda_i(j\omega_k^i I - A, A_h) = e^{-j\theta_k^i} \\ & \quad\quad \text{for some } \omega_k^i \in (0,+\infty),\ \theta_k^i \in [0, 2\pi] \\ +\infty & \text{if } \underline{\rho}(j\omega I - A, A_h) > 1,\ \text{for all } \omega \in (0,+\infty) \end{cases} \tag{G.186}$$

and $\underline{\rho}(A, B)$ denote the minimal generalized eigenvalue of the pair (A, B) (i.e. $\underline{\rho}(A, B) = \min\{|\lambda| :\ \det(A - \lambda B) = 0\}$). Then the system (G.174) is delay-dependent stable for every delay $h \in [0, \bar{h})$ where

$$\bar{h} := \min_{1\le i\le q} \bar{h}_i \tag{G.187}$$

and becomes unstable at $h = \bar{h}$.

These approaches can be extended to the case of multiple commensurate delays. In spite of their implementational simplicity, frequency-sweeping tests, by nature, cannot be executed in finite computation, and the computational accuracy hinges on the fineness of the frequency grids. Thus they are likely to be a disadvantage if high computational precision is sought.

G.4 Constant Matrix Tests

Preceding discussions lead us to search for alternate stability tests, that are both readily implementable and can be performed via finite step algorithms. In particular, for numerical precision, it will be highly desirable to eliminate any frequency sweep, while retaining the merits of computing eigenvalues and generalized eigenvalues, for the computational ease of the latter. The stability tests to be introduced below combine these advantageous featuresand only require the computation of constant matrices.

Let us consider the characteristic quasipolynomial

$$a(s, e^{-sh}) = \sum_{k=0}^{q} a_k(s) e^{-khs} \tag{G.188}$$

where

$$a_0(s) = s^n + \sum_{i=0}^{n-1} a_{0i} s^i \qquad a_k(s) = \sum_{i=0}^{n-1} a_{ki} s^i \tag{G.189}$$

The following theorem [Gu et al., 2003] conclude on both delay-independent and delay-dependent stability of quasipolynomial (G.188).

Theorem G.10 *Suppose that the latter quasipolynomial is stable at $h = 0$. Let $H_n := 0$,*

$T_n := I$, and

$$
\begin{aligned}
H_i : &= \begin{bmatrix} a_{qi} & a_{q-1,i} & \dots & a_{1i} \\ 0 & a_{qi} & \dots & a_{2i} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{qi} \end{bmatrix}, \; i = 0, 1, \dots, n-1 \\
T_i &:= \begin{bmatrix} a_{0i} & 0 & \dots & 0 \\ a_{1i} & a_{0i} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{q-1,i} & a_{q-2,i} & \dots & a_{0i} \end{bmatrix}, \; i = 0, 1, \dots, n-1 \\
P_i &:= \begin{bmatrix} j^i T_i & j^i H_i \\ (-j)^i H_i^T & (-j)^i T_i^T \end{bmatrix}, \; i = 0, 1, \dots, n
\end{aligned}
$$

Furthermore, define

$$
P := \begin{bmatrix} 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \\ -P_n^{-1}P_0 & P_n^{-1}P_1 & \dots & P_n^{-1}P_{n-1} \end{bmatrix}
$$

Then, $\bar{h} = +\infty$ if $\sigma(P) \cap \mathbb{R}_+ = \emptyset$ or $\sigma(P) \cap \mathbb{R}_+ = \{0\}$. Additionally, let

$$
\begin{aligned}
F(s) &:= \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0(s) & -a_1(s) & \dots & -a_{q-1}(s) \end{bmatrix} \\
G(s) &:= \operatorname{diag}(1,\ 1,\ \dots,\ a_q(s))
\end{aligned}
$$

Then $\bar{h} = +\infty$ if $\sigma(F(j\omega_k), G(j\omega_k)) \cap \partial\mathbb{D} = \emptyset$ for all $0 \neq \omega_k \in \sigma(P) \cap \mathbb{R}_+$. In these cases, the quasipolynomial (G.188) is delay-independent stable. Otherwise,

$$
\bar{h} = \min_{1 \le k \le 2nq} \theta_k / \omega_k \tag{G.190}
$$

where $0 \neq \omega_k \in \sigma(P) \cap \mathbb{R}_+$ and $\theta_k \in [0, 2\pi]$ satisfy the relation $e^{j\theta_k} \in \sigma(F(j\omega_k), G(j\omega_k))$. The quasipolynomial (G.188) is stable for all $h \in [0, \bar{h})$, but is unstable at $h = \bar{h}$.

This theorem suggests that a two-step procedure can be employed to test the stability of the quasipolynomial (G.188). First compute the eigenvalues of the $2nq \times 2nq$ matrix P. If P has no real eigenvalue or only one real eigenvalue at zero, we conclude that the quasipolynomial is stable independent of delay. If it is not the case, compute next the generalized eigenvalues of the $q \times q$ matrix pair $(F(j\omega_k), G(j\omega_k))$, with respect to each positive real eigenvalue ω_k of P. If for all such eigenvalues the pair $(F(j\omega_k), G(j\omega_k))$ has no generalized eigenvalue on the unit circle, we again conclude that the quasipolynomial is stable independent of delay. Otherwise, we obtain the delay margin $\bar{h}$.

H Stabilizability and Detectability of LPV Systems

This appendix is devoted to the analysis of stabilizability and detectability of LPV systems. Some recent works have been devoted to this problems [Blanchini, 2000, Blanchini et al., 2007]. Since stabilizability and detectability of LPV systems is highly related to the notion of stability of LPV systems, different variations of notions can be provided [Hespanha et al., 2001, Hespanha and Morse, 1999, Mohammadpour and Grigoriadis, 2007b, Willigenburg and Koning, To Appear]. This section is not devoted to the enumeration of these variations but rather focuses on main ideas. As an example, one can define the robust and quadratic stabilizability/detectability in which the rate of variation of parameters is taken into account or not. It is also possible to define the almost everywhere quadratic/robust stabilizability/detectability.

In this section, LPV systems of the form (H.191) will be considered.

$$\begin{array}{rcl}\dot{x} & = & A(\rho)x + B(\rho)u \\ y & = & C(\rho)x + D(\rho)u\end{array} \tag{H.191}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are respectively the state of the system, the control input and the measured output. Moreover the parameters are assumed to take values in a compact set denoted U_ρ while their derivative evolve in the set $U_d = \text{hull}[U_\nu]$ where U_ν is the set of vertices of U_d.

Before providing a discussion on different types of properties of a LPV system, it is important that these properties of controllability/stabilizability/observability/detectability are not absolute at all. Indeed, by controllability, for instance, it is generally meant that a state-feedback exists such that all the eigenvalues of the system controlled by a state-feedback can be placed arbitrarily in the complex plane (and therefore in the complex left-half plane). But there exist several controllability concepts depending on the type of controllers which is considered. The same distinction holds for observers too.

As straightforward results, the extension of rank conditions for controllability/stabilizability and observability/detectability are now provided (see Appendix B.6).

Lemma H.1 *System (H.191) is robustly controllable by state-feedback control law $u = K(\rho)x$ if and only if the following rank condition*

$$\text{rank}[\mathscr{C}(\rho)] = n \text{ with } \mathscr{C}(\rho) = \begin{bmatrix} B(\rho) & A(\rho)B(\rho) & \dots & A(\rho)^{n-1}B(\rho) \end{bmatrix} \tag{H.192}$$

holds for all $\rho \in U_\rho$.

Proof: *This the generalization of the controllability rank criterion to LPV systems (see Appendix B.6.* □

Lemma H.2 *System (H.191) is robustly stabilizable with a state-feedback control law $u = K(\rho)x$ if and only if the following rank condition*

$$\text{rank}[\mathscr{S}(s,\rho)] = n \text{ with } \mathscr{S}(s,\rho) = \begin{bmatrix} sI - A(\rho) & B(\rho) \end{bmatrix} \tag{H.193}$$

holds for all $\rho \in U_\rho$ and $s \in \mathbb{C}^+$.

Proof: *This the generalization of the stabilizability rank criterion to LPV systems (see Appendix B.6.* □

Lemma H.3 *System (H.191) without control input is robustly observable with observer* $\dot{\hat{x}} = A(\rho)\hat{x} + L(\rho)(y - C(\rho)\hat{x})$ *if and only if the following rank condition*

$$\text{rank}[\mathscr{O}(\rho)] = n \text{ with } \mathscr{O}(\rho) = \begin{bmatrix} C(\rho) \\ C(\rho)A(\rho) \\ \vdots \\ C(\rho)A(\rho)^{n-1} \end{bmatrix} \tag{H.194}$$

holds for all $\rho \in U_\rho$.

Proof: *This the generalization of the observability rank criterion to LPV systems (see Appendix B.6.* □

Lemma H.4 *System (H.191) without control input is robustly detectable with observer* $\dot{\hat{x}} = A(\rho)\hat{x} + L(\rho)(y - C(\rho)\hat{x})$ *if and only if the following rank condition*

$$\text{rank}[\mathscr{D}(s,\rho)] = n \text{ with } \mathscr{S}(s,\rho) = \begin{bmatrix} sI - A(\rho) \\ C(\rho) \end{bmatrix} \tag{H.195}$$

holds for all $\rho \in U_\rho$ *and* $s \in \mathbb{C}^+$.

Proof: *This the generalization of the detectability rank criterion to LPV systems (see Appendix B.6.* □

The latter conditions on controllability and observability allows to certificate that a suitable process (control law or observer) exists and allows to obtain both quadratic and robust stability of the closed-loop system or the estimation error. Hence Lemmas H.1 and H.3 are conditions for both quadratic and robust stability or observation. But, since it does not take into account the rate of variation it is difficult to foresee for which maximal bound the system will remain robustly controllable or observable.

Concerning stabilizability and detectability, defined by Lemmas H.2 and H.4, the quadratic stabilizability and detectability will be more difficult to obtain since quadratic properties are very sensitive to the repartition of the modes of the system, especially the uncontrollable and unobservable modes. On the other hand, robust stability is far less sensitive and would be easier to achieve.

These results only provide qualitative results by answering by 'no' or 'yes', if a system is quadratically or robustly stabilizable and this remains the main drawback. Indeed, the absence of the rate of variation into the conditions is a lack of information on the maximal admissible bound for which the system will be stabilizable or detectable. Finally no constructive approach can be efficiently derived form these conditions. This motivates the necessity of having other conditions not using rank operators, if possible.

H.1 Controllability and Stabilizability

The objectives of controllability and stabilizability is to provide necessary and sufficient conditions to the existence of controllers for a given system. These conditions may take, sometimes equivalent, different forms. In this section, we will only focus on state-feedback and full-order dynamic output feedback controllers since their existence can be analyzed through rank constraints of parameter dependent matrices and LMIs. We will show that, while observability is difficult to analyze or even impossible, stabilizability can be expressed in a nice fashion through LMIs. Three main controllabiliy/stabilizability results will deployed: quadratic and robust stabilizability/controllability and stabilizability with average dwell-time. While the quadratic stabilizability deal with quadratic stabilization, the robust controllability/stabilizability deals with robust stabilization. Finally, the stabilizability with average dwell-time deals with the possibility of stabilizing a system even in presence of loss of stabilizability.

We are interested here in determining if a state-feedback control law of the form

$$u = K(\rho)x \tag{H.196}$$

or a full-order dynamic output feedback control law of the form

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{H.197}$$

stabilizing LPV system (H.191) exist. In each case, we will aim at giving different equivalent existence conditions to the existence of each one of the control laws.

H.1.1 Quadratic stabilizability

The quadratic controllability/stabilizability aims at determining if there exist control laws of the form (H.196) or (H.197) such that system (H.191) controlled by (H.196) or (H.197) is quadratically stable. It is important to point out that it is difficult to generalize rank conditions of the controllability matrix in this case (see Appendix B.7). Indeed, the rank condition allows to determine if there exists a state-feedback control law such that all the eigenvalues of the closed-loop system lie in the complex left-half plane at an arbitrary location. However, as discussed and illustrated in Section 1.3.1, it is not sufficient for a LPV system to have all its eigenvalues in the complex left-half plane to have quadratic stability since quadratic stability allows for unbounded parameter variation rates. This is the reason for which quadratic controllability is difficult to express.

The following result provides quadratic stabilizability results for LPV system (H.191) with a state-feedback control law (H.196).

Lemma H.5 *System (H.191) is quadratically stabilizable by a state-feedback control law of the form (H.196) if and only if one the following equivalent statements holds:*

1. *There exist $X = X^T \succ 0$ and $Y(\rho)$ such that the LMI*

$$A(\rho)X + XA(\rho)^T + B(\rho)Y(\rho) + Y(\rho)^T B(\rho)^T \prec 0 \tag{H.198}$$

   hold for all $\rho \in U_\rho$.

2. *There exists $X = X^T \succ 0$ such that the LMI*

$$\mathrm{Ker}[B(\rho)^T]^T \left(XA(\rho)^T + A(\rho)X\right) \mathrm{Ker}[B(\rho)^T] \prec 0 \tag{H.199}$$

   holds for all $\rho \in U_\rho$.

3. *There exist $X = X^T \succ 0$ and a scalar function $\tau(\rho)$ such that the LMI*

$$XA(\rho)^T + A(\rho)X + \tau(\rho)B(\rho)B(\rho)^T \prec 0 \tag{H.200}$$

holds for all $\rho \in U_\rho$.

Moreover, if one of the statements holds then a suitable state-feedback control law of the form (H.196) stabilizing LPV system (H.191) is given by

- *either $u(t) = K(\rho)x$ with $K(\rho) = Y(\rho)X^{-1}$ or*
- *$u(t) = K(\rho)x$ with $K(\rho) = -\kappa B(\rho)^T X^{-1}$ where $\kappa > 0$ satisfies $\kappa B(\rho)B(\rho)^T - (XA(\rho)^T + A(\rho)X) \succ 0$ and $P^{-1} = X$ for all $\rho \in U_\rho$.*

Proof: *The proof is as follows. The closed-loop system is given by*

$$\dot{x} = (A(\rho) + B(\rho)K(\rho))x$$

and define the Lyapunov function $V(x) = x^T Px$. The closed-loop system is quadratically stable if and only if the derivative of the Lyapunov function evaluated along the trajectories solution of the closed-loop system is negative definite for all $\rho \in U_\rho$ and all $x \neq 0$. Then, differentiating V gives

$$\dot{V} = x^T[(A(\rho) + B(\rho)K(\rho))^T P + P(A(\rho) + B(\rho)K(\rho))]x$$

Assuming that $\dot{V} < 0$ we have

$$\begin{array}{lll} & (A(\rho) + B(\rho)K(\rho))^T P + P(A(\rho) + B(\rho)K(\rho)) & \prec \; 0 \\ & \text{according to a congruence tranformation w.r.t. } X = P^{-1} \text{ we get} & \\ \Rightarrow & X(A(\rho) + B(\rho)K(\rho))^T + (A(\rho) + B(\rho)K(\rho))X & \prec \; 0 \\ & \text{with the change of variable } Y(\rho) = K(\rho)X \text{ we obtain} & \\ \Rightarrow & XA(\rho) + Y(\rho)^T B(\rho)^T + A(\rho)X + B(\rho)Y(\rho) & \prec \; 0 \end{array}$$

This proves that statement 1) is a necessary condition to quadratic stabilizability. The proof of sufficiency can be done by following the proof backward: suppose statement 1) holds, then choosing the state-feedback gain $K(\rho) = Y(\rho)X^{-1}$ implies

$$(A(\rho) + B(\rho)K(\rho))^T P + P(A(\rho) + B(\rho)K(\rho)) \prec 0$$

which proves quadratic stability of the closed-loop system. The first statement is then equivalent to quadratic stabilizability.

Now rewrite the latter inequality of statement 1) in the form

$$XA(\rho)^T + A(\rho)X + B(\rho)Y(\rho) + Y(\rho)^T B(\rho)^T \prec 0 \tag{H.201}$$

Since the matrix $Y(\rho)$ is unconstrained (totally free) then the Finsler's lemma (extended to the parameter dependent case) applies (see Appendix E.16) and leads to the following equivalent inequalities:

$$\begin{array}{ll} \text{Ker}[B(\rho)^T]^T(XA(\rho)^T + A(\rho)X)\text{Ker}[B(\rho)^T] & \prec \; 0 \\ XA(\rho)^T + A(\rho)X + \tau(\rho)B(\rho)^T B(\rho) & \prec \; 0 \end{array} \tag{H.202}$$

where $\tau(\rho)$ is a scalar continuous function to be determined. To provide an expression of a suitable control law, it suffices to apply results given in Appendix A.9. This concludes the proof. □

The following result provides quadratic stabilizability results for LPV system (H.191) with a dynamic output feedback control law (H.197).

Lemma H.6 *System (H.191) is quadratically stabilizable by a dynamic output feedback control law of the form (H.197) if and only if one the following equivalent statements holds:*

1. *There exist $X_1 = X_1^T \succ 0$, $P_1 = P_1^T \succ 0$ and $(\bar{A}_c(\rho), \bar{B}_c(\rho), \bar{C}_c(\rho), \bar{D}_c(\rho))$ such that the LMI*

$$\begin{array}{ll} \begin{bmatrix} A(\rho)X_1 + B(\rho)\bar{C}_c(\rho) & A(\rho) + B(\rho)\bar{D}_c(\rho)C(\rho) \\ \bar{A}_c(\rho) & P_1 A(\rho) + \bar{B}_c(\rho)C(\rho) \end{bmatrix}^H & \prec \ 0 \\ \begin{bmatrix} X_1 & I \\ I & P_1 \end{bmatrix} & \succeq \ 0 \end{array} \tag{H.203}$$

   hold for all $\rho \in U_\rho$.

2. *There exist $X_1 = X_1^T \succ 0$ and $P_1 = P_1^T \succ 0$ such that the LMI*

$$\begin{array}{ll} \mathrm{Ker}[C(\rho)]^T \left(A(\rho)^T P_1 + P_1 A(\rho)\right) \mathrm{Ker}[C(\rho)] & \prec \ 0 \\ \mathrm{Ker}[B(\rho)^T]^T \left(X_1 A(\rho)^T + A(\rho)X_1\right) \mathrm{Ker}[B(\rho)^T] & \prec \ 0 \\ \begin{bmatrix} X_1 & I \\ I & P_1 \end{bmatrix} & \succeq \ 0 \end{array} \tag{H.204}$$

   holds for all $\rho \in U_\rho$.

3. *There exist $X_1 = X_1^T \succ 0$, $P_1 = P_1^T \succ 0$ and a scalar function $\tau(\rho)$ such that the LMI*

$$\begin{array}{ll} A(\rho)^T P_1 + P_1 A(\rho) + \tau(\rho)C(\rho)^T C(\rho) & \prec \ 0 \\ X_1 A(\rho)^T + A(\rho)X_1 + \tau(\rho)B(\rho)B(\rho)^T & \prec \ 0 \\ \begin{bmatrix} X_1 & I \\ I & P_1 \end{bmatrix} & \succeq \ 0 \end{array} \tag{H.205}$$

   holds for all $\rho \in U_\rho$.

Moreover, if one of the statements holds then a suitable state-feedback control law of the form (H.196) stabilizing LPV system (H.191) is given by

- *either the solution of algebraic equations*

$$\begin{array}{rcl} A_c & := & (P_2^{-1}\bar{A}_c - P_2^{-1}P_1 B\bar{C}_c)X_2^{-T} - (P_2^{-1}\bar{B}_c - P_2^{-1}P_1 B\bar{D}_c)CX_1X_2^{-T} \\ B_c & := & P_2^{-1}\bar{B}_c - P_2^{-1}P_1 B\bar{D}_c \\ C_c : & = & \bar{C}_c X_2^{-T} - \bar{D}_c C X_1 X_2^{-T} \\ D_c : & = & \bar{D}_c \\ P_2 X_2^T & = & I - P_1 X_1 \end{array}$$

  or

- *the computation of the following formula*

$$\Omega(\rho) := -\kappa(\bar{B}(\rho)^T P\Psi(\rho)P\bar{B}(\rho))^{-1}\bar{B}(\rho)^T P\Psi(\rho)\bar{C}(\rho)^T \tag{H.206}$$

where κ satisfies $\Psi(\rho) := (\kappa\bar{C}(\rho)^T\bar{C}(\rho) - A(\rho)^T P + PA(\rho))^{-1} \succ 0$ for all $\rho \in U_\rho$.

Proof: *First of all we need to construct the extended system whose state contains both system and controller states*

$$\left[\begin{array}{c} \dot{x} \\ \dot{x}_c \\ \hline x_c \\ y \end{array}\right] = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \left[\begin{array}{c} x \\ x_c \\ \hline \dot{x}_c \\ u \end{array}\right] \tag{H.207}$$

where $\begin{bmatrix} \bar{A}(\rho) & \bar{B}(\rho) \\ \bar{C}(\rho) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A(\rho) & 0 & 0 & B(\rho) \\ 0 & 0 & I & 0 \\ \hline 0 & I & 0 & 0 \\ C(\rho) & 0 & 0 & 0 \end{array}\right]$ *and let us denote by $\Omega(\rho)$ the matrix* $\begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix}$. *The closed-loop system is then given by*

$$\dot{\bar{x}} = (\bar{A}(\rho) + \bar{B}(\rho)\Omega(\rho)\bar{C}(\rho))\bar{x}$$

where $\bar{x} = \mathrm{col}(x, x_c)$. Introduce now the Lyapunov function $V(\bar{x}) = \bar{x}^T P\bar{x}$ and computing its derivative along trajectories solutions of the closed-loop system yields

$$\dot{V}(\bar{x}) = \bar{x}^T[P(\bar{A}(\rho) + \bar{B}(\rho)\Omega(\rho)\bar{C}(\rho)) + (\bar{A}(\rho) + \bar{B}(\rho)\Omega(\rho)\bar{C}(\rho))^T P]\bar{x} \tag{H.208}$$

To show statement 2), note that since the matrix $\Omega(\rho)$ is unconstrained which indicates that the projection lemma applies (see E.18) and hence the negative definiteness of $\dot{V}$ is equivalent to the feasibility of both underlying LMIs

$$\begin{array}{rl} \mathrm{Ker}[\bar{C}(\rho)]^T(P\bar{A}(\rho) + \bar{A}(\rho)^T P)\mathrm{Ker}[\bar{C}(\rho)] & \prec 0 \\ \mathrm{Ker}[\bar{B}(\rho)^T]^T(\bar{A}(\rho)P^{-1} + P^{-1}\bar{A}(\rho)^T)\mathrm{Ker}[\bar{B}(\rho)] & \prec 0 \end{array} \tag{H.209}$$

The conditions above are apparently not LMI due to the presence of matrices P and P^{-1}, but actually they are. This is shown hereafter. We have the following equalities:

$$\begin{array}{rcl} \mathrm{Ker}[\bar{C}(\rho)] & = & \mathrm{Ker}\begin{bmatrix} 0 & I \\ C(\rho) & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Ker}[C(\rho)] \\ 0 \end{bmatrix} \\ \mathrm{Ker}[\bar{B}(\rho)^T] & = & \mathrm{Ker}\begin{bmatrix} 0 & I \\ B(\rho)^T & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Ker}[B(\rho)^T] \\ 0 \end{bmatrix} \end{array} \tag{H.210}$$

Finally inequalities (H.209) become

$$\begin{array}{rl} \mathrm{Ker}[C(\rho)]^T(P_1 A(\rho) + A(\rho)^T P_1)\mathrm{Ker}[C(\rho)] & \prec 0 \\ \mathrm{Ker}[B(\rho)^T]^T(A(\rho)X_1 + X_1 A(\rho)^T)\mathrm{Ker}[B(\rho)] & \prec 0 \end{array} \tag{H.211}$$

where $P = \begin{bmatrix} P_1 & * \\ * & * \end{bmatrix}$ *and* $X = \begin{bmatrix} X_1 & * \\ * & * \end{bmatrix}$ *($*$ means 'do not care'). In virtue of the completion lemma (see Appendix E.19) the matrices P_1 and X_1 can be considered as independent by adding the LMI*

$$\begin{bmatrix} P_1 & I \\ I & X_1 \end{bmatrix} \succeq 0$$

ensuring that the whole matrices P and X can be constructed from these two blocks. This shows that quadratic stabilizability by full-order dynamic output feedback is equivalent to statement 2).

We show now the equivalence between statements 2) and 3). First rewrite inequalities in the compact form

$$\begin{bmatrix} \mathrm{Ker}[C(\rho)] & 0 \\ 0 & \mathrm{Ker}[B(\rho)^T] \end{bmatrix}^T \begin{bmatrix} P_1A(\rho)+A(\rho)^TP_1 & 0 \\ 0 & A(\rho)X_1+X_1A(\rho)^T \end{bmatrix} (\star)^T \prec 0 \tag{H.212}$$

and according to the Finsler's lemma extended to the parameter dependent case (see Appendix E.16) this is equivalent to the existence of a continuous scalar function $\tau(\rho)$ such that the LMI

$$\begin{bmatrix} P_1A(\rho)+A(\rho)^TP_1 & 0 \\ 0 & A(\rho)X_1+X_1A(\rho)^T \end{bmatrix} + \tau(\rho)\begin{bmatrix} C(\rho)^TC(\rho) & 0 \\ 0 & B(\rho)B(\rho)^T \end{bmatrix} \prec 0 \tag{H.213}$$

is feasible for all $\rho \in U_\rho$. This proves the equivalence of statement 2) and 3).

We aim now at showing that quadratic stabilizability is equivalent to statement 1). It is convenient to introduce here the following matrix

$$Z = \begin{bmatrix} X_1 & I \\ X_2^T & 0 \end{bmatrix} \tag{H.214}$$

which satisfies

$$\begin{aligned} Z^TP &= \begin{bmatrix} X_1 & I \\ X_2^T & 0 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} = \begin{bmatrix} X_1P_1+X_2P_2 & X_1P_2+X_2P_3 \\ P_1 & P_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ P_1 & P_2 \end{bmatrix} \\ Z^TPZ &= \begin{bmatrix} I & 0 \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} X_1 & I \\ X_2^T & 0 \end{bmatrix} = \begin{bmatrix} X_1 & I \\ I & P_1 \end{bmatrix} \end{aligned} \tag{H.215}$$

Come back to inequality (H.208) and performing a congruence transformation with respect to matrix $Z = \begin{bmatrix} X_1 & I \\ X_2^T & 0 \end{bmatrix}$ gives

$$\mathcal{M}(\rho) + \mathcal{M}(\rho)^T \prec 0 \tag{H.216}$$

where

$$\begin{aligned} \mathcal{M}(\rho) &= \begin{bmatrix} I & 0 \\ P_1 & P_2 \end{bmatrix} \left(\begin{bmatrix} A(\rho) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\rho) \\ I & 0 \end{bmatrix} \Omega(\rho) \begin{bmatrix} 0 & I \\ C(\rho) & 0 \end{bmatrix} \right) \begin{bmatrix} X_1 & I \\ X_2^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} AX_1 + BC_cX_2^T + BD_cX_1 & A + BD_cC \\ P_1AX_1 + P_2A_cX_2^T + P_1BC_cX_2^T + P_2B_cCX_1 + P_1BD_cCX_1 & P_1A + P_2B_cC + P_1BD_cC \end{bmatrix} \\ &= \begin{bmatrix} A(\rho)X_1 & A(\rho) \\ 0 & P_1A(\rho) \end{bmatrix} + \begin{bmatrix} 0 & B(\rho) \\ I & 0 \end{bmatrix} \left(\begin{bmatrix} P_1A(\rho)X_1 & 0 \\ 0 & 0 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} P_2 & P_1B(\rho) \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} X_2^T & 0 \\ C(\rho)X_1 & I \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & C(\rho) \end{bmatrix} \end{aligned} \tag{H.217}$$

Then the change of variable

$$\begin{bmatrix} \bar{A}_c(\rho) & \bar{B}_c(\rho) \\ \bar{C}_c(\rho) & \bar{D}_c(\rho) \end{bmatrix} = \begin{bmatrix} P_1A(\rho)X_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P_2 & P_1B(\rho) \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} X_2^T & 0 \\ C(\rho)X_1 & I \end{bmatrix} \tag{H.218}$$

linearizes the problem. Indeed, in virtue of the completion lemma, the matrices X_1 and P_1 can be chosen independently. Moreover the change of variable from

$$(A_c(\rho), B_c(\rho), C_c(\rho), D_c(\rho)) \rightarrow (\bar{A}_c(\rho), \bar{B}_c(\rho), \bar{C}_c(\rho), \bar{D}_c(\rho))$$

is bijective. To see this just note that matrices

$$\begin{bmatrix} P_2 & P_1 B(\rho) \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} X_2^T & 0 \\ C(\rho)X_1 & I \end{bmatrix}$$

are both invertible provided that P_2 and X_2 are nonsingular. Actually they can always be chosen to fit this constraint using a small perturbation term, this is shown below. Suppose that $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \succ 0$ in which P_2 is singular and P satisfies some LMI. Then there exists a sufficiently small scalar ε such that the LMI is also feasible with $P_\varepsilon = \begin{bmatrix} P_1 & P_2 + \varepsilon I \\ P_2^T + \varepsilon I & P_3 \end{bmatrix} \succ 0$ where $P_2 + \varepsilon I$ is invertible.

Finally, $\mathcal{M}(\rho)$ becomes

$$\mathcal{M}(\rho) = \begin{bmatrix} A(\rho)X_1 & A(\rho) \\ 0 & P_1 A(\rho) \end{bmatrix} + \begin{bmatrix} 0 & B(\rho) \\ I & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_c(\rho) & \bar{B}_c(\rho) \\ \bar{C}_c(\rho) & \bar{D}_c(\rho) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C(\rho) \end{bmatrix} \tag{H.219}$$

and hence the derivative of the Lyapunov function is expressed as a LMI in the variables $P_1 = P_1^T \succ 0$, $X_1 = X_1^T \succ 0$, $\bar{A}_c(\rho)$, $\bar{B}_c(\rho)$, $\bar{C}_c(\rho)$, $\bar{D}_c(\rho)$:

$$\begin{bmatrix} A(\rho)X_1 + B(\rho)\bar{C}_c(\rho) & A(\rho) + B(\rho)\bar{D}_c(\rho)C(\rho) \\ \bar{A}_c(\rho) & P_1 A(\rho) + \bar{B}_c(\rho)C(\rho) \end{bmatrix}^H \prec 0 \tag{H.220}$$

We aim now at proving the expression of the controller matrices A_c, B_c, C_c and D_c. A classical method is to use the reciprocal of the change of variable. In this case we have

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} := \begin{bmatrix} P_2 & P_1 B(\rho) \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} \bar{A}_c(\rho) & \bar{B}_c(\rho) \\ \bar{C}_c(\rho) & \bar{D}_c(\rho) \end{bmatrix} - \begin{bmatrix} P_1 A(\rho) X_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} X_2^T & 0 \\ C(\rho)X_1 & I \end{bmatrix}^{-1} \tag{H.221}$$

We have the following relations:

$$\begin{aligned} \begin{bmatrix} P_2 & P_1 B(\rho) \\ 0 & I \end{bmatrix}^{-1} &= \begin{bmatrix} P_2^{-1} & -P_2^{-1} P_1 B \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} X_2^T & 0 \\ C(\rho)X_1 & I \end{bmatrix}^{-1} &= \begin{bmatrix} X_2^{-T} & 0 \\ -CX_1 X_2^{-T} & I \end{bmatrix} \end{aligned}$$

and using these latter expressions into (H.221) and expanding the results equations yields the explicit expression for A_c, B_c, C_c and D_c.

In view of proving the second explicit expression, apply the results of Appendix A.9 on inequality (H.208). □

We have shown that the quadratic stabilizability of LPV systems in both state-feedback and dynamic output feedback cases can be cast as a LMI problem. Of course, all statements are not equivalent from a computational point of view. Indeed, in each lemma, the statements 1) and 3) involve at least one parameter dependent matrix as a variable of the SDP. As explained in Section 1.3.3.1, such variables cannot be computed exactly and is then a source of conservatism. On the other hand, statement 2) does not involve such terms and allows for an efficient and nonconservative stabilization tests.

On the other hand, if for some reasons, the controller should have a specific parameter dependence (e.g. a quadratic dependence on the parameter $K(\rho) = K_0 + K_1\rho + K_2\rho^2$), then statements 2) should be considered instead.

H.1.2 Robust controllability and stabilizability

Since quadratic stability is rather conservative, except when the parameters actually admit discontinuities in their trajectories, it would be more convenient to consider the problem of robust controllability and robust stabilizability in order to provide less conservative conditions. The results will be given without proofs since they are roughly similar to proofs of the previous section. In the following, both robust controllability and stabilizability will be addressed. Indeed, while the quadratic controllability has not been presented for technical reasons, the robust controllability (for state-feedback control laws) has been in Lemma H.1. The essential reason is the simplicity of the conditions involved in robust controllability. Finally, robust stabilizability will be introduced and expressed in terms of parameter dependent LMIs.

Lemma H.7 *System (H.191) is robustly stabilizable by a state-feedback control law of the form (H.196) if and only if one the following equivalent statements holds:*

1. *There exist a continuously differentiable matrix function $X(\rho) = X(\rho)^T \succ 0$ and a continuous matrix function $Y(\rho)$ such that the LMI*

$$A(\rho)X + XA(\rho)^T + B(\rho)Y(\rho) + Y(\rho)^T B(\rho)^T - \sum_i \nu_i \frac{\partial X(\rho)}{\partial \rho_i} \prec 0 \tag{H.222}$$

   hold for all $\rho \in U_\rho$ and all $\nu = \text{col}_i(\nu_i) \in U_\nu$.

2. *There exists a continuously differentiable matrix function $X(\rho) = X(\rho)^T \succ 0$ such that the LMI*

$$\text{Ker}[B(\rho)^T]^T \left(X(\rho)A(\rho)^T + A(\rho)X(\rho) - \sum_i \nu_i \frac{\partial X(\rho)}{\partial \rho_i} \right) \text{Ker}[B(\rho)^T] \prec 0 \tag{H.223}$$

   holds for all $\rho \in U_\rho$ and all $\nu = \text{col}_i(\nu_i) \in U_\nu$.

3. *There exist a continuously differentiable matrix function $X(\rho) = X(\rho)^T \succ 0$ and a scalar function $\tau(\rho)$ such that the LMI*

$$X(\rho)A(\rho)^T + A(\rho)X(\rho) + \tau(\rho)B(\rho)B(\rho)^T - \sum_i \nu_i \frac{\partial X(\rho)}{\partial \rho_i} \prec 0 \tag{H.224}$$

   holds for all $\rho \in U_\rho$ and all $\nu = \text{col}_i(\nu_i) \in U_\nu$.

Moreover, if one of the statements holds then a suitable state-feedback control law of the form (H.196) stabilizing LPV system (H.191) is given by

- *either $u(t) = Y(\rho)X(\rho)^{-1}$ or*

- $u(t) = -\kappa B(\rho)^T X(\rho)^{-1} x(t)$ *with* $P(\rho)^{-1} = X(\rho)$ *and where* $\kappa > 0$ *satisfies*

$$\kappa B(\rho)B(\rho)^T - (X(\rho)A(\rho)^T + A(\rho)X(\rho) - \sum_i \nu_i \frac{\partial X(\rho)}{\partial \rho_i}) \succ 0$$

for all $\rho \in U_\rho$ *and all* $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

Proof: *The proof is identical as for Lemma H.5 using formulae* $X(\rho) = P(\rho)^{-1}$, $\dot{X}(\rho) = -X(\rho)\dot{P}(\rho)X(\rho)$ *and* $\dot{X}(\rho) = \sum_i \frac{\partial X(\rho)}{\partial \rho_i}\dot{\rho}_i$. □

Lemma H.8 *System (H.191) is robustly stabilizable by a dynamic output feedback control law of the form (H.197) if and only if there exist continuously differentiable matrix functions* $X_1(\rho) = X_1(\rho)^T \succ 0$ *and* $P_1(\rho) = P_1(\rho)^T \succ 0$ *such that the LMI*

$$\begin{array}{lcl} \mathrm{Ker}[C(\rho)]^T \left(A(\rho)^T P_1(\rho) + P_1(\rho)A(\rho) + \sum_i \nu_i \frac{\partial P_1(\rho)}{\partial \rho_i} \right) \mathrm{Ker}[C(\rho)] & \prec & 0 \\ \mathrm{Ker}[B(\rho)^T]^T \left(X_1(\rho)A(\rho)^T + A(\rho)X_1(\rho) - \sum_i \nu_i \frac{\partial X_1(\rho)}{\partial \rho_i} \right) \mathrm{Ker}[B(\rho)^T] & \prec & 0 \\ \begin{bmatrix} P_1(\rho) & I \\ I & X_1(\rho) \end{bmatrix} & \succeq & 0 \end{array} \tag{H.225}$$

holds for all $\rho \in U_\rho$ *and* $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$. *In this case, the controller satisfies the following LMI*

$$(\bar{A}(\rho) + \bar{B}(\rho)\Omega(\rho)\bar{C}(\rho))^T P(\rho) + P(\rho)(\bar{A}(\rho) + \bar{B}(\rho)\Omega(\rho)\bar{C}(\rho)) + \sum_i \nu_i \frac{\partial P(\rho)}{\partial \rho_i} \prec 0 \tag{H.226}$$

for all $\rho \in U_\rho$ *and* $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

Proof: *The proof is similar as for Lemma H.6.* □

Note that the robust stabilizability by dynamic output feedback, only the projected conditions have been provided. The main reason is the computational complexity of this result compared to method of change of variable. But the controller computation cannot be given explicitly since, in this case, it would depend on the derivative of the parameters[Apkarian and Adams, 1998, Lu et al., 2006], which are generally considered as difficult to measure or estimate. The method using the change of variables also leads to such a controller. In [Apkarian and Adams, 1998], these controllers are considered to be practically invalid (unimplementable) and simplifications are provided in order to avoid such complications.

In the current discussion, annoying terms involving parameters derivative are avoided by considering the controller depending exclusively on the parameters as a solution of LMI (H.226). In this case, the controller matrix must be decomposed on a basis in order to get a tractable LMI problem. If a too simple basis is chosen then the problem may become unfeasible while the projected conditions are. In this case, the basis should be extended in order to reach a feasible LMI problem.

H.1.3 Stabilizability with average dwell-time

We provide here stabilizability in the context of average-dwell time which was initially provided for switched systems [Hespanha and Morse, 1999]. As explained in Section 1.3.1 for the stability problem, the problem of stabilization with average dwell-time addresses the problem of existence of a controller which cannot place all the eigenvalues of the closed-loop system in the complex left-half plane (i.e. there exists parameter values for which the system is not stabilizable). In such a case, the stability of the closed-loop system can be ensured provided that some constraints on parameter trajectories are fulfilled. For more details on dwell-time results, the readers should refer to [Hespanha et al., 2001, Hespanha and Morse, 1999, Mohammadpour and Grigoriadis, 2007b] and Section 1.3.1. The following discussion has not been provided in the literature and is a personal development. Moreover, only the quadratic stabilizability by state-feedback with average dwell-time will be introduced for sake of brevity but robust stabilizability with average dwell-time is possible to define.

First consider that the compact set of values of the parameters, denoted here U_ρ can be separated into two subsets U_u and U_s such that $U_u \cup U_s = U_\rho$ and $U_u \cap U_s = \emptyset$. These two subsets characterize respectively the set of parameters for which the system is not stabilizable and stabilizable respectively. Introduce the characteristic measure $\delta_u(\alpha)$ of the set U_u such that

$$\delta_u(\alpha) = \begin{cases} 1 & \text{if } \alpha \in U_u \\ 0 & \text{if } \alpha \in U_s \end{cases} \tag{H.227}$$

and the quantity

$$T_p(\tau, t) = \int_\tau^t \delta_u(\rho(t))dt \tag{H.228}$$

The following lemmas provide the main result of that section.

Lemma H.9 *The LPV system (H.191) is quadratically stabilizable with average dwell-time by parameter-dependent state-feedback (H.196) if and only there exist a matrix $X = X^T \succ 0$, scalars $\beta, \varpi > 0$ such that the LMIs*

$$\begin{array}{lcll} \mathrm{Ker}[B(\rho)^T]^T[XA(\rho)^T + XA(\rho)^T]\mathrm{Ker}[B(\rho)^T] & \prec & -2\varpi \mathrm{Ker}[B(\rho)^T]^T X \mathrm{Ker}[B(\rho)^T] & \text{if } \rho \in U_s \\ \mathrm{Ker}[B(\rho)^T]^T[XA(\rho)^T + XA(\rho)^T]\mathrm{Ker}[B(\rho)^T] & \prec & 2\beta \mathrm{Ker}[B(\rho)^T]^T X \mathrm{Ker}[B(\rho)^T] & \text{if } \rho \in U_u \end{array} \tag{H.229}$$

are feasible. In this case a suitable control law of the form (H.196) stabilizing LPV system (H.191) with (average dwell-time) decay rate $\varpi-(\varpi+\beta)\alpha$ and maximal ratio $\alpha < \alpha^* = \dfrac{\varpi}{\varpi+\beta}$ is given by is given by the equations:

$$u = K(\rho)x \text{where } K(\rho) = -\kappa B(\rho)^T X^{-1} \tag{H.230}$$

where κ is chosen satisfying

$$\begin{array}{lcl} \kappa B(\rho)B(\rho)^T - XA(\rho)^T - A(\rho)X + 2\varpi X & \succ & 0 \text{ if } \rho \in U_s \\ \kappa B(\rho)B(\rho)^T - XA(\rho)^T - A(\rho)X - 2\beta X & \succ & 0 \text{ if } \rho \in U_u \end{array}$$

Proof: *Let us consider the Lyapunov function*

$$V(x) = x^T P e^{-2\delta(t)t} x \tag{H.231}$$

where

$$\delta(t) := \begin{cases} -\varpi & \text{if } \rho(t) \in U_s \\ \beta & \text{if } \rho(t) \in U_u \end{cases}$$

with $\beta, \varpi > 0$. *The computation of the time derivative of* V *along trajectories solutions of the closed-loop system yields*

$$\dot{V}(t) < 0 \Leftrightarrow (A(\rho) + B(\rho))^T P + P(A(\rho) + B(\rho)) \prec 2\delta(t)P \tag{H.232}$$

It is possible to rewrite it into two parts

$$\begin{array}{rcll} (A(\rho) + B(\rho)K(\rho))^T P + P(A(\rho) + B(\rho)K(\rho)) & \prec & -2\varpi P & \text{if } \rho \in U_s \\ (A(\rho) + B(\rho)K(\rho))^T P + P(A(\rho) + B(\rho)K(\rho)) & \prec & 2\beta P & \text{if } \rho \in U_u \end{array} \tag{H.233}$$

Performing a congruence transformation with respect to $X = P^{-1}$ *and applying the Finsler's lemma (see Appendix E.16) yields*

$$\begin{array}{rcll} \mathrm{Ker}[B(\rho)^T]^T[XA(\rho)^T + A(\rho)X]\mathrm{Ker}[B(\rho)^T] & \prec & -2\varpi \mathrm{Ker}[B(\rho)^T]^T X \mathrm{Ker}[B(\rho)^T] & \text{if } \rho \in U_s \\ \mathrm{Ker}[B(\rho)^T]^T[XA(\rho) + A(\rho)X]\mathrm{Ker}[B(\rho)^T] & \prec & 2\beta \mathrm{Ker}[B(\rho)^T]^T X \mathrm{Ker}[B(\rho)^T] & \text{if } \rho \in U_u \end{array} \tag{H.234}$$

If the two following LMIs are satisfied, then it is possible to find a state-feedback matrix such that (H.233) are satisfied. In this case, we have

$$\dot{V}(t) < \begin{cases} -2\varpi V(t) & \text{if } \rho \in U_s \\ 2\beta V(t) & \text{if } \rho \in U_u \end{cases} \tag{H.235}$$

Solving this linear differential inequality leads to

$$\begin{array}{rcl} V(t) & < & \exp[-2\varpi(t - \tau - T_p(\tau, t)) + 2\beta T_p(\tau, t)]V(\tau) \\ & < & \exp[-2\varpi(t - \tau) + 2(\varpi + \beta)T_p(\tau, t)]V(\tau) \\ & < & \exp\left([2(\varpi + \beta)\alpha - 2\varpi](t - \tau)\right)\exp[2(\varpi + \beta)T_0]V(\tau) \end{array} \tag{H.236}$$

Hence, the Lyapunov function is strictly decreasing if the inequality $2(\varpi + \beta)\alpha - 2\varpi < 0$ *holds. This is satisfied for any* $\alpha < \alpha^* = \dfrac{\varpi}{\varpi + \beta}$ *and in this case, the decay rate of the closed-loop system is at least of* $\varpi - (\varpi + \beta)\alpha$. *Finally the explicit construction of the control law is made using relations of Appendix A.8.* $\square$

The following example shows an application of the latter lemma.

Example H.10

$$\dot{x}(t) = \begin{bmatrix} 1 & \rho(t) \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{H.237}$$

where $\rho(t) = f_{th}(\sin(t))$ *where* $f_{th}(x)$ *is the dead-band function such that*

$$f_{th}(\eta) := \begin{cases} 0 & \text{if } |\eta| \leq th \\ \eta - th & \text{if } \eta > th \\ -\eta + th & \text{if } \eta < -th \end{cases} \tag{H.238}$$

with $th \geq 0$. *Note that for* $th = 0$, *the parameter reduces to the expression* $\rho(t) = \sin(t)$. *It is clear that there are a loss of stabilizability whenever* $\rho(t)$ *reaches 0. With the help of the* *dead-band function, we will modulate the duration of the unstabilizability and show that from a certain threshold* th_0 *the system looses parametric global stabilizability.*

Due to the periodicity of the parameter, it is sufficient to evaluate the quantity T_p *over one period only and hence we find*

$$T_p(2k\pi, 2(k+1)\pi) = 4\arcsin(th) \tag{H.239}$$

and hence the instability ration $\alpha = \dfrac{2\arcsin(th)}{\pi}$.

We retrieve the fact that for $th = 0$ *then* $T_p(2k\pi, 2(k+1)\pi) = 0$ *and the system is stabilizable almost everywhere. It is clear that the minimal value for* β *is* 1 *(the eigenvalue corresponding to unstabilizable mode whenever* $\rho = 0$*). Hence the idea now is to provide a bound on the decay rate of the stable part guaranteeing the parametric global stability of the closed-loop system. We must have*

$$\alpha < \alpha^* = \frac{\varpi}{\varpi + \beta} = \frac{\varpi}{\varpi + 1} \tag{H.240}$$

Hence we get

$$\frac{2\arcsin(th)}{\pi} < \frac{\varpi}{\varpi + 1} \tag{H.241}$$

and finally

$$\varpi > \frac{2\arcsin(th)}{\pi - 2\arcsin(th)} \tag{H.242}$$

In the general case, ϖ *must obey*

$$\varpi > \frac{2\arcsin(th)}{\pi\beta - 2\arcsin(th)} \tag{H.243}$$

Figure 6.7 shows the evolution of the minimal bound of ϖ *for which the system is parametrically globally stable. For any* ϖ *belonging to surface above the curve, the system is parametrically globally stable.*

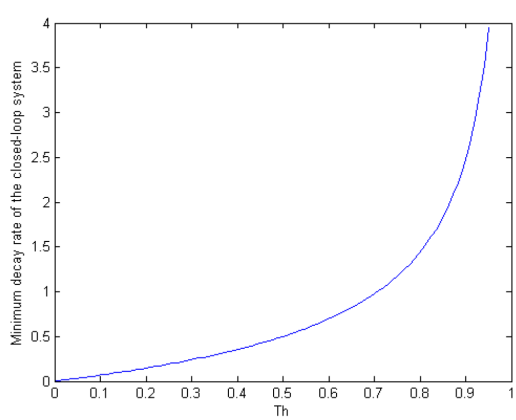

Figure 6.7: Evolution of the lower bound of ϖ *with respect to the dead-band width th*

Finally the problem remains to find a suitable control law such that for every parameter value of $\mathcal{P}$, the decay rate of the closed-loop system is at least of $\dfrac{2\arcsin(th)}{\pi\beta - 2\arcsin(th)}$.

H.2 Observability and Detectability

Similar results on observability and detectability are provided in this section. Whenever proofs are qualitatively identical, they will be omitted. This section addresses the problem of finding full-order observers of the form

$$\dot{\hat{x}} = A(\rho)\hat{x} + L(\rho)(y - C(\rho)\hat{x}) \tag{H.244}$$

or the form

$$\dot{\hat{x}} = M(\rho)\hat{x} + N(\rho)y \tag{H.245}$$

for LPV systems of the form (H.191). For simplicity system (H.191) is considered uncontrolled (i.e. $u \equiv 0$).

It is clear that observer (H.244) is a particular case of (H.245) by choosing $M(\rho) = A(\rho) - L(\rho)C(\rho)$ and $N(\rho) = L(\rho)$. The difference between these observers is the greater flexibility offered by observer (H.245) which has a larger number of degree of freedom, leading then to a more powerful design technique; this is actually more visible when dealing with systems with disturbances and performances specifications. It would be also possible to define detectability for low-order observers but this is omitted for brevity. Such observers will be designed in Section 4.1.

H.2.1 Quadratic observability and detectability

This section is devoted to the existence of quadratic observers. By quadratic observers we mean that an observer such that the estimation error is quadratically stable exists. The problem of existence of both types of observers (i.e. (H.244) and (H.244)) will be addressed in which necessary and sufficient conditions will be provided.

Lemma H.11 *There exists a quadratic observer of the form (H.244) for system (H.191) with no control input if and only if the following equivalent statements holds:*

1. *There exists $P = P^T \succ 0$ and $Y(\rho)$ such that*

$$A(\rho)^T P + PA(\rho) + Y(\rho)C(\rho) + C(\rho)^T Y(\rho)^T \prec 0 \tag{H.246}$$

   holds for all $\rho \in U_\rho$.

2. *There exists $P = P^T \succ 0$ such that*

$$\mathrm{Ker}[C(\rho)]^T (A(\rho)^T P + PA(\rho))\mathrm{Ker}[C(\rho)] \prec 0 \tag{H.247}$$

   holds for all $\rho \in U_\rho$.

3. *There exists $P = P^T \succ 0$ and a scalar function $\tau(\rho)$ such that*

$$A(\rho)^T P + PA(\rho) - \tau(\rho)C(\rho)^T C(\rho) \prec 0 \tag{H.248}$$

   holds for all $\rho \in U_\rho$.

In this case, a suitable observer gain $L(\rho)$ is given by

- *either $L(\rho) = P^{-1}Y(\rho)$; or*
- *$L(\rho) = \kappa P^{-1}C(\rho)$ where κ is chosen such that*

$$(\kappa C(\rho)^T C(\rho) - A(\rho)^T P + PA(\rho))^{-1} \succ 0$$

Proof: *The proof is similar as for quadratic stabilizability by state-feedback.* □

Lemma H.12 *There exists a quadratic observer of the form (H.245) for system (H.191) with no control input if and only if the following equivalent statements holds in which*

$$\begin{array}{rcl} \Psi(\rho) & = & A(\rho)(I + C(\rho)^T C(\rho))^{-1} \begin{bmatrix} I & C(\rho)^T \end{bmatrix} \\ \Phi(\rho) & = & I - \begin{bmatrix} I \\ C(\rho) \end{bmatrix} (I + C(\rho)^T C(\rho))^{-1} \begin{bmatrix} C(\rho)^T & I \end{bmatrix} \end{array}$$

1. *There exists a matrix $P = P^T \succ 0$ such that*
$$M(\rho)^T P + PM(\rho) \succ 0 \tag{H.249}$$
holds for all $\rho \in U_\rho$ and $M(\rho) + N(\rho)C(\rho) - A(\rho) = 0$.
2. *There exists a matrix $P = P^T \succ 0$ and $Y(\rho)$ such that*
$$J_1^T (P\Psi(\rho) + Y(\rho)\Phi(\rho))^T P + (P\Psi(\rho) + Y(\rho)\Phi(\rho))J_1 \prec 0 \tag{H.250}$$
holds for all $\rho \in U_\rho$ and $J_1 = \begin{bmatrix} I & 0 \end{bmatrix}^T$.
3. *There exists a matrix $P = P^T \succ 0$ and $Y(\rho)$ such that*
$$\mathrm{Ker}[\Phi(\rho)J_1]^T \left[J_1^T \Psi(\rho)^T P + P\Psi(\rho)J_1\right] \mathrm{Ker}[\Phi(\rho)J_1] \prec 0 \tag{H.251}$$
holds for all $\rho \in U_\rho$ and $J_1 = \begin{bmatrix} I & 0 \end{bmatrix}^T$.

In these cases, the observer matrices can be computed by

- *either $M(\rho) = (\Psi(\rho) + Y(\rho)P^{-1}\Phi(\rho))J_1$ and $N(\rho) = (\Psi(\rho) + Y(\rho)P^{-1}\Phi(\rho))J_2$; or*
- *$M(\rho) = (\Psi(\rho) - \kappa\Phi(\rho)P^{-1}\Phi(\rho))J_1$ and $N(\rho) = (\Psi(\rho) - \kappa\Phi(\rho)P^{-1}\Phi(\rho))J_2$ where κ is chosen such that*
$$\left[\kappa J_1^T \Phi(\rho)^T \Phi(\rho) J_1 - (J_1^T \Psi(\rho)^T P + P\Psi(\rho)J_1)\right]^{-1} \succ 0$$
with $J_1 = \begin{bmatrix} I & 0 \end{bmatrix}^T$ and $J_1 = \begin{bmatrix} 0 & I \end{bmatrix}^T$.

Proof: *The dynamic of the error $e = x - \hat{x}$ is given by*

$$\begin{array}{rcl} \dot{e} & = & A(\rho)x - M(\rho)\hat{x} - N(\rho)C(\rho)x \\ & = & (A(\rho) - N(\rho)C(\rho))x - M(\rho)\hat{x} \\ & = & M(\rho)e + (A(\rho) - N(\rho)C(\rho) - M(\rho))x \end{array}$$

Since the error is desired to be independent of the state x, then matrices $M(\rho)$ and $N(\rho)$ are chosen to fulfill the conditions

1. *$M(\rho)$ quadratically stable*
2. *$A(\rho) - N(\rho)C(\rho) - M(\rho) = 0$ for all $\rho \in U_\rho$*

In this case, the error e is autonomous and quadratically stable. Rewrite the equality into the compact sform $\begin{bmatrix} M(\rho) & N(\rho) \end{bmatrix} \begin{bmatrix} I \\ C(\rho) \end{bmatrix} = A(\rho)$ and since $\begin{bmatrix} I \\ C(\rho) \end{bmatrix}$ has full-column rank then all solutions are given by (see Appendix A.8)

$$\begin{bmatrix} M(\rho) & N(\rho) \end{bmatrix} = A(\rho) \begin{bmatrix} I \\ C(\rho) \end{bmatrix}^+ + Z(\rho) \left(I - \begin{bmatrix} I \\ C(\rho) \end{bmatrix} \begin{bmatrix} I \\ C(\rho) \end{bmatrix}^+ \right) \tag{H.252}$$

where $Z(\rho)$ is an arbitrary matrix.

Denoting

$$\begin{aligned} \Psi(\rho) &= A(\rho) \begin{bmatrix} I \\ C(\rho) \end{bmatrix}^+ \\ \Phi(\rho) &= I - \begin{bmatrix} I \\ C(\rho) \end{bmatrix} \begin{bmatrix} I \\ C(\rho) \end{bmatrix}^+ \\ J_1 &= \begin{bmatrix} I \\ 0 \end{bmatrix} \\ J_2 &= \begin{bmatrix} 0 \\ I \end{bmatrix} \end{aligned}$$

such that $M(\rho) = \begin{bmatrix} M(\rho) & N(\rho) \end{bmatrix} J_1$ and $N(\rho) = \begin{bmatrix} M(\rho) & N(\rho) \end{bmatrix} J_2$.

The explicit expression of the pseudoinverse is given by

$$\begin{bmatrix} I \\ C(\rho) \end{bmatrix}^+ = (C(\rho)^T C(\rho) + I)^{-1} \begin{bmatrix} I & C(\rho)^T \end{bmatrix} \tag{H.253}$$

and hence we have

$$\begin{aligned} \Psi(\rho) &= A(\rho)(C(\rho)^T C(\rho) + I)^{-1} \begin{bmatrix} I & C(\rho)^T \end{bmatrix} \\ \Phi(\rho) &= I - \begin{bmatrix} I \\ C(\rho) \end{bmatrix} (I - C(\rho)^T C(\rho))^{-1} \begin{bmatrix} I & C(\rho)^T \end{bmatrix} \end{aligned}$$

Then the error evolution is governed by

$$\dot{e} = (\Psi(\rho) + Z(\rho)\Phi(\rho))J_1 \tag{H.254}$$

Define the Lyapunov function $V(e) = e^T P e$ and computing the time derivative along trajectories solutions of (H.254) gives

$$\dot{V} = e^T [J_1^T (\Psi(\rho) + Z(\rho)\Phi(\rho))^T P + P(\Psi(\rho) + Z(\rho)\Phi(\rho))J_1] e < 0 \tag{H.255}$$

in which the change of variable $Y(\rho) = PZ(\rho)$ linearizes the problem and yields

$$J_1^T (P\Psi(\rho) + Y(\rho)\Phi(\rho))^T P + (P\Psi(\rho) + Y(\rho)\Phi(\rho))J_1 \prec 0 \tag{H.256}$$

This shows statements 1) and 2).

Now applying the Finsler's lemma (see Appendix E.16) we get

$$\mathrm{Ker}[\Phi(\rho)J_1]^T \left[J_1^T \Psi(\rho)^T P + P\Psi(\rho)J_1\right] \mathrm{Ker}[\Phi(\rho)J_1] \prec 0 \tag{H.257}$$

which shows statement 3). Finally applying results of Appendix A.8, the observer gains expressions are computed. □

The latter quadratic detectability conditions involve LMIs and algebraic equalities which make their verification a simple task. However, quadratic detectability remains a conservative property since it allows for unbounded parameter variation rates.

H.2.2 Robust observability and detectability

Robust observability and detectability provide less conservative condition by taking into account bounds on rate of parameter variation. In the robust framework, both observability and detectability will be addressed. The main reason for which observability is addressed in the robust framework stems from the fact that for a system to be robustly stable (i.e. the estimation error) it suffices that all the eigenvalues of the system matrix have negative real part for all parameters values (sufficient condition) while for quadratic stability this is necessary condition only.

The main drawbacks of the latter conditions come from the use of rank operators of parameter dependent matrices. Such conditions are computationally difficult to verify. Moreover, even if it possible to state that a robust observer effectively exists, then it is not possible to foresee for which bound on parameter derivatives it works. This motivates the following results, based on LMIs depending on rate of parameter variations.

Lemma H.13 *There exists a robust observer of the form (H.244) for system (H.191) with no control input if and only if the following equivalent statements holds:*

1. *There exist a continuously differentiable matrix function $P(\rho) = P(\rho)^T \succ 0$ and a continuous matrix function $Y(\rho)$ such that*

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) + Y(\rho)C(\rho) + C(\rho)^T Y(\rho)^T + \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i} \prec 0 \tag{H.258}$$

holds for all $\rho \in U_\rho$ and $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

2. *There exists a continuously differentiable matrix function $P(\rho) = P(\rho)^T \succ 0$ such that*

$$\mathrm{Ker}[C(\rho)]^T (A(\rho)^T P(\rho) + P(\rho)A(\rho) + \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i})\mathrm{Ker}[C(\rho)] \prec 0 \tag{H.259}$$

holds for all $\rho \in U_\rho$ and $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

3. *There exist a continuously differentiable matrix function $P(\rho) = P(\rho)^T \succ 0$ and a scalar function $\tau(\rho)$ such that*

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) + \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i} - \tau(\rho)C(\rho)^T C(\rho) \prec 0 \tag{H.260}$$

holds for all $\rho \in U_\rho$ and $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

In this case, a suitable observer gain $L(\rho)$ is given by

- *either* $L(\rho) = P(\rho)^{-1}Y(\rho)$*; or*
- $L(\rho) = \kappa P(\rho)^{-1}C(\rho)$ *where* κ *is chosen such that*

$$(\kappa C(\rho)^T C(\rho) - A(\rho)^T P(\rho) + P(\rho)A(\rho) - \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i})^{-1} \succ 0$$

Proof: *The proof is similar as for robust stabilizability by state-feedback.* □

Lemma H.14 *There exists a robust observer of the form (H.245) for system (H.191) with no control input if and only if the following equivalent statements holds:*

1. *There exists a continuously differentiable matrix function* $P(\rho) = P(\rho)^T \succ 0$ *such that*

$$M(\rho)^T P(\rho) + P(\rho)M(\rho) + \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i} \succ 0 \tag{H.261}$$

holds for all $\rho \in U_\rho$ *and* $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$ *such that* $M(\rho) + N(\rho)C(\rho) - A(\rho) = 0$.

2. *There exists a continuously differentiable matrix function* $P(\rho) = P(\rho)^T \succ 0$ *and a continuous function* $Y(\rho)$ *such that*

$$J_1^T (P(\rho)\Psi(\rho) + Y(\rho)\Phi(\rho))^T P + (P(\rho)\Psi(\rho) + Y(\rho)\Phi(\rho))J_1 + \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i} \prec 0 \tag{H.262}$$

holds for all $\rho \in U_\rho$ *and* $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

3. *There exists a matrix a continuously differentiable matrix function* $P(\rho) = P(\rho)^T \succ 0$ *such that*

$$\mathrm{Ker}[\Phi(\rho)J_1]^T \left[J_1^T P\Psi(\rho)P + P\Psi(\rho)J_1 + \sum_i \nu_i \frac{P(\rho)}{\partial \rho_i} \right] \mathrm{Ker}[\Phi(\rho)J_1] \prec 0 \tag{H.263}$$

holds for all $\rho \in U_\rho$ *and* $\nu = \mathrm{col}_i(\nu_i) \in U_\nu$.

In these cases, the observer matrices can be computed by

- *either* $M(\rho) = (\Psi(\rho) + Y(\rho)P(\rho)^{-1}\Phi(\rho))J_1$ *and* $N(\rho) = (\Psi(\rho) + Y(\rho)P(\rho)^{-1}\Phi(\rho))J_2$*; or*
- $M(\rho) = (\Psi(\rho) - \kappa\Phi(\rho)P(\rho)^{-1}\Phi(\rho))J_1$ *and* $N(\rho) = (\Psi(\rho) - \kappa\Phi(\rho)P(\rho)^{-1}\Phi(\rho))J_2$ *where* κ *is chosen such that*

$$\left[\kappa J_1^T \Phi(\rho)^T \Phi(\rho)J_1 - (J_1^T \Psi(\rho)^T P(\rho) + P(\rho)\Psi(\rho)J_1)\right]^{-1} \succ 0$$

Proof: *The proof is similar as for quadratic observability by observer of the form (H.245).* □

I Controllability and Observability of TDS

The controllability and observability of time-delay systems have been studied in Eising [1982], Kamen [1978], Lee and Olbrot [1981, 1982], Morse [1976], Sontag [1976] in the module framework. See Delfour and Mitter [1972], Lee and Olbrot [1981], Manitius and Triggiani [1978] for the representation over Banach functional spaces and see Lee and Olbrot [1981], Olbrot [1972, 1973], Weiss [1967] for models expressed as functional differential equations. Controllability of time-delay systems is rather more difficult than their finite dimensional counterpart: several nonequivalent controllability/observability properties can be defined and depend on the type of representation used to model time-delay systems. In Lee and Olbrot [1981], bridges are lighted between these different notions related to different time-delay system representations.

In this section, we will focus on time-delay systems of the form Let us consider system

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + A_h x(t-h) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \tag{I.264}$$

whose corresponding Laplace transform is given by

$$\begin{array}{rcl} [sI - A - A_h e^{-sh}]X(s) & = & BU(s) \\ Y(s) & = & CX(s) \end{array} \tag{I.265}$$

I.1 Spectral Controllability and Stabilizability

Among all different fundamental definitions of controllability and observability, only the so-called *spectral controllability* will be introduced here. An important property related to spectral controllability is that all the modes of the closed-loop system (observer) can be placed at any chosen values in a finite number (finite spectrum assignment).

In the finite dimensional case, it is well known that the system $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{n\times n}$ is controllable (see Appendix B.7) if and only if one of the following statement holds:

1. $\operatorname{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = n$ for all $s \in \lambda(A)$
2. $\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$

The controllability of a LTI system is equivalent to the existence of a state-feedback control law such that all the poles of the closed-loop system can be placed arbitrarily in the complex plane. A direct idea would be to generalize this notion directly to time-delay systems by mean of the condition

$$\operatorname{rank}\begin{bmatrix} sI - A - A_h e^{-sh} & B \end{bmatrix} = n \iff \operatorname{rank}\begin{bmatrix} B & (A + A_h e^{-sh})B & \dots & (A + A_h e^{-sh})^{n-1}B \end{bmatrix} = n \tag{I.266}$$

holds for all $s \in \{\tau \in \mathbb{C} : \det(\tau I - A - A_h e^{-\tau h}) = 0\}$. But actually such a criterion is wrong according to [Spong and Tarn, 1981] where the following counterexample is provided.

Example I.1

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t-h) \\ x_2(t-h) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{I.267}$$

For this system we have

$$\begin{aligned} A + A_h e^{-sh} &= \begin{bmatrix} 1 & 1/2 - e^{-sh} \\ 0 & e^{-sh} \end{bmatrix} \\ sI - A - A_h e^{-sh} &= \begin{bmatrix} s-1 & -1/2 + e^{-sh} \\ 0 & s - e^{-sh} \end{bmatrix} \end{aligned} \tag{I.268}$$

Letting $h = 2\ln 2$*, it is easily verified that*

$$\operatorname{rank} \begin{bmatrix} sI - A - 4^{-s} A_h & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} s-1 & -1/2 + 4^{-s} & 0 \\ 0 & s - 4^{-s} & 1 \end{bmatrix} = n$$

since it is impossible to find $s \in \{\tau \in \mathbb{C} : \det(\tau I - A - 4^{-\tau} A_h) = 0\}$ *such that the terms (1,1) and (1,2) or (1,2) and (2,2) vanish simultaneously. On the other hand,*

$$\operatorname{rank} \begin{bmatrix} B & (A + A_h 4^{-s})B \end{bmatrix} = \begin{bmatrix} 0 & 1/2 + 4^{-s} \\ 1 & 4^{-s} \end{bmatrix} < 2$$

for all $s \in \{\tau \in \mathbb{C} : \det(\tau I - A - 4^{-\tau} A_h) = 0\}$ *since for* $s = 1/2$ *we have* $1/2 + 4^{-1/2} = 0$.

This counterexample emphasized that the controllability of a time-delay system cannot be viewed as a simple extension of controllability of LTI systems. This leads us to the notion of *spectral controllability* leaning on the the representation of time-delay system over a ring of operator [Morse, 1976, Sontag, 1976]. The following theorem, proved in [Spong and Tarn, 1981], provides a necessary and sufficient condition for spectral controllability of (I.264).

Theorem I.2 *Time-delay (I.264) is spectrally controllable if and only if the following statements hold:*

1. $\operatorname{rank} \begin{bmatrix} sI - A - A_h^{-sh} & B \end{bmatrix} = n$ *forall* $s \in \{\tau \in \mathbb{C} : \det(\tau I - A - A_h^{-\tau h}) = 0\} \cap \{\tau \in \mathbb{C} : \operatorname{rank} \begin{bmatrix} B & \tilde{A}(e^{-\tau h})B & \dots & \tilde{A}(e^{-\tau h})^{n-1}B \end{bmatrix} < n\}$
2. $\operatorname{rank} \begin{bmatrix} B & \tilde{A}(\nabla)B & \dots & \tilde{A}(\nabla)^{n-1}B \end{bmatrix} = n$ *for all* $\nabla \in \mathbb{R}[\nabla]$ *is the ring of polynomials (weak R-controllability).*

The interest of this test resides in its simplicity of application, it involves nothing more than solving a polynomial equation (weak R-controllability) and then verifying that the roots verify a given property. For more details on different controllability definitions, please refer to [Morse, 1976, Spong and Tarn, 1981, **?**].

It seems obvious that spectral controllability may depend on the values of delay h.

Definition I.3 *If stabilizability/detectability condition holds for any delay, the system is said to be delay-independent stabilizable/detectable and otherwise, it is delay-dependent stabilizable/detectable.*

The following example provides a delay-dependent spectrally controllable time-delay system:

Example I.4

$$\dot{x}(t) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} x(t-h) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \tag{I.269}$$

Then we have

$$\begin{bmatrix} B & \tilde{A}(\nabla)B \end{bmatrix} = \begin{bmatrix} 1 & \nabla + 2 \\ 1 & 2\nabla - 2 \end{bmatrix}$$

Therefore the system is weakly-controllable (the matrix is of full rank over the ring of polynomial $\mathbb{R}[\nabla]$. Hence the system is spectrally controllable if and only if

$$\begin{bmatrix} \lambda_0 + 2 & -8 & 1 \\ -4 & \lambda_0 - 2 & 1 \end{bmatrix} = 2 \tag{I.270}$$

for all $\lambda_0 \in \{\tau \in \mathbb{C} : \; e^{-\tau h} = 4$. There is a loss of rank if and only if $\lambda_0 \neq -6$, i.e. if and only if $h \neq \ln(4)/6$.

From a control point of view, if a system is spectrally controllable then there exists a state-feedback control law with distributed delay of the form

$$u(t) = Kx(t) + \int_{-h}^{0} G(t+\theta)x(t+\theta)d\theta \tag{I.271}$$

allowing to assign the number of poles to a finite value and placed arbitrarily in the complex plane. It is important to note that the implementation of such a control law is not trivial. Indeed, the approximation of the integral into a finite sum may destabilize the closed-loop system (and even loose the finite spectrum assignment) this is the reason why the discretization of the control law should be handled with care [Mondié and Michiels, 2003].

These results cannot be directly extended to systems with time-varying delays due to the use of Laplace transform. It is explained below how it is possible to determine if a system with time-varying delays is stabilizable using a simple Lyapunov-Krasovskii functional. However, it is worth noting that the stabilizability, while determined using LMIs, depends on the considered type of controller (recall that the spectral controllability considers a distributed state-feedback control law): for finite-dimensional linear system the controllability always refer to the existence of a state-feedback control law. On the other hand, in the time-delay system framework, it is possible to consider several type of control law by considering a state-feedback structure according to the dependence on delayed or instantaneous information.

Theorem I.5 *System (I.264) is asymptotically stabilizable by instantaneous state-feedback $u(t) = Kx(t)$ if there exist a nonsingular matrix $X \in \mathbb{R}^{n\times n}$ and symmetric matrices $P, Q, R \succ 0$ such that the LMIs*

$$\begin{aligned} \begin{bmatrix} -(Y+Y)^T & A_hY & Y & h_{max}\tilde{R} \\ \star & -(1-\mu)\tilde{Q} - \tilde{R} & 0 & 0 \\ \star & \star & -\tilde{P} & h_{max}\tilde{R} \\ \star & \star & \star & -\tilde{R} \end{bmatrix} &\prec 0 \\ \left[\begin{array}{c|c} (B^T)_\perp & 0 \\ \hline 0 & I \end{array}\right]^T \tilde{\Xi} \left[\begin{array}{c|c} (B^T)_\perp & 0 \\ \hline 0 & I \end{array}\right] &\prec 0 \end{aligned} \tag{I.272}$$

hold where $(B^T)_\perp$ is basis of the null-space of B^T.

Proof: *It will be shown later, in Section (3.5-lemma 3.5.2, that (I.264) is asymptotically stable for all delays in* $[0, h_{max}]$ *if there exist* $X \in \mathbb{R}^{n\times n}$ *and* $P, Q, R \succ 0$ *such that*

$$\begin{bmatrix} -(X+X)^T & P+X^TA & X^TA_h & X^T & h_{max}R \\ \star & -P+Q-R & R & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & \star & -P & h_{max}R \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \qquad \text{(I.273)}$$

Assume now that the control law is $u(t) = Kx(t)$ *then the closed-loop system is then given by*

$$\dot{x}(t) \quad = \quad (A+BK)x(t) + A_hx(t-h) \qquad \text{(I.274)}$$

Inject the latter expression into LMI (I.284) yields

$$\Xi + U^TKV + V^TK^TU \prec 0 \qquad \text{(I.275)}$$

with Ξ *is (I.284),* $U = \begin{bmatrix} B^TX^T & 0 & 0 & 0 & 0 \end{bmatrix}$ *and* $V = \begin{bmatrix} 0 & I & 0 & 0 & 0 \end{bmatrix}$. *Perform a congruence transformation with respect to* $\mathrm{diag}(Y,Y,Y,Y,Y)$ *with* $Y = X^{-1}$ *(X is nonsingular since from the left-upper block of LMI (I.284) we must have* $X + X^T \succ 0$*) leads to inequality*

$$\tilde{\Xi} + \tilde{U}^TK\tilde{V} + \tilde{V}^TK^T\tilde{U} \prec 0 \qquad \text{(I.276)}$$

where $\tilde{\Xi} = \begin{bmatrix} -(Y+Y)^T & \tilde{P}+AY & A_hY & Y & h_{max}\tilde{R} \\ \star & -\tilde{P}+\tilde{Q}-\tilde{R} & R & 0 & 0 \\ \star & \star & -(1-\mu)\tilde{Q}-\tilde{R} & 0 & 0 \\ \star & \star & \star & -\tilde{P} & h_{max}\tilde{R} \\ \star & \star & \star & \star & -\tilde{R} \end{bmatrix}$, $U^T = \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
and $V = \begin{bmatrix} 0 & Y & 0 & 0 & 0 \end{bmatrix}$ *with* $\tilde{P} = Y^TPY$, $\tilde{Q} = Y^TQY$ *and* $\tilde{R} = Y^TRY$.

Finally applying projection lemma onto the latter inequality leads to these two underlying LMIs:

$$\begin{aligned} \begin{bmatrix} -(Y+Y)^T & A_hY & Y & h_{max}\tilde{R} \\ \star & -(1-\mu)\tilde{Q}-\tilde{R} & 0 & 0 \\ \star & \star & -\tilde{P} & h_{max}\tilde{R} \\ \star & \star & \star & -\tilde{R} \end{bmatrix} &\prec 0 \\ \left[\begin{array}{c|c} (B^T)_\perp & 0 \\ \hline 0 & I \end{array}\right]^T \tilde{\Xi} \left[\begin{array}{c|c} (B^T)_\perp & 0 \\ \hline 0 & I \end{array}\right] &\prec 0 \end{aligned} \qquad \text{(I.277)}$$

□

The same reasoning can be applied to the stabilization using delayed state-feedback

Theorem I.6 *System (I.264) is asymptotically stabilizable by delayed state-feedback* $u(t) = Kx(t-h)$ *if there exist a nonsingular matrix* $X \in \mathbb{R}^{n\times n}$ *and symmetric matrices* $P, Q, R \succ 0$ *such that the LMIs*

$$\begin{aligned} \begin{bmatrix} -(Y+Y)^T & \tilde{P}+AY & Y & h_{max}\tilde{R} \\ \star & -\tilde{P}+\tilde{Q}-\tilde{R} & 0 & 0 \\ \star & \star & -\tilde{P} & h_{max}\tilde{R} \\ \star & \star & \star & -\tilde{R} \end{bmatrix} &\prec 0 \\ \left[\begin{array}{c|c} (B^T)_\perp & 0 \\ \hline 0 & I \end{array}\right]^T \tilde{\Xi} \left[\begin{array}{c|c} (B^T)_\perp & 0 \\ \hline 0 & I \end{array}\right] &\prec 0 \end{aligned} \qquad \text{(I.278)}$$

hold where $(B^T)_\perp$ is basis of the null-space of B^T.

Proof: *The proof is identical as for the instantaneous state feedback case.* □

Since the stabilizability conditions are expressed in terms of LMIs obtained from a specific Lyapunov-Krasovskii functional, stabilizability conditions are sufficient only. More accurate results can be obtained by choosing better functionals (i.e. more 'complete' ones).

I.2 Spectral Observability and Detectability

Similar results are provided here on spectral observability and detectability. Spectral observability is equivalent to the existence of an observer such that the poles of the observation error are in finite number and all located in the left-half complex plane.

The theorem on spectral observability is a direct extension of theorem (I.2) by duality.

Theorem I.7 *Time-delay (I.264) is spectrally observable if and only if the following statements hold:*

1. $\text{rank}\begin{bmatrix} sI - A - A_h^{-sh} \\ C \end{bmatrix} = n$ *forall* $s \in \ \{\tau \in \mathbb{C} : \ \det(\tau I - A - A_h^{-\tau h}) = 0\} \cap \{\tau \in \mathbb{C} :$ $\text{rank}\begin{bmatrix} C \\ C\tilde{A}(e^{-\tau h}) \\ \vdots \\ C\tilde{A}(e^{-\tau h})^{n-1} \end{bmatrix} < n\}$

2. $\text{rank}\begin{bmatrix} C \\ C\tilde{A}(\nabla) \\ \vdots \\ C\tilde{A}(\nabla)^{n-1} \end{bmatrix} = n$ *for all* $\nabla \in \mathbb{R}[\nabla]$ *is the ring of polynomials (weak R-observability).*

Whenever a time-delay system of the form (I.264) is spectrally observability then an observer of the form [Bhat and Koivo, 1976]

$$\dot{\hat{x}}(t) = A\hat{x}(t) + A_h\hat{x}(t-h) + \Phi(0)K(Cx(t) - y(t)) + A_h \int_0^h \Phi(\xi - h)K[Cx(t-\xi) - y(t-\xi)]d\xi \tag{I.279}$$

where K is the observer gain and $\Phi(\xi)$ is the matrix of eigenfunctions corresponding to operator $\mathscr{C} : x_t \to Ax(t) + A_hx(t-h)$. As for a distributed control law, the implementation of such an observer is difficult task. Moreover, spectral observability is the stronger result on observability. Indeed, many systems are not spectrally observable but are observable in the sense of the spectrum of the observation error is stable but infinite.

Since there exist a wide variety of observers we will consider here a simple extension of the Luenberger's observer taking the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + A_h\hat{x}(t-h) + K(y - C\hat{x}(t)) \tag{I.280}$$

where K is the observer gain.

In this case, the observation error $e(t) = x(t) - \hat{x}(t)$ is governed by the expression

$$\dot{e}(t) = (A - KC)e(t) + A_he(t-h) \tag{I.281}$$

which is a time-delay system itself.

In this case, we have the following theorem:

Theorem I.8 *There exists an observer of the form (I.280) if there exist* X, $P = P^T, Q = Q^T, R = R^T \succ 0$ *such that the LMIs*

$$\begin{bmatrix} -P+Q-R & R & 0 & 0 \\ \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & -P & h_{max}R \\ \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{I.282}$$

$$\mathcal{Z}^T \begin{bmatrix} -(X+X)^T & P+X^T(A-KC) & X^TA_h & X^T & h_{max}R \\ \star & -P+Q-R & R & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & \star & -P & h_{max}R \\ \star & \star & \star & \star & -R \end{bmatrix} \mathcal{Z} \prec 0 \tag{I.283}$$

hold with $\mathcal{Z} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Ker}[C] & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$.

Proof: *It will be shown later in Section (3.5-lemma 3.5.2, that (I.264) is asymptotically stable for all delays in* $[0, h_{max}]$ *if there exist* $X \in \mathbb{R}^{n\times n}$ *and* $P, Q, R \succ 0$ *such that*

$$\begin{bmatrix} -(X+X)^T & P+X^TA & X^TA_h & X^T & h_{max}R \\ \star & -P+Q-R & R & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & \star & -P & h_{max}R \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{I.284}$$

Then by substituting the expression of the estimation error (I.281) into the latter LMI, we get

$$\begin{bmatrix} -(X+X)^T & P+X^T(A-KC) & X^TA_h & X^T & h_{max}R \\ \star & -P+Q-R & R & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & \star & -P & h_{max}R \\ \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{I.285}$$

The latter LMI can be rewritten as

$$\begin{bmatrix} -(X+X)^T & P+X^TA & X^TA_h & X^T & h_{max}R \\ \star & -P+Q-R & R & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & \star & -P & h_{max}R \\ \star & \star & \star & \star & -R \end{bmatrix} + \begin{bmatrix} -I \\ 0 \\ 0 \\ 0 \\ 0 \\ \\ \end{bmatrix} (X^TK) \begin{bmatrix} 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} + (\star)^T \prec 0 \tag{I.286}$$

In virtue of the projection lemma, the existence of K is equivalent to the feasibility of the two underlying LMIs:

$$\begin{bmatrix} -P+Q-R & R & 0 & 0 \\ \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & -P & h_{max}R \\ \star & \star & \star & -R \end{bmatrix} \prec 0 \tag{I.287}$$

$$\mathcal{Z}^T \begin{bmatrix} -(X+X)^T & P+X^T(A-KC) & X^TA_h & X^T & h_{max}R \\ \star & -P+Q-R & R & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 \\ \star & \star & \star & -P & h_{max}R \\ \star & \star & \star & \star & -R \end{bmatrix} \mathcal{Z} \prec 0 \tag{I.288}$$

with $\mathcal{Z} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Ker}[C] & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$ $\square$

J Complements on Observation and Filtering of LPV Time-Delay Systems

This appendix aims to providing extra results on observation and filtering of (uncertain) LPV time-delay systems. These results complete the contents of Section 4.

J.1 Observation of unperturbed LPV Time-Delay Systems

This section provides the extension of Theorem 4.1.5 in the case of a discretized Lyapunov-Krasovskii functional.

J.1.1 Observer with exact delay value - discretized Lyapunov-Krasovskii functional case

Since the Lyapunov-Krasovskii functional used in Section 4.1.1 is rather simple, it may lead to conservative results. However, the obtained results are encouraging and motivates the results of this section in which the functional in use is a discretized version of this functional. We have the following result when using observer expression of Section 4.1.1 with the stability/performance result obtained from the use of the discretized Lyapunov-Krasovskii functional introduced in Section 3.6. The proof, residing in the application of Lemma 3.6.4 instead of Lemma 3.5.2, is very similar and is hence omitted.

Theorem J.1 *There exists a parameter dependent observer of the form (4.6) for system (4.1) such that theorem 4.1.2 for all $h \in \mathscr{H}_1^\circ$ is satisfied if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^r$, a matrix function $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, constant matrices $Q_i, R_i \in \mathbb{S}_{++}^r$, $i = 0, \ldots, N-1$, $X \in \mathbb{R}^{r\times r}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and a positive scalar $\gamma > 0$ such that the following matrix inequality*

$$\left[\begin{array}{cccc|ccc} -X^H & U_{12}(\rho) & 0 & X^T & \bar{h}_1 R_0 & \ldots & \bar{h}_1 R_{N-1} \\ \star & U_{22}(\rho,\nu) & U_{23}(\rho) & 0 & 0 & \ldots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \ldots & 0 \\ \star & \star & \star & -P(\rho) & -\bar{h}_1 R_0 & \ldots & -\bar{h}_1 R_{N-1} \\ \hline \star & \star & \star & \star & \multicolumn{3}{c}{-\operatorname{diag}_i R_i} \end{array}\right] \prec 0 \tag{J.289}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ and where

$$U_{22} = \left[\begin{array}{cccccc|c} U_{11}' & R_0 & 0 & 0 & \ldots & 0 & 0 \\ \star & N_1^{(1)} & R_1 & 0 & \ldots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \ldots & 0 & 0 & -\gamma I \end{array}\right] \tag{J.290}$$

$$\begin{array}{rcl} U_{11}' &=& \partial_\rho P(\rho)\dot{\rho} - P(\rho) + Q_0 - R_0 \\ N_i^{(1)} &=& -(1-i\mu_N)Q_{i-1} + (1+i\mu_N)Q_i - R_{i-1} - R_i, \quad i = 1,\ldots,N-1 \\ N^{(2)} &=& -(1-\mu)Q_{N-1} - R_{N-1} \\ U_{12}(\rho) &=& \left[\begin{array}{cccccc} P(\rho) + X^T\Theta(\rho) - \bar{L}(\rho)\Xi(\rho) & 0 & \ldots & 0 & X^T\Psi(\rho) - \bar{L}(\rho)\Omega(\rho) & (X^T T - \bar{H}C)E \end{array}\right] \\ U_{23}(\rho) &=& \left[\begin{array}{ccccc|c} I_r & 0 & \ldots & 0 & 0 & 0 \end{array}\right]^T \\ \bar{h} &=& h_{max}/N \\ \mu_N &=& \mu/N \end{array}$$

Moreover, the gain is given by $L(\rho) = X(\rho)^{-T}\bar{L}(\rho)$ and we have $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

J.2 Filtering of Observation of unperturbed LPV Time-Delay Systems

In order to filter or estimate internal signals and output signals of a system, it is also possible to design a filter. The main difference between a filter and an observer is that in the observer it is generally sought that the dynamical model of the observation error is independent of the system state (at least in the ideal case). Moreover, in the control framework, it is generally difficult to elaborate a control law from a filter while using an observer it is not. However, the design of a filter is generally more simple than an observer for different reasons.

The first one in the smaller number of computations by hand that are needed to obtain the extended system. In the observer design, solutions of algebraic matrix equations have to be expressed while for the filter design, no matrices need to be computed.

The second one is that the structure of the extended system containing both system state and filter state allows for a genuine non conservative change of variables [Tuan et al., 2001, 2003].

The third and last reason, is the wide adaptability of filters to any types of time-delay systems. Indeed, in the observer approach presented in Section 4.1, the output matrix C must be parameter independent for theoretical reason, it is difficult to consider a disturbance on the measured output and the delayed state must not affect the measurements. This restricts considerably the field of action of the approach by imposing a strong structure for the measurement process.

Among the types of filters it is possible to isolate two main classes, as for the observers, according to the dependence or not of the filter on the delay. Filters involving a delayed-state are called *filters with memory* while others are called *memoryless*. Both cases will be detailed in what follows.

In the filtering section, the following general LPV time-delay system will be considered:

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho)x(t) + A_h(\rho)x(t-h(t)) + E(\rho)w(t) \\ z(t) & = & C(\rho)x(t) + C_h(\rho)x(t-h(t)) + F(\rho)w(t) \\ y(t) & = & C_y(\rho)x(t) + C_{yh}(\rho)x(t-h(t)) + F_y(\rho)w(t) \end{array} \tag{J.291}$$

where $x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ w \in \mathbb{R}^q,\ z \in \mathbb{R}^r$ are respectively the system state, the observer state, the system measurements, the system control input, the system exogenous inputs, the signal to be estimated and its estimate. The time-varying delay is assumed to belong to the set $\mathscr{H}_1^\circ$.

The general filter is given by

$$\begin{array}{rcl} \dot{x}_F(t) & = & A_F(\rho)x_F(t) + A_{Fh}(\rho)x_{Fh}(t-d(t)) + B_F(\rho)y(t) \\ z_F(t) & = & C_F(\rho)x_F(t) + C_{Fh}(\rho)x_{Fh}(t-d(t)) + D_F(\rho)y(t) \end{array} \tag{J.292}$$

where $x_F \in \mathbb{R}^k$ and $z_F \in \mathbb{R}^t$. When $k = n$ the filter is said to be of *full-order* while when $k < n$, the filter is said to be of *reduced-order*. Moreover, it may happen that the delay used in the filter is not identical to the real delay involved in the dynamical model of the system, this motivates the use of the delay $d(t)$ in the filter model. If the filter involves a delay part, it is called *filter with memory* while if it does not, it is called *memoryless filter*.

Problem J.2 *Find all matrices $A_F(\rho), B_F(\rho), C_F(\rho), F_F(\rho)$ (with $A_{Fh}(\rho)$ and $C_{Fh}(\rho)$ in the memory case) which minimizes the impact of the disturbances over the error $e(t) = z(t) - z_F(t)$ in a $\mathcal{L}_2$ framework, that is, we aim at minimizing $\gamma > 0$ such that*

$$||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

holds.

J.2.1 Design of a filter with memory with exact delay-value

The more general filter is the 'complete' filter involving information on the delay through a delayed part in its model and we suppose here that $d(t) = h(t)$ all the time. In this case the extended system, which is the combination of the system and the observer, is given by:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_F(t) \\ \hline e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) \\ \hline \mathcal{C}(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + \begin{bmatrix} \mathcal{A}_h(\rho) \\ \hline \mathcal{C}_h(\rho) \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} + \begin{bmatrix} \mathcal{E}(\rho) \\ \hline \mathcal{F}(\rho) \end{bmatrix} w(t) \quad \text{(J.293)}$$

where

$$\begin{aligned} \mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ B_F(\rho)C_y(\rho) & A_F(\rho) \end{bmatrix} \\ \mathcal{C}(\rho) &= \begin{bmatrix} C(\rho) - D_F(\rho)C_y(\rho) & -C_F(\rho) \end{bmatrix} \\ \mathcal{A}_h(\rho) &= \begin{bmatrix} A_F(\rho) & 0 \\ B_F(\rho)C_{yh}(\rho) & A_{Fh}(\rho) \end{bmatrix} \\ \mathcal{C}_h(\rho) &= \begin{bmatrix} C_h(\rho) - D_F(\rho)C_{yh}(\rho) & -C_{Fh}(\rho) \end{bmatrix} \\ \mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) \\ B_F(\rho)F_y(\rho) \end{bmatrix} \\ \mathcal{F}(\rho) &= \begin{bmatrix} -D_F(\rho)F_y(\rho) \end{bmatrix} \end{aligned}$$

From this extended model, it is possible to derive sufficient conditions for the existence of a suitable filters. To obtain them, it suffices to substitute the extended system expression into the LMI conditions of Lemma 3.5.2 which has been obtained by using a simple Lyapunov-Krasovskii functional.

Theorem J.3 *There exists a full-order filter of the form (J.292) with $d(t) = h(t)$, $h(t) \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $\tilde{P} : U_\rho \to \mathbb{S}_{++}^{2n}$, symmetric matrices $\tilde{Q}, \tilde{R} \in \mathbb{S}_{++}^{2n}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, matrix functions $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ and a scalar $\gamma > 0$ such that the LMI*

$$\begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho) + \tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}\tilde{R} \\ \star & \tilde{\Psi}_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)\tilde{Q} - \tilde{R} & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\ \star & \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} \prec 0 \quad \text{(J.294)}$$

holds for all $\rho \in U_\rho$ *with*

$$\begin{array}{lcl}
\Psi_{22}(\rho,\nu) & = & \partial_\rho \tilde{P}(\rho)\nu - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} \\
\tilde{P}(\rho) & = & \tilde{X}^T P(\rho)\tilde{X} \\
\tilde{Q} & = & \tilde{X}^T Q\tilde{X} \\
\tilde{R} & = & \tilde{X}^T R\tilde{X} \\
\hat{X} & = & \left[\begin{array}{cc} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{array}\right] \\
\tilde{\mathcal{A}}(\rho) & = & \left[\begin{array}{cc} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{array}\right] \\
\tilde{\mathcal{A}}_h(\rho) & = & \left[\begin{array}{cc} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{array}\right] \\
\tilde{\mathcal{E}}(\rho) & = & \left[\begin{array}{c} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \end{array}\right] \\
\tilde{\mathcal{C}}(\rho)^T & = & \left[\begin{array}{c} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{array}\right] \\
\tilde{\mathcal{C}}_h(\rho)^T & = & \left[\begin{array}{c} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{array}\right] \\
\hat{X}_3 & = & U^T \Sigma U \qquad \text{(using SVD see Appendix A.6)}
\end{array}$$

Moreover the filter matrices are computed using

$$\left[\begin{array}{ccc} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{array}\right] = \left[\begin{array}{ccc} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{array}\right]$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *Substituting the model (J.293) into LMI (3.95) we get*

$$\left[\begin{array}{ccccccc}
-X^H & P(\rho) + X^T\mathcal{A}(\rho) & X^T\mathcal{A}_h(\rho) & X^T\mathcal{E}(\rho) & 0 & X^T & h_{max}R \\
\star & \Psi_{22}(\rho,\nu) & R & 0 & \mathcal{C}(\rho)^T & 0 & 0 \\
\star & \star & -(1-\mu)Q - R & 0 & \mathcal{C}_h(\rho)^T & 0 & 0 \\
\star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I_r & 0 & 0 \\
\star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\
\star & \star & \star & \star & \star & \star & -R
\end{array}\right] \prec 0 \qquad \text{(J.295)}$$

with

$$\Psi_{22}(\rho,\nu) = \partial_\rho P(\rho)\nu - P(\rho) + Q - R \qquad \text{(J.296)}$$

Define the matrix $\tilde{X} = \begin{bmatrix} I_n & 0 \\ 0 & X_4^{-1}X_3 \end{bmatrix}$ *then we have*

$$\begin{array}{rclcl}
\tilde{\mathcal{A}}(\rho) & = & \tilde{X}^T X^T \mathcal{A}(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) & = & \tilde{X}^T X^T \mathcal{A}_h(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) & = & \tilde{X}^T X^T \mathcal{E}(\rho)\tilde{X} & = & \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T & = & \tilde{X}^T \mathcal{C}(\rho) & = & \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T & = & \tilde{X}^T \mathcal{C}_h(\rho) & = & \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
 & & \text{where} \hat{X} & = & \tilde{X}^T X \tilde{X} \quad = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_3^T X_4^{-1} X_3 \end{bmatrix} \\
\tilde{A}_F(\rho) & = & X_3^T A_F(\rho) X_4^{-1} X_3 & & \\
\tilde{A}_{Fh}(\rho) & = & X_3^T A_{Fh}(\rho) X_4^{-1} X_3 & & \\
\tilde{B}_F(\rho) & = & X_3^T B_F(\rho) & & \\
\tilde{C}_F(\rho) & = & C_F(\rho) X_4^{-1} X_3 & & \\
\tilde{C}_{Fh}(\rho) & = & C_{Fh}(\rho) X_4^{-1} X_3 & &
\end{array}$$

Then perform a congruence transformation on (J.295) with respect to $\mathrm{diag}(\tilde{X}, \tilde{X}, \tilde{X}, I_q, I_r, \tilde{X}, \tilde{X})$ *we get LMI*

$$\begin{bmatrix}
-\hat{X}^H & \tilde{P}(\rho) + \tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}\tilde{R} \\
\star & \tilde{\Psi}_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\
\star & \star & -(1-\mu)\tilde{Q} - \tilde{R} & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\
\star & \star & \star & -\gamma I_q & \mathcal{F}(\rho)^T & 0 & 0 \\
\star & \star & \star & \star & -\gamma I_r & 0 & 0 \\
\star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R} \\
\star & \star & \star & \star & \star & \star & -\tilde{R}
\end{bmatrix} \prec 0$$

with

$$\begin{array}{rcl}
\tilde{\Psi}_{22}(\rho,\nu) & = & \partial_\rho \tilde{P}(\rho)\nu - \tilde{P}(\rho) + \tilde{Q} - \tilde{R} \\
\tilde{P}(\rho) & = & \tilde{X}^T P(\rho)\tilde{X} \\
\tilde{Q} & = & \tilde{X}^T Q \tilde{X} \\
\tilde{R} & = & \tilde{X}^T R \tilde{X}
\end{array}$$

which is exactly (J.294). Now let us focus on the computation of the filter matrices. Note that

$$\begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ \tilde{C}_F(\rho) & \tilde{C}_{Fh}(\rho) & \tilde{D}_F(\rho) \end{bmatrix} = \begin{bmatrix} X_3^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} X_4^{-1}X_3 & 0 & 0 \\ 0 & X_4^{-1}X_3 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{J.297}$$

Thus it suffices to construct back the matrix X *in order to compute the observer gain. A singular values decomposition (SVD, see Appendix A.6) on* $\hat{X}_3$ *allows to compute the matrices*

X_3 and X_4 which are necessary to construct the filter matrices. Indeed, we have $\hat{X}_3 = U^T \Sigma U$ and hence

$$\begin{array}{rcl} X_4 & = & \Sigma^{-1} \\ X_3 & = & U \end{array}$$

and finally we have

$$\begin{aligned} \begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} &= \begin{bmatrix} U^T & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_F(\rho) & \tilde{A}_{Fh}(\rho) & \tilde{B}_F(\rho) \\ \tilde{C}_F(\rho) & \tilde{C}_{Fh}(\rho) & \tilde{D}_F(\rho) \end{bmatrix} \begin{bmatrix} \Sigma U & 0 & 0 \\ 0 & \Sigma U & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix} \end{aligned}$$

□

In order to improve the latter result, we give here its extension using a discretized Lyapunov-Krasovskii functional. The following result is obtained using the same method as for Theorem J.3 but using lemma 3.6.4.

Theorem J.4 *There exists a full-order filter of the form (J.292) with $d(t) = h(t)$, $h(t) \in \mathscr{H}_1^\circ$ if there exists a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^{2n}$, symmetric matrices $\tilde{Q}_i, \tilde{R}_i \in \mathbb{S}_{++}^{2n}$, $i = 0, \ldots, N-1$, $\hat{X} \in \mathbb{R}^{2n \times 2n}$, matrix functions $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ and a scalar $\gamma > 0$ such that the LMI*

$$\left[\begin{array}{cccc|ccc} -\hat{X}^H & \tilde{U}_{12}(\rho) & 0 & \hat{X}^T & \bar{h}_1\tilde{R}_0 & \ldots & \bar{h}_1\tilde{R}_{N-1} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{U}_{23}(\rho) & 0 & 0 & \ldots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \ldots & 0 \\ \star & \star & \star & -\tilde{P}(\rho) & -\bar{h}_1\tilde{R}_0 & \ldots & -\bar{h}_1\tilde{R}_{N-1} \\ \hline \star & \star & \star & \star & \multicolumn{3}{c}{-\operatorname{diag}_i \tilde{R}_i} \end{array}\right] \prec 0 \qquad \text{(J.298)}$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ and where

$$U_{22} = \left[\begin{array}{cccccc|c} U'_{11} & \tilde{R}_0 & 0 & 0 & \ldots & 0 & 0 \\ \star & \tilde{N}_1^{(1)} & \tilde{R}_1 & 0 & \ldots & 0 & 0 \\ \star & \star & \tilde{N}_2^{(1)} & \tilde{R}_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \tilde{R}_{N-1} & 0 \\ & & & & & \tilde{N}^{(2)} & 0 \\ \hline \star & \star & \star & \ldots & 0 & 0 & -\gamma I \end{array}\right] w \qquad \text{(J.299)}$$

$$
\begin{array}{rcl}
\tilde{U}'_{11} & = & \partial_\rho \tilde{P}(\rho)\dot{\rho} - \tilde{P}(\rho) + \tilde{Q}_0 - \tilde{R}_0 \\
\tilde{N}_i^{(1)} & = & -(1-i\mu_N)\tilde{Q}_{i-1} + (1+i\mu_N)\tilde{Q}_i - \tilde{R}_{i-1} - \tilde{R}_i \\
\tilde{N}^{(2)} & = & -(1-\mu)\tilde{Q}_{N-1} - \tilde{R}_{N-1} \\
\tilde{U}_{12}(\rho) & = & \left[\begin{array}{cccccccc} P(\rho)+\tilde{\mathcal{A}}(\rho) & 0 & 0 & \tilde{\mathcal{A}}_h(\rho) & \dots & 0 & \tilde{\mathcal{E}}(\rho) \end{array}\right] \\
\tilde{U}_{23}(\rho) & = & \left[\begin{array}{ccccc|c} \tilde{\mathcal{C}}(\rho) & 0 & \dots & 0 & \tilde{\mathcal{C}}_h(\rho) & \mathcal{F}(\rho) \end{array}\right]^T \\
\hat{X} & = & \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) & = & \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) & = & \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) & = & \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T & = & \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T & = & \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\hat{X}_3 & = & U^T \Sigma U \qquad \text{(SVD)}
\end{array}
$$

Moreover the filter matrices are computed using

$$
\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}
$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

J.2.2 Design of a filter with memory with approximate delay-value

It is of interest to deal with a filter involving a delay which is different from the plant, that is $d(t) \neq h(t)$ most of the time. According to this assumption, the extended system takes the following form

$$
\begin{aligned}
\left[\begin{array}{c} \dot{x}(t) \\ \dot{x}_F(t) \\ \hline e(t) \end{array}\right] = {} & \left[\begin{array}{c} \mathcal{A}(\rho) \\ \hline \mathcal{C}(\rho) \end{array}\right] \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + \left[\begin{array}{c} \mathcal{A}_h(\rho) \\ \hline \mathcal{C}_h(\rho) \end{array}\right] \begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} \\
& + \left[\begin{array}{c} \mathcal{A}_d(\rho) \\ \hline \mathcal{C}_d(\rho) \end{array}\right] \begin{bmatrix} x(t-d(t)) \\ x_F(t-d(t)) \end{bmatrix} + \left[\begin{array}{c} \mathcal{E}(\rho) \\ \hline \mathcal{F}(\rho) \end{array}\right] w(t)
\end{aligned} \tag{J.300}
$$

where

$$
\begin{aligned}
\mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ B_F(\rho)C_y(\rho) & A_F(\rho) \end{bmatrix} \\
\mathcal{C}(\rho) &= \begin{bmatrix} C(\rho) - F_F(\rho)C_y(\rho) & -C_F(\rho) \end{bmatrix} \\
\mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho) & 0 \\ B_F(\rho)C_{yh}(\rho) & 0 \end{bmatrix} \\
\mathcal{C}_h(\rho) &= \begin{bmatrix} C_h(\rho) - F_F(\rho)C_{yh}(\rho) & 0 \end{bmatrix} \\
\mathcal{A}_d(\rho) &= \begin{bmatrix} 0 & 0 \\ 0 & A_{Fh}(\rho) \end{bmatrix} \\
\mathcal{C}_d(\rho) &= \begin{bmatrix} 0 & -C_{Fh}(\rho) \end{bmatrix} \\
\mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) \\ B_F(\rho)F_y(\rho) \end{bmatrix} \\
\mathcal{F}(\rho) &= \begin{bmatrix} -F_F(\rho)F_y(\rho) \end{bmatrix}
\end{aligned}
$$

The following result is provided through the use of Theorem 3.7.2 which is the relaxation of 3.7.1 and considers the stability and $\mathcal{L}_2$ performances of systems of the form (J.300).

Theorem J.5 *There exists a filter of the form (J.300) for system (J.291) with* $d(t) = h(t)$, $h(t) \in \mathscr{H}_1^\circ$ *if there exists a continuously differentiable matrix function* $\tilde{P} : U_\rho \to \mathbb{S}_{++}^{2n}$, *symmetric matrices* $\tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2 \in \mathbb{S}_{++}^{2n}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, *matrix functions* $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ *and a scalar* $\gamma > 0$ *such that the LMIs*

$$
\begin{bmatrix}
-\hat{X}^H & \tilde{P}(\rho) + \tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) + \tilde{\mathcal{A}}_d(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}(\rho)^T & h_{max}\tilde{R}_1 & \tilde{R}_2 \\
\star & \tilde{\Theta}_{11}(\rho,\nu) & \tilde{R}_1 & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 & 0 \\
\star & \star & \tilde{\Theta}_{22} & 0 & \tilde{\mathcal{C}}_h(\rho)^T + \tilde{\mathcal{C}}_d(\rho)^T & 0 & 0 & 0 \\
\star & \star & \star & -\gamma I & \mathcal{F}(\rho)^T & 0 & 0 & 0 \\
\star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 \\
\star & \star & \star & \star & \star & \star & -\tilde{R}_1 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta}
\end{bmatrix} \prec 0 \tag{J.301}
$$

and

$$
\begin{bmatrix} \tilde{\Pi}_{11}(\rho,\nu) & \tilde{\Pi}_{12}(\rho) \\ \star & \tilde{\Pi}_{22}(\rho) \end{bmatrix} \prec 0 \tag{J.302}
$$

hold for all $(\rho,\nu)\in U_\rho\times U_\nu$ *and where*

$$\begin{aligned}
\tilde{\Pi}_{11}(\rho,\nu) &= \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho)+\tilde{\mathcal{A}} & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{A}}_d(\rho) & \tilde{\mathcal{E}}(\rho) \\ \star & \tilde{\Theta}_{11}(\rho,\nu) & \tilde{R}_1 & 0 & 0 \\ \star & \star & \tilde{\Psi}_{22} & (1-\mu)\tilde{R}_2/\delta & 0 \\ \star & \star & \star & \tilde{\Psi}_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} \\
\tilde{\Pi}_{12}(\rho) &= \begin{bmatrix} 0 & \hat{X}^T & h_{max}\tilde{R}_1 & \tilde{R}_2 \\ \tilde{\mathcal{C}}(\rho)^T & 0 & 0 & 0 \\ \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 & 0 \\ \tilde{\mathcal{C}}_d(\rho)^T & 0 & 0 & 0 \\ \mathcal{F}(\rho)^T & 0 & 0 & 0 \\ & & & \end{bmatrix} \\
\tilde{\Pi}_{22}(\rho) &= \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 \\ \star & \star & -\tilde{R}_1 & 0 \\ \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} \end{bmatrix} \\
\tilde{\Theta}_{11}(\rho,\nu) &= -\tilde{P}(\rho)+\tilde{Q}_1+\tilde{Q}_2+\sum_{i=1}^{N}\frac{\partial\tilde{P}}{\partial\rho_i}\nu_i-\tilde{R}_1 \\
\tilde{\Theta}_{22} &= -(1-\mu)(\tilde{Q}_1+\tilde{Q}_2)-\tilde{R}_1 \\
\tilde{\Psi}_{22} &= -(1-\mu)(\tilde{Q}_1+\tilde{R}_2/\delta)-\tilde{R}_1 \\
\tilde{\Psi}_{33} &= -(1-\mu_c)\tilde{Q}_2-(1-\mu)\tilde{R}_2/\delta \\
\hat{X} &= \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} \hat{X}_1^T A(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho)+\tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} \hat{X}_1^T A_h(\rho)+\tilde{B}_F(\rho)C_{yh}(\rho) & 0 \\ \hat{X}_2^T A_h(\rho)+\tilde{B}_F(\rho)C_{yh}(\rho) & 0 \end{bmatrix} \\
\tilde{\mathcal{A}}_d(\rho) &= \begin{bmatrix} 0 & \tilde{A}_{Fh}(\rho) \\ 0 & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) &= \begin{bmatrix} \hat{X}_1 E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho)+\tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T &= \begin{bmatrix} C(\rho)^T-C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T &= \begin{bmatrix} C_h(\rho)^T-C_{hy}(\rho)^T D_F(\rho)^T \\ 0 \end{bmatrix} \\
\tilde{\mathcal{C}}_d(\rho)^T &= \begin{bmatrix} 0 \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\hat{X}_3 &= U^T\Sigma U \qquad \text{(SVD)}
\end{aligned}$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

J.2.3 Design of memoryless filter

A memoryless filter is a filter embedding no information on the delay involved in the system. Even if such a filter leads to worse performances than the filter with memory, it is of interest whenever no information on the current delay value is available. A memoryless filter can be obtained by setting $A_{Fh}(\cdot) = 0$ and $C_{Fh}(\cdot) = 0$ in filter model (J.293).

In this case, the extended system writes:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_F(t) \\ \hline e(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) \\ \hline \mathcal{C}(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + \begin{bmatrix} \mathcal{A}_h(\rho) \\ \hline \mathcal{C}_h(\rho) \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} + \begin{bmatrix} \mathcal{E}(\rho) \\ \hline \mathcal{F}(\rho) \end{bmatrix} w(t) \quad \text{(J.303)}$$

where

$$\begin{aligned} \mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ B_F(\rho)C_y(\rho) & A_F(\rho) \end{bmatrix} \\ \mathcal{C}(\rho) &= \begin{bmatrix} C(\rho) - F_F(\rho)C_y(\rho) & -C_F(\rho) \end{bmatrix} \\ \mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho) \\ B_F(\rho)C_{yh} \end{bmatrix} Z \\ \mathcal{C}_h(\rho) &= [C_h(\rho) - F_F(\rho)C_{yh}(\rho)]\, Z \\ \mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) \\ B_F(\rho)F_y(\rho) \end{bmatrix} \\ \mathcal{F}(\rho) &= \begin{bmatrix} -F_F(\rho)F_y(\rho) \end{bmatrix} \\ Z &= \begin{bmatrix} I_n & 0 \end{bmatrix} \end{aligned}$$

Theorem J.6

$$\begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}^T & h_{max}Z^TR \\ \star & \tilde{\Psi}'_{22}(\rho,\nu) & R & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 \\ \star & \star & \star & -\gamma I_m & \mathcal{F}(\rho)^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_p & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}Z^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \prec 0 \quad \text{(J.304)}$$

holds for all $(\rho, \nu) \in U_\rho \times U_\nu$ *where*

$$\begin{array}{rcl}
\tilde{\Psi}'_{22}(\rho,\nu) & = & \partial_\rho \tilde{P}(\rho)\nu - \tilde{P}(\rho) + Z^T(Q(\rho) - R)Z \\
\hat{X} & = & \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) & = & \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) & = & \begin{bmatrix} \hat{X}_1^T A_h + \tilde{B}_F(\rho)C_{yh}(\rho) \\ \hat{X}_2^T A_h + \tilde{B}_F(\rho)C_{yh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) & = & \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T & = & \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T & = & C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\
\hat{X}_3 & = & U^T \Sigma U \qquad \text{(SVD)}
\end{array}$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

Proof: *Due to the structure of matrices* $\mathcal{A}_h$ *and* $\mathcal{C}_h$, *this suggests that lemma 3.5.4 would be appropriate for this type of problem as shown in [Zhang and Han, 2008]. This lemma is the immediate LMI result obtained from the Lyapunov-Krasovskii functional. Actually, due to the specific structure of matrices acting on the delayed state, no multiple coupling and congruence transformations [Tuan et al., 2001, 2003, Zhang and Han, 2008] can be directly applied without creating nonlinear terms. Since the congruence transformation is performed using blocks of the* P *matrix involving products and inverse of the, it cannot be applied on time-varying matrices. This is the reason why lemma 3.5.5, which is the relaxed version of Lemma 3.5.4, should be considered instead.*

The remaining of the proof is similar to proofs of previous results, except that the congruence transformation is performed here with respect to the matrix

$$\mathrm{diag}(\tilde{X}, \tilde{X}, \tilde{X}, I_m, I_p, \tilde{X}, I_n)$$

and we have

$$\tilde{\mathcal{A}}_h = \tilde{X}\mathcal{A}_h \begin{bmatrix} A_h(\rho) \\ -B_F(\rho)C_{yh}(\rho) \end{bmatrix} = \begin{bmatrix} \hat{X}_1^T A_h + \tilde{B}_F(\rho)C_{yh}(\rho) \\ \hat{X}_2^T A_h + \tilde{B}_F(\rho)C_{yh}(\rho) \end{bmatrix}$$

Other matrices remain unchanged. □

J.3 Observation of uncertain LPV Time-Delay Systems

Although observation and filtering of certain systems is an interesting and still an open problem, it is more relevant, from a practical point of view, to consider the wide class of uncertain LPV time-delay systems. Indeed, systems are generally not known completely since some modeling errors, nonlinear terms (and so on...) remain and are neglected. The robust observation and filtering problems aim at providing conditions for which an observer or filter exist, achieve some performances even in presence of such uncertainties. This section is devoted to such a problem. According to the type of process to be designed, the class of systems under consideration differs and hence specific types of problems will be introduced at the beginning of each section.

In this section the following class of uncertain LPV time-delay systems is considered

$$\begin{array}{rcl} \dot{x}(t) & = & (A(\rho)+\Delta A(\rho,t))x(t)+(A_h(\rho)+\Delta A_h(\rho,t))x(t-h(t)) \\ & & +(E(\rho)+\Delta E(\rho,t))w(t) \\ y(t) & = & Cy(t) \\ z(t) & = & Tx(t) \end{array} \tag{J.305}$$

for which the the following observer aims to be designed

$$\begin{array}{rcl} \dot{\xi}(t) & = & M_0(\rho)\xi(t)+M_h(\rho)\xi(t-d(t))+N_0(\rho)y(t)+N_h(\rho)y(t-h(t))+S(\rho)u(t) \\ \hat{z}(t) & = & \xi(t)+Hy(t) \end{array} \tag{J.306}$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^r$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$, $w \in \mathbb{R}^q$, $z \in \mathbb{R}^r$ and $\hat{z} \in \mathbb{R}^r$ are respectively the system state, the observer state, the system measurements, the system control input, the system exogenous inputs, the signal to be estimated and its estimate. The time-varying delay is assumed to belong to the set $\mathscr{H}_1^\circ$.

The uncertain terms are assumed to obey the following relations

$$\left[\begin{array}{ccc} \Delta A(\rho,t) & \Delta A_h(\rho,t) & \Delta E(\rho,t) \end{array}\right] = G(\rho)\Delta(t)\left[\begin{array}{ccc} H_A(\rho) & H_{A_h}(\rho) & H_E(\rho) \end{array}\right] \tag{J.307}$$

where $\Delta(t)^T\Delta(t) \preceq I$ for all $t \geq 0$.

J.3.1 Robust observer with exact delay value - simple Lyapunov-Krasovskii functional

The key idea for designing such an observer for uncertain systems is roughly the same as for the certain system case. The observer matrices are sought in order to compensate known matrices of the systems acting on the state of the system. The main difference compared to the certain system case is the presence of uncertain terms that will remain in the equation of the dynamical model of the observation error. That is we obtain the following extended model:

$$\begin{array}{rcl} \left[\begin{array}{c} \dot{x}(t) \\ \dot{e}(t) \end{array}\right] & = & (\mathcal{A}(\rho)+\Delta(t)\mathcal{A}(\rho))\left[\begin{array}{c} x(t) \\ e(t) \end{array}\right]+(\mathcal{A}_h(\rho)+\Delta(t)\mathcal{A}_h(\rho))\left[\begin{array}{c} x(t-h(t)) \\ e(t-h(t)) \end{array}\right] \\ & & +(\mathcal{E}(\rho)+\Delta(t)\mathcal{E}(\rho))w(t) \\ z_{obs}(t) & = & \mathcal{I}\left[\begin{array}{c} x(t) \\ e(t) \end{array}\right] \end{array} \tag{J.308}$$

with

$$\begin{array}{rcl}
\mathcal{A}(\rho) & = & \begin{bmatrix} A(\rho) & 0 \\ 0 & M_0(\rho) \end{bmatrix} \\
\Delta\mathcal{A}(\rho) & = & \begin{bmatrix} \Delta A(\rho) & 0 \\ (T-HC)\Delta A(\rho) & 0 \end{bmatrix} = \begin{bmatrix} G(\rho) \\ (T-HC)G(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_A(\rho) & 0 \end{bmatrix} \\
\mathcal{A}_h(\rho) & = & \begin{bmatrix} A_h(\rho) & 0 \\ 0 & M_h(\rho) \end{bmatrix} \\
\Delta\mathcal{A}_h(\rho) & = & \begin{bmatrix} \Delta A_h(\rho) & 0 \\ (T-HC)\Delta A_h(\rho) & 0 \end{bmatrix} = \begin{bmatrix} G(\rho) \\ (T-HC)G(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_{A_h}(\rho) & 0 \end{bmatrix} \\
\mathcal{E}(\rho) & = & \begin{bmatrix} E(\rho) \\ (T-HC)E(\rho)) \end{bmatrix} \\
\Delta\mathcal{E}(\rho) & = & \begin{bmatrix} \Delta E(\rho) \\ (T-HC)\Delta E(\rho) \end{bmatrix} = \begin{bmatrix} G(\rho) \\ (T-HC)G(\rho) \end{bmatrix} \Delta(t) H_E(\rho) \\
\mathcal{I} & = & \begin{bmatrix} 0 & I_r \end{bmatrix}
\end{array} \tag{J.309}$$

and under conditions

$$\begin{array}{rcl}
(T-HC)A(\rho) - M_0(\rho)(T-HC) - N_0(\rho)C & = & 0 \\
(T-HC)A_h(\rho) - M_h(\rho)(T-HC) - N_h(\rho)C & = & 0
\end{array}$$

The conditions of existence of such an observer are the same as for the observer designed in Section 4.1.1 and hence intermediate results will not be recalled here. The only condition differing from observer of Section 4.1.1, is the LMI condition which is given in the following theorem:

Theorem J.7 *There exists a robust observer of the form (J.306) if there exist a continuously differentiable matrix $P : U_\rho \to \mathbb{S}_{++}^{n+t}$, matrix functions $X_1 : U_\rho \to \mathbb{R}^{n\times n}$, $X_2 : U_\rho \to \mathbb{R}^{t\times n}$ and $X_3 : U_\rho \to \mathbb{R}^{t\times t}$, $Z : U_\rho \to \mathbb{R}^{r\times 2r+3m}$, constant matrices $Q, R \in \mathbb{S}_{++}^{n+t}$, $\bar{H} \in \mathbb{R}^{r\times m}$ and scalars $\gamma, \varepsilon > 0$ such that the LMI*

$$\begin{bmatrix} \tilde{\mathcal{M}}(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \tilde{\mathcal{G}}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *where*

$$
\begin{array}{rcl}
\tilde{\mathcal{M}}(\rho,\nu) & = & \left[\begin{array}{ccccccc}
-X(\rho)^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & X(\rho)^T & h_{max}R \\
\star & U_{22}(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & \\
\star & \star & U_{33} & 0 & 0 & 0 & \\
\star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\
\star & \star & \star & \star & -\gamma I_r & 0 & 0 \\
\star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\
\star & \star & \star & \star & \star & \star & -R
\end{array}\right] \\
\mathcal{G}(\rho)^T & = & \left[\begin{array}{c}
X_1(\rho)^T G(\rho) \\
X_1(\rho)^T G(\rho) + (X_3(\rho)^T T - \bar{H}C)G(\rho) \\ \hline
0 \\ 0 \\ \hline
0 \\ 0 \\ \hline
0 \\ \hline
0 \\ \hline
0 \\ 0 \\ \hline
0 \\ 0
\end{array}\right] \quad \mathcal{H}(\rho)^T = \left[\begin{array}{c}
0 \\ 0 \\ \hline
H_A(\rho)^T \\ 0 \\ \hline
H_{A_h}(\rho)^T \\ 0 \\ \hline
H_E(\rho)^T \\ \hline
0 \\ \hline
0 \\ 0 \\ \hline
0 \\ 0
\end{array}\right] \\
X(\rho) & = & \left[\begin{array}{cc} X_1(\rho) & X_2(\rho) \\ 0 & X_3(\rho) \end{array}\right] \\
\tilde{\mathcal{A}}(\rho) & = & X(\rho)^T \mathcal{A}(\rho) = \left[\begin{array}{cc} X_1(\rho)^T A(\rho) & 0 \\ X_2(\rho)^T A(\rho) & X_3(\rho)^T \Theta(\rho) - \bar{L}(\rho)\Xi(\rho) \end{array}\right] \\
\tilde{\mathcal{A}}_h(\rho) & = & X(\rho)^T \mathcal{A}_h(\rho) = \left[\begin{array}{cc} X_1(\rho)^T A_h(\rho) & 0 \\ X_2(\rho)^T A_h(\rho) & X_3(\rho)^T \Upsilon(\rho) - \bar{L}(\rho)\Omega(\rho) \end{array}\right] \\
\tilde{\mathcal{E}}(\rho) & = & X(\rho)^T \mathcal{E}(\rho) = \left[\begin{array}{c} X_1(\rho)^T E(\rho) \\ X_2(\rho)^T E(\rho) + (X_3(\rho)^T T^T - C^T \bar{H}^T)E(\rho) \end{array}\right] \\
\bar{L}(\rho) & = & (X_3(\rho)^T \Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \\
U_{22}(\rho,\nu) & = & -P(\rho) + Q - R + \nu\partial_\rho P(\rho) \\
U_{33} & = & -(1-\mu)Q - R
\end{array}
$$

and matrices $\Phi(\rho), \Psi(\rho), \Omega(\rho), \Xi(\rho), \Upsilon(\rho), \Theta(\rho)$ *are defined in Lemma 4.1.4.*

Moreover, the generalized observer matrix gain $L(\rho)$ *is given by*

$$L(\rho) = X_3^{-T} \bar{L}(\rho) \tag{J.310}$$

and the extended system satisfies

$$||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$$

for all $\Delta(t)$ *such that* $\Delta(t)^T \Delta(t) \preceq I$.

Proof: *By substitution of the matrices of system (J.308) into LMI (3.95) of Lemma 3.5.2, which is a relaxed version of stability/performance lemma developed using a simple Lyapunov-Krasovskii functional, and choosing* $X = \left[\begin{array}{cc} X_1(\rho) & X_2(\rho) \\ 0 & X_3(\rho) \end{array}\right]$, *we obtain the following matrix inequality*

$$\mathcal{M}(\rho,\nu) + \mathcal{G}(\rho)^T \Delta(t) \mathcal{H}(\rho) + \mathcal{H}(\rho)^T \Delta(t) \mathcal{G}(\rho) \prec 0 \tag{J.311}$$

where

$$\mathcal{M}(\rho,\nu) = \begin{bmatrix} -(X+X^T) & P(\rho)+X(\rho)^T\mathcal{A}(\rho) & X(\rho)^T\mathcal{A}_h(\rho) & X(\rho)^T\mathcal{E}(\rho) & 0 & X(\rho)^T & h_{max}R \\ \star & U_{22}(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & \\ \star & \star & U_{33} & 0 & 0 & 0 & \\ \star & \star & \star & -\gamma I_q & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix}$$

$$\mathcal{G}(\rho)^T = \begin{bmatrix} X_1(\rho)^T G(\rho) \\ X_1(\rho)^T G(\rho) + X_3(\rho)^T(T-HC)G(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix} \qquad \mathcal{H}(\rho)^T = \begin{bmatrix} 0 \\ 0 \\ \hline H_A(\rho)^T \\ 0 \\ \hline H_{A_h}(\rho)^T \\ 0 \\ \hline H_E(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix}$$

and

$$\begin{array}{lcl} U_{22}(\rho,\nu) & = & -P(\rho)+Q-R+\nu\partial_\rho P(\rho) \\ U_{33} & = & -(1-\mu)Q-R \end{array}$$

In virtue of the bounding lemma (see Appendix E.14), inequality (J.311) is feasible if and only if there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} \mathcal{M}(\rho)+\varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{J.312}$$

holds.

The uncertainties have been removed from the conditions, it suffices now to linearize the inequality in order to get a SDP problem. According to Lemma 4.1.4, the matrices $M_0(\rho)$, $M_h(\rho)$ and H can be written in the following form

$$\begin{array}{lcl} M_0(\rho) & = & \Theta(\rho)-L(\rho)\Xi(\rho) \\ M_h(\rho) & = & \Upsilon(\rho)-L(\rho)\Omega(\rho) \\ H(\rho) & = & \Phi(\rho)-L(\rho)\Psi(\rho) \\ T-HC & = & T-\Phi(\rho)C+L(\rho)\Psi(\rho)C \end{array}$$

where $L(\rho)$ is an uncertain matrix to be designed.

Through the change of variable $\bar{L} = X_3^T L(\rho)$ the condition is linearized and thus becomes a LMI. On the other hand, since H is a constant matrix (see proof of Lemma 4.1.5) then $\bar{L}(\rho)$ must be defined using the equality

$$\bar{L}(\rho) = (X^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(\rho)(I-\Psi(\rho)\Psi(\rho)^+)$$

which concludes the proof. □

J.3.2 Robust observer with exact delay value - discretized Lyapunov-Krasovskii functional

Due to the form of the Lyapunov-Krasovskii functional, even if the latter result yields interesting observers, the design LMI remains conservative. This motivates the use of a discretized version of such a functional described in Section 3.6, Lemma 3.6.4. Since the proof is similar to the proof of Theorem J.7 it will be omitted.

Theorem J.8 *There exists a parameter dependent observer of the form (4.3) such that theorem 4.1.2 for all $h \in \mathscr{H}_1^\circ$ is satisfied if there exist a continuously differentiable matrix function $P : U_\rho \to \mathbb{S}_{++}^r$, matrix functions $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$, $X_1 : U_\rho \to \mathbb{R}^{n\times n}$, $X_2 : U_\rho \to \mathbb{R}^{t\times n}$ and $X_3 : U_\rho \to \mathbb{R}^{t\times t}$, constant matrices $Q_i, R_i \in \mathbb{S}_{++}^r$, $\bar{H} \in \mathbb{R}^{r\times m}$ and positive scalars $\gamma, \varepsilon > 0$ such that the following matrix inequality*

$$\begin{bmatrix} \tilde{M}(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \tilde{\mathcal{G}}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{J.313}$$

where

$$\tilde{\mathcal{M}}(\rho,\nu) = \left[\begin{array}{cccc|ccc} -X^H & U_{12}(\rho) & 0 & X^T & \bar{h}_1 R_0 & \dots & \bar{h}_1 R_{N-1} \\ \star & U_{22}(\rho,\nu) & U_{23}(\rho) & 0 & 0 & \dots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \dots & 0 \\ \star & \star & \star & -P(\rho) & -\bar{h}_1 R_0 & \dots & -\bar{h}_1 R_{N-1} \\ \hline \star & \star & \star & \star & \multicolumn{3}{c}{-\operatorname{diag}_i R_i} \end{array}\right] \prec 0 \tag{J.314}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ and where

$$U_{22} = \left[\begin{array}{cccccc|c} U'_{11} & R_0 & 0 & 0 & \dots & 0 & 0 \\ \star & N_1^{(1)} & R_1 & 0 & \dots & 0 & 0 \\ \star & \star & N_2^{(1)} & R_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & R_{N-1} & 0 \\ & & & & & N^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right] \tag{J.315}$$

$$\tilde{\mathcal{G}}(\rho)^T = \left[\begin{array}{c} X_1(\rho)^T G(\rho) \\ X_1(\rho)^T G(\rho) + (X_3(\rho)^T T - \bar{H}C)G(\rho) \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] \qquad \tilde{\mathcal{H}}(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_A(\rho)^T \\ 0 \\ 0 \\ 0 \\ \vdots \\ H_{A_h}(\rho)^T \\ 0 \\ H_E(\rho) \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

$$\begin{array}{rcl} U'_{11} & = & \partial_\rho P(\rho)\nu - P(\rho) + Q_0 - R_0 \\ N_i^{(1)} & = & -(1 - i\mu_N)Q_{i-1} + (1 + i\mu_N)Q_i - R_{i-1} - R_i \\ N^{(2)} & = & -(1-\mu)Q_{N-1} - R_{N-1} \\ U_{12}(\rho) & = & \left[\begin{array}{cccccc} P(\rho) + \tilde{\mathcal{A}}(\rho) & 0 & \dots & 0 & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{E}}(\rho) \end{array}\right] \\ U_{23}(\rho) & = & \left[\begin{array}{ccccc|c} I_r & 0 & \dots & 0 & 0 & 0 \end{array}\right]^T \\ \bar{h} & = & h_{max}/N \\ \mu_N & = & \mu/N \\ \tilde{\mathcal{A}}(\rho) & = & \left[\begin{array}{cc} X_1(\rho)^T A(\rho) & 0 \\ X_2(\rho)^T A(\rho) & X_3(\rho)^T\Theta(\rho) - \bar{L}(\rho)\Xi(\rho) \end{array}\right] \\ \tilde{\mathcal{A}}_h(\rho) & = & \left[\begin{array}{cc} X_1(\rho)^T A_h(\rho) & 0 \\ X_2(\rho)^T A_h(\rho) & X_3(\rho)^T\Upsilon(\rho) - \bar{L}(\rho)\Omega(\rho) \end{array}\right] \\ \tilde{\mathcal{E}}(\rho) & = & \left[\begin{array}{c} X_1(\rho)^T E(\rho) \\ X_2(\rho)^T E(\rho) + (X_3(\rho)^T T^T - C^T \bar{H}^T)E(\rho) \end{array}\right] \\ \bar{L}(\rho) & = & (X_3(\rho)^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \\ X(\rho) & = & \left[\begin{array}{cc} X_1(\rho) & X_2(\rho) \\ 0 & X_3(\rho) \end{array}\right] \end{array}$$

Moreover, the gain is given by $L(\rho) = X(\rho)^{-T}\bar{L}(\rho)$ *and we have* $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

J.3.3 Robust observer with approximate delay value - simple Lyapunov-Krasovskii functional

When the delay-used in the observer is different from the delay of the system, the latter results are not applicable. In such a case, the extended system writes

$$\begin{array}{rcl}\left[\begin{array}{c}\dot{x}(t)\\ \dot{e}(t)\end{array}\right] & = & (\mathcal{A}(\rho)+\Delta(t)\mathcal{A}(\rho))\left[\begin{array}{c}x(t)\\ e(t)\end{array}\right]+(\mathcal{A}_h(\rho)+\Delta(t)\mathcal{A}_h(\rho))\left[\begin{array}{c}x(t-h(t))\\ e(t-h(t))\end{array}\right]\\ & & \mathcal{A}_d(\rho)\left[\begin{array}{c}x(t-d(t))\\ e(t-d(t))\end{array}\right]+(\mathcal{E}(\rho)+\Delta(t)\mathcal{E}(\rho))w(t)\\ z_{obs}(t) & = & \mathcal{I}\left[\begin{array}{c}x(t)\\ e(t)\end{array}\right]\end{array} \tag{J.316}$$

with

$$\begin{array}{rcl}\mathcal{A}(\rho) & = & \left[\begin{array}{cc}A(\rho) & 0\\ 0 & M_0(\rho)\end{array}\right]\\ \Delta\mathcal{A}(\rho) & = & \left[\begin{array}{cc}\Delta A(\rho) & 0\\ (T-HC)\Delta A(\rho) & 0\end{array}\right]=\left[\begin{array}{c}G(\rho)\\ (T-HC)G(\rho)\end{array}\right]\Delta(t)\left[\begin{array}{cc}H_A(\rho) & 0\end{array}\right]\\ \mathcal{A}_h(\rho) & = & \left[\begin{array}{cc}A_h(\rho) & 0\\ (T-HC)A_h(\rho) & 0\end{array}\right]\\ \Delta\mathcal{A}_h(\rho) & = & \left[\begin{array}{cc}\Delta A_h(\rho) & 0\\ (T-HC)\Delta A_h(\rho) & 0\end{array}\right]=\left[\begin{array}{c}G(\rho)\\ (T-HC)G(\rho)\end{array}\right]\Delta(t)\left[\begin{array}{cc}H_{A_h}(\rho) & 0\end{array}\right]\\ \mathcal{A}_d(\rho) & = & \left[\begin{array}{cc}0 & 0\\ 0 & M_h(\rho)\end{array}\right]\\ \Delta\mathcal{A}_d(\rho) & = & \left[\begin{array}{cc}0 & 0\\ -(T-HC)\Delta A_h(\rho) & 0\end{array}\right]=\left[\begin{array}{c}0\\ -(T-HC)G\end{array}\right]\Delta(t)\left[\begin{array}{cc}H_{A_h}(\rho) & 0\end{array}\right]\\ \mathcal{E}(\rho) & = & \left[\begin{array}{c}E(\rho)\\ (T-HC)E(\rho))\end{array}\right]\\ \Delta\mathcal{E}(\rho) & = & \left[\begin{array}{c}\Delta E(\rho)\\ (T-HC)\Delta E(\rho)\end{array}\right]=\left[\begin{array}{c}G(\rho)\\ (T-HC)G(\rho)\end{array}\right]\Delta(t)H_E(\rho)\\ \mathcal{I} & = & \left[\begin{array}{cc}I_t & 0\end{array}\right]\end{array} \tag{J.317}$$

and under conditions

$$\begin{array}{rcl}(T-HC)A(\rho)-M_0(\rho)(T-HC)-N_0(\rho)C & = & 0\\ (T-HC)A_h(\rho)-M_h(\rho)(T-HC)-N_h(\rho)C & = & 0\end{array}$$

The following result aims at providing a constructive sufficient condition of existence of the observer (J.321). It is based on the application of Theorem 3.7.2 which the relaxed version of Theorem 3.7.1 considering the stability and $\mathcal{L}_2$ performances of LPV systems with two delays coupled through an algebraic equation. Since the methodology remains the same as for the other results, the proof will be omitted.

Theorem J.9 *There exists a parameter dependent observer of the form (J.321) such that theorem 4.1.8 holds for all* $h \in \mathscr{H}_1^\circ$, $d(t) = h(t) + \varepsilon(t)$ *with* $\varepsilon(t) \in [-\delta, \delta]$ *is satisfied if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}^r_{++}$*, a matrix function* $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$*, constant matrices* $Q_i, R_i \in \mathbb{S}^{r+n}_{++}$*,* $i = 1, 2$*,* $X_1 \in \mathbb{R}^{n\times n}$*,* $X_2 \in \mathbb{R}^{n\times r}$*,* $X_3 \in \mathbb{R}^{r\times r}$*,* $\bar{H} \in \mathbb{R}^{r\times m}$ *and a positive scalar* $\gamma > 0$ *such that the following LMIs*

$$\left[\begin{array}{cc}\mathcal{M}(\rho,\nu)+\varepsilon\mathcal{H}^1(\rho)^T\mathcal{H}^1(\rho) & \mathcal{G}^1(\rho)^T\\ \star & -\varepsilon I\end{array}\right]\prec 0 \tag{J.318}$$

and

$$\begin{bmatrix} \Pi(\rho,\nu) + \varepsilon\mathcal{H}^2(\rho)^T\mathcal{H}^2(\rho) & \mathcal{G}^2(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{J.319}$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ *and where*

$$
\begin{aligned}
\Pi(\rho,\nu) &= \begin{bmatrix} \Pi_{11}(\rho,\nu) & \Pi_{12}(\rho) \\ \star & \Pi_{22}(\rho) \end{bmatrix} \\
\mathcal{M}(\rho,\nu) &= \begin{bmatrix} -X^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_d(\rho)+\tilde{\mathcal{A}}_h(\rho) & \bar{\mathcal{E}}(\rho) & 0 & X^T & h_{max}R_1 & R_2 \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & \mathcal{I}^T & 0 & 0 & 0 \\ \star & \star & \Theta_{22} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & \star & \star & \star & \star & -R_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \\
\Pi_{11}(\rho,\nu) &= \begin{bmatrix} -X^H & P(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{A}}_d(\rho) & \bar{\mathcal{E}}(\rho) \\ \star & \Theta_{11}(\rho,\nu) & R_1 & 0 & 0 \\ \star & \star & \Psi_{22} & (1-\mu)R_2/\delta & 0 \\ \star & \star & \star & \Psi_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} \\
\Pi_{12}(\rho) &= \begin{bmatrix} 0 & X^T & h_{max}R_1 & R_2 \\ \mathcal{I}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\Pi_{22}(\rho) &= \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -P(\rho) & -h_{max}R_1 & -R_2 \\ \star & \star & -R_1 & 0 \\ \star & \star & \star & -\dfrac{R_2}{2\delta} \end{bmatrix} \\
\Theta_{11}(\rho,\nu) &= -P(\rho) + Q_1 + Q_2 + \sum_{i=1}^{N} \frac{\partial P}{\partial \rho_i}\nu_i - R_1 \\
\Theta_{22} &= -(1-\mu)(Q_1+Q_2) - R_1 \\
\Psi_{22} &= -(1-\mu)(Q_1+R_2/\delta) - R_1 \\
\Psi_{33} &= -(1-\mu_c)Q_2 - (1-\mu)R_2/\delta \\
\mathcal{I} &= \begin{bmatrix} 0 & I_r \end{bmatrix} \\
\bar{L}(\rho) &= (X_3^T\Phi(\rho) - \bar{H})\Psi(\rho)^+ + Z(\rho)(I - \Psi(\rho)\Psi(\rho)^+) \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} X_1^T A(\rho) & 0 \\ X_2^T A(\rho) & X_3^T\Theta(\rho) - \bar{L}(\rho)\Xi(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} X_1^T A_h(\rho) & 0 \\ X_2^T A_h(\rho) + X_3^T(T - \Phi(\rho)C)A_h(\rho) + \bar{L}(\rho)\Psi(\rho)CAh(\rho) & 0 \end{bmatrix} \\
\tilde{\mathcal{A}}_d(\rho) &= \begin{bmatrix} 0 & 0 \\ -X_3^T(T-\Phi(\rho)C) - \bar{L}(\rho)\Psi(\rho)C & X_3^T\Upsilon(\rho) - \bar{L}(\rho)\Omega(\rho) \end{bmatrix} \\
\bar{\mathcal{E}}(\rho) &= \begin{bmatrix} X_1^T E(\rho)^T \\ (X_2^T T - \bar{H}C)E(\rho)^T \end{bmatrix}
\end{aligned}
$$

$$
\mathcal{H}^1(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_A(\rho)^T \\ 0 \\ \hline H_{A_h}(\rho)^T \\ 0 \\ \hline H_E(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad
\mathcal{G}^1(\rho)^T = \left[\begin{array}{c} X_1(\rho)^T G(\rho) \\ X_1(\rho)^T G(\rho) + (X_3(\rho)^T T - \bar{H}C)G(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]
$$

$$
\mathcal{H}^2(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_A(\rho)^T \\ 0 \\ \hline H_{A_h}(\rho)^T \\ 0 \\ \hline H_{A_h}(\rho)^T \\ 0 \\ \hline H_E(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad
\mathcal{G}^2(\rho)^T = \left[\begin{array}{c} X_1(\rho)^T G(\rho) \\ X_1(\rho)^T G(\rho) + (X_3(\rho)^T T - \bar{H}C)G(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]
$$

Moreover, the gain is given by $L(\rho) = X_3^{-T}\bar{L}(\rho)$ *and we have* $||e||_{\mathcal{L}_2} < \gamma||w||_{\mathcal{L}_2}$

J.3.4 Robust memoryless Observer

We consider now that no information on the delay is available in real time and thus only a memoryless observer of the form

$$
\begin{array}{rcl} \dot{\xi}(t) & = & M(\rho)\xi(t) + N(\rho)y(t) \\ \hat{z}(t) & = & \xi(t) + Hy(t) \end{array} \tag{J.320}
$$

is seeked.

In this case, the extended system is given by the expression

$$
\begin{array}{rcl} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} & = & (\mathcal{A}(\rho) + \Delta\mathcal{A}(\rho)) \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + (\mathcal{A}_h(\rho) + \Delta\mathcal{A}_h(\rho))Y \begin{bmatrix} x(t-h(t)) \\ e(t-h(t)) \end{bmatrix} \\ & & +(\mathcal{E}(\rho) + \Delta\mathcal{E}(\rho))w(t) \\ z_{obs}(t) & = & \mathcal{I} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \end{array} \tag{J.321}
$$

where

$$
\begin{aligned}
\mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ 0 & M(\rho) \end{bmatrix} \\
\Delta\mathcal{A}(\rho) &= \begin{bmatrix} \Delta A(\rho,t) & 0 \\ (T-HC)\Delta A(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G(\rho) \\ (T-HC)G(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_A(\rho) & 0 \end{bmatrix} \\
\mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho) \\ (T-HC)A_h(\rho) \end{bmatrix} \\
\Delta\mathcal{A}_h(\rho) &= \begin{bmatrix} \Delta A_h(\rho,t) & 0 \\ (T-HC)\Delta A_h(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G(\rho) \\ (T-HC)G(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_{A_h}(\rho) & 0 \end{bmatrix} \\
\mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) \\ (T-HC)E(\rho) \end{bmatrix} \\
\Delta\mathcal{E}(\rho,t) &= \begin{bmatrix} \Delta E(\rho,t) \\ (T-HC)\Delta E\rho,t) \end{bmatrix} = \begin{bmatrix} G(\rho) \\ (T-HC)G(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_E(\rho) \end{bmatrix}
\end{aligned}
$$

under the assumption that Lemma 4.1.11, which is a sufficient condition for the existence of observer matrices, is satisfied.

The following theorem gives a sufficient condition on the existence of a robust memoryless observer of the form (J.320). It is derived from Lemma 3.7.2 which considers the stability of time-delay system in which the delay affects only a specific part of the state.

Theorem J.10 *There exists a robust memoryless parameter dependent observer of the form (J.320) such that theorem 4.1.10 holds for all* $h \in \mathscr{H}_1^\circ$ *is satisfied if there exist a continuously differentiable matrix function* $P : U_\rho \to \mathbb{S}_{++}^r$*, a matrix function* $Z : U_\rho \to \mathbb{R}^{r\times(2r+3m)}$*, constant matrices* $Q, R \in \mathbb{S}_{++}^{r+n}$*,* $X_1 \in \mathbb{R}^{n\times n}$*,* $X_2 \in \mathbb{R}^{n\times r}$*,* $X_3 \in \mathbb{R}^{r\times r}$*,* $\bar{H} \in \mathbb{R}^{r\times m}$ *and a positive scalar* $\gamma > 0$

$$
\begin{bmatrix} \Psi(\rho,\nu) + \varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{J.322}
$$

holds for all $(\rho,\nu)\in U_\rho\times U_\nu$ *where*

$$\begin{array}{rcl}\Psi(\rho,\nu) & = & \begin{bmatrix} -X^H & P(\rho)+X^T\tilde{\mathcal{A}}(\rho) & X^T\tilde{\mathcal{A}}_h(\rho) & X^T\tilde{\mathcal{E}}(\rho) & 0 & X^T & h_{max}Y^TR \\ \star & \Psi_{22}'(\rho,\nu) & R & 0 & \mathcal{I}^T & 0 & 0 \\ \star & \star & -(1-\mu)Q-R & 0 & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I_m & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I_r & 0 & 0 \\ \star & \star & \star & \star & \star & -P(\rho) & -h_{max}Y^TR \\ \star & \star & \star & \star & \star & \star & -R \end{bmatrix} \\ \Psi_{22}'(\rho,\nu) & = & \partial_\rho P(\rho)\nu - P(\rho) + Y^T(Q-R)Y \\ R & \in & \mathbb{S}_{++}^n \\ Z & = & \begin{bmatrix} I_n & 0 \end{bmatrix} \\ \mathcal{I} & = & \begin{bmatrix} 0 & I_r \end{bmatrix} \\ \bar{L}(\rho) & = & (X_3^T\Phi(\rho)-\bar{H})\Psi(\rho)^+ + Z(I-\Psi(\rho)\Psi(\rho)^+) \\ \mathcal{H}(\rho)^T & = & \begin{bmatrix} 0 \\ 0 \\ \hline H_A(\rho)^T \\ 0 \\ \hline H_{A_h}(\rho) \\ \hline H_E(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \end{bmatrix} \qquad \mathcal{G}(\rho)^T = \begin{bmatrix} X_1(\rho)^TG(\rho) \\ X_1(\rho)^TG(\rho)+(X_3(\rho)^TT-\bar{H}C)G(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \end{bmatrix}\end{array}$$

Moreover the generalized observer gain $L(\rho)$ *is given by the relation* $L(\rho)=X_3^{-T}\bar{L}(\rho)$ *and we have* $||e||_{\mathcal{L}_2}<\gamma||w||_{\mathcal{L}_2}$.

J.4 Filtering of uncertain LPV Time-Delay Systems

This subsection aims at providing several results on the filtering of uncertain LPV systems.

J.4.1 Design of robust filters with exact delay-value - discretized Lyapunov-Krasovskii functional

Since Theorem 4.2.1 may result in conservative results due to the use of a simple Lyapunov-Krasovskii functional, the next theorem extends the result in the case of a discretized Lyapunov-Krasovskii functional. It is based on the application of Theorems of Section 3.6 which address the problem of stability/performances of LPV time-delay systems using a discretized functional.

Theorem J.11 *There exists a full-order filter of the form (J.292) with* $d(t)=h(t)$, $h(t)\in\mathscr{H}_1^\circ$ *if there exists a continuously differentiable matrix function* $P:U_\rho\to\mathbb{S}_{++}^{2n}$, *symmetric matrices* $\tilde{Q}_i,\tilde{R}_i\in\mathbb{S}_{++}^{2n}$, $\hat{X}\in\mathbb{R}^{2n\times 2n}$, *matrix functions* $\tilde{A}_F,\tilde{A}_{Fh}:U_\rho\to\mathbb{R}^{n\times n}$, $\tilde{B}_F:U_\rho\to\mathbb{R}^{n\times m}$, $\tilde{C}_F,\tilde{C}_{Fh}:U_\rho\to\mathbb{R}^{t\times n}$, $\tilde{D}_F:U_\rho\to\mathbb{R}^{n\times m}$ *and a scalar* $\gamma,\varepsilon>0$ *such that the LMI*

$$\begin{bmatrix} \Psi_{11}(\rho,\nu)+\varepsilon\mathcal{H}(\rho)^T\mathcal{H}(\rho) & \mathcal{G}(\rho)^T \\ \star & -\varepsilon I \end{bmatrix}\prec 0 \tag{J.323}$$

holds for all $(\rho,\nu) \in U_\rho \times U_\nu$ *and where*

$$\Psi_{11}(\rho,\nu) = \left[\begin{array}{cccc|ccc} -\hat{X}^H & \tilde{U}_{12}(\rho) & 0 & \hat{X}^T & \bar{h}_1\tilde{R}_0 & \dots & \bar{h}_1\tilde{R}_{N-1} \\ \star & \tilde{U}_{22}(\rho,\nu) & \tilde{U}_{23}(\rho) & 0 & 0 & \dots & 0 \\ \star & \star & -\gamma I & 0 & 0 & \dots & 0 \\ \star & \star & \star & -\tilde{P}(\rho) & -\bar{h}_1\tilde{R}_0 & \dots & -\bar{h}_1\tilde{R}_{N-1} \\ \hline \star & \star & \star & \star & & -\operatorname{diag}_i \tilde{R}_i & \end{array}\right]$$

$$U_{22} = \left[\begin{array}{cccccc|c} U'_{11} & \tilde{R}_0 & 0 & 0 & \dots & 0 & 0 \\ \star & \tilde{N}_1^{(1)} & \tilde{R}_1 & 0 & \dots & 0 & 0 \\ \star & \star & \tilde{N}_2^{(1)} & \tilde{R}_2 & & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \tilde{R}_{N-1} & 0 \\ & & & & & \tilde{N}^{(2)} & 0 \\ \hline \star & \star & \star & \dots & 0 & 0 & -\gamma I \end{array}\right]$$

$$\begin{aligned}
\tilde{U}'_{11} &= \partial_\rho \tilde{P}(\rho)\dot{\rho} - \tilde{P}(\rho) + \tilde{Q}_0 - \tilde{R}_0 \\
\tilde{N}_i^{(1)} &= -(1 - i\mu_N)\tilde{Q}_{i-1} + (1 + i\mu_N)\tilde{Q}_i - \tilde{R}_{i-1} - \tilde{R}_i \\
\tilde{N}^{(2)} &= -(1-\mu)\tilde{Q}_{N-1} - \tilde{R}_{N-1} \\
\tilde{U}_{12}(\rho) &= \left[\begin{array}{cccccccc} P(\rho) + \tilde{\mathcal{A}}(\rho) & 0 & 0 & \tilde{\mathcal{A}}_h(\rho) & \dots & 0 & \tilde{\mathcal{E}}(\rho) \end{array}\right] \\
\tilde{U}_{23}(\rho) &= \left[\begin{array}{ccccc|c} \tilde{\mathcal{C}}(\rho) & 0 & \dots & 0 & \tilde{\mathcal{C}}_h(\rho) & \mathcal{F}(\rho) \end{array}\right]^T \\
\hat{X} &= \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho)C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho)C_{yh}(\rho) & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) &= \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho)C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T &= \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T &= \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\hat{X}_3 &= U^T \Sigma U \qquad \text{(SVD)}
\end{aligned}$$

$$\mathcal{H}(\rho)^T = \begin{bmatrix} 0 \\ 0 \\ \hline H_x(\rho)^T 0 \\ \vdots \\ 0 \\ H_{xh}(\rho)^T \\ 0 \\ H_w(\rho)^T \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathcal{G}(\rho)^T = \begin{bmatrix} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho) G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho) G_y(\rho) \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline G_z(\rho) - D_F(\rho) G_y(\rho) \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

Moreover the filter matrices are computed using

$$\begin{bmatrix} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{bmatrix} = \begin{bmatrix} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{bmatrix}$$

and we have $||e||_{\mathcal{L}_2} \le \gamma ||w||_{\mathcal{L}_2}$.

J.4.2 Design of robust filters with approximate delay-value

This section provides a result on the existence of a robust filter implementing a delay which is different from the system one. In this case, the extended system describing the evolution of the system and the filter is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_F(t) \end{bmatrix} &= (\mathcal{A}(\rho) + \Delta\mathcal{A}(\rho,t)) \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + (\mathcal{A}_h(\rho) + \Delta\mathcal{A}_h(\rho,t)) \begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} \\ &\quad + \mathcal{A}_d(\rho) \begin{bmatrix} x(t-d(t)) \\ x(t-d(t)) \end{bmatrix} + (\mathcal{E}(\rho) + \Delta\mathcal{E}(\rho,t))w(t) \\ e(t) &= z(t) - z_F(t) \\ &= (\mathcal{C}(\rho) + \Delta\mathcal{C}(\rho,t)) \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix} + (\mathcal{C}_h(\rho) + \Delta\mathcal{C}_h(\rho,t)) \begin{bmatrix} x(t-h(t)) \\ x_F(t-h(t)) \end{bmatrix} \\ &\quad + (\mathcal{F}(\rho) + \Delta\mathcal{F}(\rho,t))w(t) \end{aligned} \tag{J.324}$$

where

$$
\begin{aligned}
\mathcal{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ B_F(\rho)C_y(\rho) & A_F(\rho) \end{bmatrix} \\
\Delta\mathcal{A}(\rho,t) &= \begin{bmatrix} \Delta A(\rho,t) & 0 \\ B_F(\rho)\Delta C_y(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_x(\rho) & 0 \end{bmatrix} \\
\mathcal{A}_h(\rho) &= \begin{bmatrix} A_h(\rho) & 0 \\ B_F(\rho)C_{yh}(\rho) & 0 \end{bmatrix} \\
\Delta\mathcal{A}_h(\rho,t) &= \begin{bmatrix} \Delta A_h(\rho,t) & 0 \\ B_F(\rho)\Delta C_{yh}(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} \Delta(t) \begin{bmatrix} H_{xh}(\rho) & 0 \end{bmatrix} \\
\mathcal{A}_d(\rho) &= \begin{bmatrix} 0 & 0 \\ 0 & A_{Fh}(\rho) \end{bmatrix} \\
\mathcal{E}(\rho) &= \begin{bmatrix} E(\rho) \\ B_F(\rho)F_y(\rho) \end{bmatrix} \\
\Delta\mathcal{E}(\rho,t) &= \begin{bmatrix} \Delta E(\rho,t) & 0 \\ B_F(\rho)\Delta E(\rho,t) & 0 \end{bmatrix} = \begin{bmatrix} G_x(\rho) \\ B_F(\rho)G_y(\rho) \end{bmatrix} \Delta(t) H_w(\rho) \\
\mathcal{C}(\rho) &= \begin{bmatrix} C(\rho) - D_F(\rho)C_y(\rho) & -C_F(\rho) \end{bmatrix} \\
\Delta\mathcal{C}(\rho,t) &= \begin{bmatrix} \Delta C(\rho,t) - D_F(\rho)\Delta C_y(\rho,t) & 0 \end{bmatrix} = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t) \begin{bmatrix} H_x(\rho) & 0 \end{bmatrix} \\
\mathcal{C}_h(\rho) &= \begin{bmatrix} C_h(\rho) - D_F(\rho)C_{yh}(\rho) & -C_{Fh}(\rho) \end{bmatrix} \\
\Delta\mathcal{C}_h(\rho,t) &= \begin{bmatrix} \Delta C_h(\rho,t) - D_F(\rho)\Delta C_{yh}(\rho,t) & 0 \end{bmatrix} = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t) \begin{bmatrix} H_{xh}(\rho) & 0 \end{bmatrix} \\
\mathcal{F}(\rho) &= F(\rho) - D_F(\rho)F_y(\rho) \\
\Delta\mathcal{F}(\rho,t) &= \Delta F(\rho) - D_F(\rho)\Delta F_y(\rho) = (G_z(\rho) - D_F(\rho)G_y(\rho))\Delta(t)H_w(\rho)
\end{aligned}
$$

In this case, we obtain the following theorem which is derived from the application of Theorem 3.7.2 addressing the difficult problem of stability analysis of LPV time-delay systems with two coupled delays.

Theorem J.12 *There exists a filter of the form (J.300) for system (J.291) with* $d(t) = h(t)$, $h(t) \in \mathscr{H}_1^\circ$ *if there exists a continuously differentiable matrix function* $\tilde{P} : U_\rho \to \mathbb{S}_{++}^{2n}$, *symmetric matrices* $\tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2 \in \mathbb{S}_{++}^{2n}$, $\hat{X} \in \mathbb{R}^{2n\times 2n}$, *matrix functions* $\tilde{A}_F, \tilde{A}_{Fh} : U_\rho \to \mathbb{R}^{n\times n}$, $\tilde{B}_F : U_\rho \to \mathbb{R}^{n\times m}$, $\tilde{C}_F, \tilde{C}_{Fh} : U_\rho \to \mathbb{R}^{t\times n}$, $\tilde{D}_F : U_\rho \to \mathbb{R}^{n\times m}$ *and a scalar* $\gamma > 0$ *such that the LMIs*

$$
\begin{bmatrix} \Theta(\rho,\nu) + \varepsilon\mathcal{H}_1(\rho)^T\mathcal{H}_1(\rho) & \mathcal{G}_1(\rho)^T \\ \star & -\varepsilon i \end{bmatrix} \prec 0 \tag{J.325}
$$

and

$$
\begin{bmatrix} \Pi(\rho,\nu) + \varepsilon\mathcal{H}_2(\rho)^T\mathcal{H}_2(\rho) & \mathcal{G}_2(\rho)^T \\ \star & -\varepsilon I \end{bmatrix} \prec 0 \tag{J.326}
$$

hold for all $(\rho,\nu) \in U_\rho \times U_\nu$ and where

$$\Pi(\rho,\nu) = \begin{bmatrix} \tilde{\Pi}_{11}(\rho,\nu) & \tilde{\Pi}_{12}(\rho) \\ \star & \tilde{\Pi}_{22}(\rho) \end{bmatrix}$$

$$\Theta(\rho,\nu) = \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho)+\tilde{\mathcal{A}}(\rho) & \tilde{\mathcal{A}}_{hd}(\rho) & \tilde{\mathcal{E}}(\rho) & 0 & \hat{X}(\rho)^T & h_{max}\tilde{R}_1 & \tilde{R}_2 \\ \star & \tilde{\Theta}_{11}(\rho,\nu) & \tilde{R}_1 & 0 & \tilde{\mathcal{C}}(\rho)^T & 0 & 0 & 0 \\ \star & \star & \tilde{\Theta}_{22} & 0 & \tilde{\mathcal{C}}_{hd}(\rho)^T & 0 & 0 & 0 \\ \star & \star & \star & -\gamma I & \mathcal{F}(\rho)^T & 0 & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 \\ \star & \star & \star & \star & \star & \star & -\tilde{R}_1 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} \end{bmatrix}$$

$$\tilde{\Pi}_{11}(\rho,\nu) = \begin{bmatrix} -\hat{X}^H & \tilde{P}(\rho)+\tilde{\mathcal{A}} & \tilde{\mathcal{A}}_h(\rho) & \tilde{\mathcal{A}}_d(\rho) & \tilde{\mathcal{E}}(\rho) \\ \star & \tilde{\Theta}_{11}(\rho,\nu) & \tilde{R}_1 & 0 & 0 \\ \star & \star & \tilde{\Psi}_{22} & (1-\mu)\tilde{R}_2/\delta & 0 \\ \star & \star & \star & \tilde{\Psi}_{33} & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix}$$

$$\tilde{\Pi}_{12}(\rho) = \begin{bmatrix} 0 & \hat{X}^T & h_{max}\tilde{R}_1 & \tilde{R}_2 \\ \tilde{\mathcal{C}}(\rho)^T & 0 & 0 & 0 \\ \tilde{\mathcal{C}}_h(\rho)^T & 0 & 0 & 0 \\ \tilde{\mathcal{C}}_d(\rho)^T & 0 & 0 & 0 \\ \mathcal{F}(\rho)^T & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\Pi}_{22}(\rho) = \begin{bmatrix} -\gamma I & 0 & 0 & 0 \\ \star & -\tilde{P}(\rho) & -h_{max}\tilde{R}_1 & -\tilde{R}_2 \\ \star & \star & -\tilde{R}_1 & 0 \\ \star & \star & \star & -\dfrac{\tilde{R}_2}{2\delta} \end{bmatrix}$$

$$\begin{aligned} \tilde{\Theta}_{11}(\rho,\nu) &= -\tilde{P}(\rho)+\tilde{Q}_1+\tilde{Q}_2+\sum_{i=1}^{N}\frac{\partial\tilde{P}}{\partial\rho_i}\nu_i-\tilde{R}_1 \\ \tilde{\Theta}_{22} &= -(1-\mu)(\tilde{Q}_1+\tilde{Q}_2)-\tilde{R}_1 \\ \tilde{\Psi}_{22} &= -(1-\mu)(\tilde{Q}_1+\tilde{R}_2/\delta)-\tilde{R}_1 \\ \tilde{\Psi}_{33} &= -(1-\mu_c)\tilde{Q}_2-(1-\mu)\tilde{R}_2/\delta \\ \tilde{C}_{hd} &= \tilde{C}_h(\rho)+\tilde{C}_d(\rho) \\ \tilde{A}_{hd} &= \tilde{A}_h(\rho)+\tilde{A}_d(\rho) \end{aligned}$$

$$\begin{aligned}
\hat{X} &= \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_3 \end{bmatrix} \\
\tilde{\mathcal{A}}(\rho) &= \begin{bmatrix} \hat{X}_1^T A(\rho) + \tilde{B}_F(\rho) C_y(\rho) & \tilde{A}_F(\rho) \\ \hat{X}_2^T A(\rho) + \tilde{B}_F(\rho) C_y(\rho) & \tilde{A}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{A}}_h(\rho) &= \begin{bmatrix} \hat{X}_1^T A_h(\rho) + \tilde{B}_F(\rho) C_{yh}(\rho) & 0 \\ \hat{X}_2^T A_h(\rho) + \tilde{B}_F(\rho) C_{yh}(\rho) & 0 \end{bmatrix} \\
\tilde{\mathcal{A}}_d(\rho) &= \begin{bmatrix} 0 & \tilde{A}_{Fh}(\rho) \\ 0 & \tilde{A}_{Fh}(\rho) \end{bmatrix} \\
\tilde{\mathcal{E}}(\rho) &= \begin{bmatrix} \hat{X}_1 E(\rho) + \tilde{B}_F(\rho) C_y(\rho) \\ \hat{X}_2^T E(\rho) + \tilde{B}_F(\rho) C_y(\rho) \\ \end{bmatrix} \\
\tilde{\mathcal{C}}(\rho)^T &= \begin{bmatrix} C(\rho)^T - C_y(\rho)^T D_F(\rho)^T \\ -\tilde{C}_F(\rho) \end{bmatrix} \\
\tilde{\mathcal{C}}_h(\rho)^T &= \begin{bmatrix} C_h(\rho)^T - C_{hy}(\rho)^T D_F(\rho)^T \\ 0 \end{bmatrix} \\
\tilde{\mathcal{C}}_d(\rho)^T &= \begin{bmatrix} 0 \\ -\tilde{C}_{Fh}(\rho) \end{bmatrix} \\
\hat{X}_3 &= U^T \Sigma U \qquad \text{(SVD)}
\end{aligned}$$

$$\mathcal{G}_1(\rho)^T = \left[\begin{array}{c} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho) G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho) G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline G_z(\rho) - \tilde{D}_F(\rho) G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad \mathcal{H}_1(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho)^T \\ 0 \\ \hline 0 \\ \hline H_w(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$$

$$\mathcal{G}_2(\rho)^T = \left[\begin{array}{c} \hat{X}_1^T G_x(\rho) + \tilde{B}_F(\rho)G_y(\rho) \\ \hat{X}^T G_x + \tilde{B}_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline G_z(\rho) - \tilde{D}_F(\rho)G_y(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad \mathcal{H}_2(\rho)^T = \left[\begin{array}{c} 0 \\ 0 \\ \hline H_x(\rho)^T \\ 0 \\ \hline H_{xh}(\rho)^T \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline H_w(\rho) \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$$

Moreover the filter matrices are computed using

$$\left[\begin{array}{ccc} A_F(\rho) & A_{Fh}(\rho) & B_F(\rho) \\ C_F(\rho) & C_{Fh}(\rho) & D_F(\rho) \end{array}\right] = \left[\begin{array}{ccc} U^{-T}\tilde{A}_F(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{A}_{Fh}(\rho)U^{-1}\Sigma^{-1} & U^{-T}\tilde{B}_F(\rho) \\ \tilde{C}_F(\rho)U^{-1}\Sigma^{-1} & \tilde{C}_{Fh}(\rho)U^{-1}\Sigma^{-1} & \tilde{D}_F(\rho) \end{array}\right]$$

and we have $||e||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2}$.

K New Delay-SIR Model for pulse Vaccination

A New Delay-SIR Model for Pulse Vaccination

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Abstract: This paper introduces a new model for disease outbreaks. This model describes the disease evolution through a system of nonlinear differential equations with distributed-delay. The main difference between classical SIR-model resides in the fact that the recovery rate of the population is expressed as a distributed-delay term modeling the time spent being sick by infected people. This model is identified to fit realistic measurements which shows the effectiveness of the model. Finally, we develop an optimal campaign vaccination strategy based on recent results on the impulsive control of time-delay systems.

Keywords: Intensive and chronic therapy; Model formulation, experiment design; Decision support and control

1. INTRODUCTION

Nowadays, due to the large mobility of people within a country or even worldwide, the risk of being infected by a virus is relatively higher than several decades ago. That is why it is interesting to elaborate models of the evolution of diseases in order to develop strategies to decrease the impact of the outbreak.

The first model of an epidemic was suggested by Bernoulli in 1970. He used this model to explain the basic control effects obtained through population immunization, and the advantages of vaccination in order to prevent an epidemic. Simple mathematical models are governed by action-mass laws [Daley and Gani, 1999, Hethcote, 2002]. The rate of spread of infection is hereby assumed to be proportional to the density of susceptible people and the density of infected people (strong homogenous mixing). Simpler models, based on weak homogenous mixing (rate of new infections proportional to the number of susceptibles) are explored in [Anderson and May, 2002]. One parameter stands out in these models: the ratio of the rate of infection to the rate of recovery, denoted by r_0, called the basic reproduction number. It is the average number of new cases produced when one infective is introduced into a completely susceptible host population. A basic result in modern epidemiology is the existence of a threshold value for the reproduction number. If r_0 is below the threshold, an epidemic outbreak does not follow the introduction of a few infectious individuals in the community. For example measles has a r_0 on the order of 12-15 Anderson and May [1982].

The biological processes of sudden and severe epidemics are inherently nonlinear, and exhibit fundamentally different dynamic behaviors from linear systems (e.g. multiple equilibria, limit cycles, and chaos). In addition, more complex nonlinear models encompassing spatial variation (i.e. mixing locally within households and globally throughout the population, temporal variation (age structure) and delays [den Driessche] are also required to give added realism, which makes the control problem even harder. Hence, regarding the control of epidemics, few analytical results exist. A notable and recent exception is the work of Behncke [2000]. Nonetheless, analytical or numerical approximations for infection control measures such as vaccination, dose profile and timing in pulse vaccination regimes [Stone et al., 2000], isolation and quarantine, screening or other public health interventions are vital for controlling severe epidemics. When using finite dimensional models, it is clear that when the initial state is reached again through the action of the control, the process will be periodic [Bainov and Simeonov, 1996]. This is the principle behind pulse vaccination [Nokes and Swinton, 1997], although true periodicity of the state is not assumed. However, when delays are present the system is inherently infinite dimensional, and it is unlikely that the same state may be reached twice. Hence the premise that a periodic pulse vaccination strategy is optimal is false. Techniques recently developed by the authors Verriest et al. [2004] for optimal impulsive control for systems with delays will be applied in order to overcome this problem.

We propose in this paper a new model embedding further information such as the minimal time spent sick by the infected population. This model considers that infected people remain sick for a certain amount of time greater to a threshold τ. This time is defined by a distribution over $[\tau, +\infty)$. The model is validated while identifying its parameters using real epidemic measurements reported

[1] The authors contributed equally to this work

in a medicine journal Lancet [March 4 1978]. Finally we develop an optimal pulse vaccination strategy [E.I. Verriest and Egerstedt, 2005, Verriest, 2005] minimizing a certain criterium.

In section 2, we recall classical epidemiological models and in section 3 some vaccination strategies are discussed. Then a new model is given to represent the dynamic of the disease among the population in section 4. Finally, in section 5 the optimal pulse vaccination strategy is teated and illustrated through an example based on a real scenario.

2. SIR MODELS

To understand how basic epidemics models work, we will first describe some open loop models. The most basic model is the SIR model. People who are susceptible (S) become infected with a force of infection proportional to the number of infected (I). After the infection people become immune and are removed (the R in the SIR model), or become infected again (SIS model). The simplest of these models are closed in the sense that total population remains fixed, either by disregarding immigration (for short duration outbreaks) or assuming that birth and death rates are equal (for long duration models) or both. These are the basic Kermack-McKendrick models described in all introductory books on mathematical epidemiology and they will be briefly described below:

$$\begin{aligned} \dot{S} &= -f_1(S,I) + f_2(I,R) \\ \dot{I} &= f_2(S,I) - f_3(I,S) \\ \dot{R} &= f_3(I,S) - f_2(I,R) \end{aligned} \tag{1}$$

where $f_1 > 0$ models the rate of infection, $f_2 > 0$ the rate at which recovered people become susceptible again and $f_3 > 0$ the rate of recovery for $(R, I, S) \in \mathbb{R}^3_+$

However, diseases such as measles do not fit such a description, and call for an extended model sporting a compartment of exposed but not yet infectious (E) individuals. It is assumed that individuals stay in this class for a fixed period of time and hence such a model involves delays. Once the tools for the study of epidemic models have been produced and the models themselves understood, the real test of their validity is to use these models in predicting the outcome of various interventions Wickwire [1977], and ultimately in optimizing such interventions.

A natural question is: "What can be done to prevent a predicted epidemic from occurring?" The above models cast in various level of detail the evolution of epidemics as dynamical systems and first question one has to ask here is: "How can the dynamics be influenced by external factors?" and as standard in control theory, "What closed loop control strategies can be used?"

3. MATHEMATICAL FRAMEWORK

In order to be able to solve optimal immunization problems, some results on optimal impulse control must be recalled Verriest et al. [2004], and for the sake of easy reference, we restate them here.

3.1 Optimal Impulse Control for Point Delay Systems

To fix ideas, let the autonomous system under consideration be modeled by

$$\dot{x} = f(x) + g(x_\tau), \tag{2}$$

where $x_\tau = x(t-\tau)$, and where $x(\theta)$ is given for $-\tau < \theta < 0$. Moreover, let the effect of the impulsive inputs be given by

$$x(T_i^+) = x(T_i^-) + G(x(T_i^-), u_i, T_i). \tag{3}$$

The amplitudes, u_i, and instants, T_i, are to be chosen such that a performance index

$$J = \int_0^{t_f} L(x(t))dt + \sum_{i=1}^{N-1} K(x(T_i^-), u_i, T_i) \tag{4}$$

is optimized. Now, in Verriest et al. [2004], the following result (that will provide a basis for the developments on this proposal) was derived:

Theorem 3.1. The impulsive system in Equations (2) and 3 minimizes the performance index (4) if the magnitudes u_i and times T_i are chosen as follows:
Define:

$$H_i = L(x) + \lambda_i^T (f(x) + g(x_\tau)) \tag{5}$$

$$M_i = K(x(T_i^-), u_i, T_i) + \mu_i G(x(T_i^-), u_i, T_i). \tag{6}$$

Euler-Lagrange Equations:

$$\dot{\lambda}_i = -\left(\frac{\partial L}{\partial x}\right)^T - \left(\frac{\partial f}{\partial x}\right)^T \lambda_i - \chi_i^+ \left(\frac{\partial g}{\partial x}\right)^T \lambda_i^\tau - \chi_{i+1}^- \left(\frac{\partial g}{\partial x}\right)^T \lambda_{i+1}^\tau, \tag{7}$$

with $T_{i-1} < t < T_i$, $i = 1, \ldots, N-1$, and where $\chi_i^+(t) = 1$ if $t \in [T_{i-1}, T_i - \tau]$ and 0 otherwise, $\chi_{i+1}^-(t) = 1$ if $t \in [T_i - \tau, T_i]$ and 0 otherwise, and $\lambda_i^\tau = \lambda_i(t+\tau)$. Moreover,

$$\dot{\lambda}_N = -\left(\frac{\partial L}{\partial x}\right)^T - \left(\frac{\partial f}{\partial x}\right)^T \lambda_N - \chi_N^+ \left(\frac{\partial g}{\partial x}\right)^T \lambda_N^\tau. \tag{8}$$

Boundary Conditions:

$$\lambda_N(T_N) = 0 \tag{9}$$

$$\lambda_i(T_i^-) = \lambda_{i+1}(T_i^+) + \left(\frac{\partial M_i}{\partial x}\right)^T. \tag{10}$$

Multipliers:

$$\mu_i = \lambda_{i+1}(T_i^+), \ i = 1, \ldots, N-1 \tag{11}$$

$$\mu_N = -\left(\frac{\partial M_N}{\partial x}\right)^T. \tag{12}$$

Optimality Conditions:

$$\frac{dJ}{du_i} = \frac{\partial M_i}{\partial u_i} = 0 \tag{13}$$

$$\begin{aligned} \frac{dJ}{dT_i} &= H_i(T_i^-) - H_{i+1}(T_i^+) + \frac{\partial M_i}{\partial T_i} \\ &+ \lambda_{i+1}(T_i + \tau)^T (g(x(T_i^+)) - g(x(T_i^-))) = 0. \end{aligned} \tag{14}$$

These necessary optimality conditions will have to be massaged in some manner in order to be numerically effective.

3.2 Gradient Descent

The reason why the formulas derived here above are particularly easy to work with is that they give us access to a very straight-forward numerical algorithm.

For each iteration k, let $\theta(k) = (T, v)^T$ be the vector of control variables, and compute the following:

(1) Compute $x(t)$ forward in time on $[t_0, t_f]$ by integrating from $x(t_0) = x_0$.
(2) Compute the costate $\lambda(t)$ backward in time from t_f to t_0 by integrating the costate dynamics.
(3) Use the equations above to compute $\nabla_\theta J = (\frac{dJ}{dT}, \frac{dJ}{dv})$.
(4) Update θ as follow :
$$\theta(k+1) = \theta(k) - l(k)\nabla_\theta J^T,$$
where $l(k)$ is the step size, e.g. given by the Armijo algorithm Armijo [1996].
(5) Repeat.

Note that the cost function J may be non-convex which means that we can only expect the method to reach a local minimum. But, as we will see, it still can give quite significant reductions in cost.

4. DELAY-SIR MODEL

In this section a new model for an epidemic, the delay SIR, is proposed and matched against real epidemic data. While both the new and the old (standard SIR) model corroborate the data, the delay-SIR may be more adapted to a physical model of the disease.

The main ingredient in this model is the fact that we assume that once infected, a person is instantaneously infectious, and this for at least a time τ. After this initial lapse, we assume that the person remains infectious for an additional random time span, characterized by a density function $\rho(\theta)$. Such a model seems more reasonable to us, than the assumption that the infectious people are removed at a rate α used in the standard SIR model. Hence the delay-SIR model is described by

$$\dot{S}(t) = -\beta S(t)I(t) \tag{15}$$
$$\dot{I}(t) = \beta S(t)I(t) - Q(t) \tag{16}$$

with $Q(t)$ the removal rate. As in the classic SIR, in a time Δt the number of newly infected is given by the mass action law $\beta S(t)I(t)\Delta t$. Meanwhile from the newly infected between time $t-\theta$ and $t+\Delta t-\theta$, a fraction $\rho(\theta)$ becomes immune and is removed from the infected. The support of ρ is contained in (τ, ∞). Hence,

$$Q(t) = \int_\tau^\infty \rho(\theta)S(t-\theta)I(t-\theta)\mathrm{d}\theta. \tag{17}$$

If ρ has a rational Laplace transform, the above equations may be extended to a pure (crisp) delay system by further differentiation, as explained in Verriest [1999]. This leads to a general delay model of the form

$$\dot{S}(t) = -\beta S(t)I(t) \tag{18}$$
$$p(t) = S(t)I(t) \tag{19}$$
$$\dot{I}(t) = \beta S(t)I(t) - hq(t) \tag{20}$$
$$\dot{q}(t) = Fq(t) + gp(t-\tau). \tag{21}$$

A block diagram is given in Figure 1. where $\dim(q) = n$

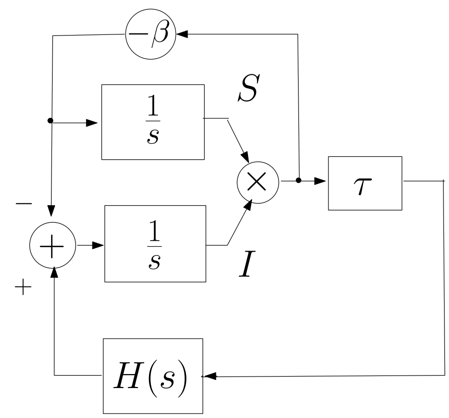

Fig. 1. Generic Delay-SIR

and (F, g, h) is an n-th SISO system with transfer function $H(s) = h(sI - F)^{-1}g$. It can be reorganized as an input-delayed linear system with nonlinear dynamic feedback. In this form the delay τ may be identifiable from the data using techniques from Belkoura et al. [2006].

We used the data published in Lancet [March 4 1978], (the Lancet, March 4, 1978) because it has been used by other authors. This data pertains to an influenza epidemic in a boys' boarding school. The population in this model is $N = 763$. Assuming that the onset was due to one infected individual, we set $S(0) = 762$ and $I(0)$. The best fit for the SIR model was obtained. The original data and the best fitting SIR model ($\beta := 0.00218; \alpha := 0.44$) are displayed in Figure 2. We considered the delay-SIR

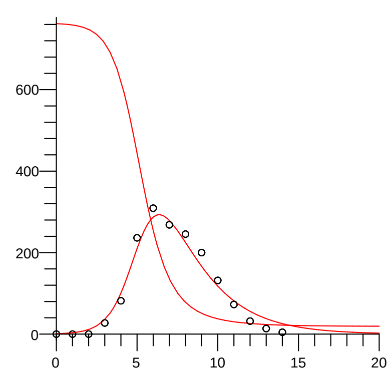

Fig. 2. SIR model fit to raw data

model with second order distribution (meaning that the corresponding $H(s)$ is a second order system).

$$\rho(\theta) = \mathcal{N}(1 + \gamma\theta)\,\mathrm{e}^{-\lambda\theta}$$

Here $\mathcal{N}$ is a normalization factor, ensuring that

$$\int_\tau^\infty \rho(\theta)\mathrm{d}\theta = 1$$

We find

$$\mathcal{N} = \frac{\lambda^2}{\gamma + \lambda + \gamma\lambda\tau}\,\mathrm{e}^{\lambda\tau}\,.$$

This distribution corresponds to a second order Jordan block. A good fit to the data was found for $\tau = 0.69, \beta = 0.00177, \gamma = 0.3$, and $\lambda = 0.75$. The evolution of $I(t)$ and $S(t)$ is given in Figure 3. The detailed description is thus

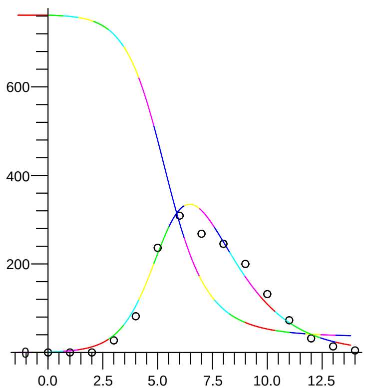

Fig. 3. Delay SIR model (Jordan) fit to raw data

$$\begin{array}{rcl} \dot{S}(t) &=& -\beta S(t)I(t) \\ \dot{I}(t) &=& \beta S(t)I(t) - \beta\mathcal{N}(q_1(t) + \gamma q_2(t) \\ \dot{R}(t) &=& \beta\mathcal{N}(q_1(t) + \gamma q_2(t) \\ p(t) &=& S(t)I(t) \\ \dot{q}_1(t) &=& -\lambda q_1(t) + \mathrm{e}^{-\lambda\tau}\, p(t-\tau) \\ \dot{q}_2(t) &=& q_1(t) - \lambda q_2(t) + \tau\,\mathrm{e}^{-\lambda\tau}\, p(t-\tau) \end{array} \quad (22)$$

5. OPTIMAL PULSE VACCINATION

We present here the application of the optimal pulse control to the proposed model of dynamic outbreak.

5.1 Necessary and Sufficient Conditions for optimal pulse vaccination

Model (22) needs to be augmented by the set of equations (23)-(27) to capture the whole impulsive control framework.

$$S(T_k^+) = S(T_k^-) - v_k \quad (23)$$
$$I(T_k^+) = I(T_k^-) \quad (24)$$
$$R(T_k^+) = R(\tau_k^-) + v_k \quad (25)$$
$$q_1(T_k^+) = q_1(T_k^-) \quad (26)$$
$$q_2(T_k^+) = q_2(T_k^-) \quad (27)$$

The vaccination takes place at certain times T_k, $k = 1, \ldots$ and have magnitude v_k. These decision variables must be determined in order to minimize the objective function (we consider a one pulse vaccination strategy for sake of simplicity)

$$J(v, T) = cv^2 + \int_0^{t_f} I(t)dt. \quad (28)$$

The integral term measures the burden of disease (total time spent sick in the population) during the epidemic, and the quadratic control cost reflects the added logistical burden when large populations need to be vaccinated. Note that a purely linear vaccination cost, without imposing the constraint $v \geq 0$, may lead to inadmissible controls Ogren and Martin [2000].

We present here a simple result with one pulse vaccination strategy

Lemma 5.1. Consider system (22) with (23)-(27), there exist an (locally) optimal one pulse vaccination strategy minimizing cost (28) if the following necessary and sufficient conditions are satisfied:

Necessary Conditions

$$\dot{\lambda} = -\left(\frac{\partial L}{\partial x}\right)^T - \left(\frac{\partial f}{\partial x}\right)\lambda - \left(\frac{\partial g}{\partial x}\right)\lambda^\tau$$
$$\lambda(\eta) = 0,\ \eta \in [t_f, t_f + \tau]$$
$$t \in [T, t_f]$$

Sufficient Conditions

$$2cv + \lambda(T)^T \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\beta\lambda(T)^T \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \mathrm{e}^{-\lambda\tau}\,\lambda(T+\tau)^T \begin{bmatrix} 0 \\ 0 \\ 1 \\ \tau \end{bmatrix} = 0,$$
$$\lambda(\eta) = 0, \eta \in [t_f, t_f + \tau]$$

with

$$\frac{df}{dx} = \begin{pmatrix} -\beta I & -\beta S & 0 & 0 \\ \beta I & \beta S & -\beta\mathcal{N} & -\beta\mathcal{N}\gamma \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{pmatrix} \quad (29)$$

$$\frac{dg}{dx} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{e}^{-\lambda\tau} I & -\mathrm{e}^{-\lambda\tau} S & 0 & 0 \\ \tau\,\mathrm{e}^{-\lambda\tau} I & \tau\,\mathrm{e}^{-\lambda\tau} S & 0 & 0 \end{pmatrix} \quad \frac{dL}{dx} = [0\ 1\ 0\ 0] \quad (30)$$

Proof : Identifying system (22) with template system (2) we have $L(x) = I$,

$$f(x) = \begin{pmatrix} -\beta SI \\ \beta SI - \beta\mathcal{N}(q_1 + \gamma q_2) \\ -\lambda q_1 \\ q_1 - \lambda q_2 \end{pmatrix} g(x) = \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{-\lambda\tau} SI \\ \tau\,\mathrm{e}^{-\lambda\tau} SI \end{pmatrix} \quad (31)$$

It is then straightforward to compute the Jacobian matrices.

We have removed the equation governing R since its value can be trivially retrieved from S and I (i.e. we have the relation $S + I + R = cst$).

Note we have

$$\begin{array}{l} L(x) = I \\ K(x(T^-), v, T) = cv^2 \end{array} \quad \frac{\partial L}{\partial x} = [0\ 1\ 0\ 0] \quad (32)$$

As $M = K + \mu^T G$ satisfies $\frac{\partial M}{\partial x} = 0$, it means that $\lambda_1(T^-) = \lambda_2(T^+)$ (i.e. the costate is continuous). Now let λ denote this single continuous costate and note that we only need to solve for λ on the time interval $[T, t_f]$ with $\lambda(t_f) = 0$. After solving for λ, we get that $\mu = \lambda(T)$ and hence that the first optimality conditions implies

$$\frac{dJ}{dv} = 2cv + \lambda(T)^T \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \tag{33}$$

The second optimality condition is a bit more involved since the function $g(x)$ depends on S which will experience an impulse at time T. First note that $g(x_\tau(T^-)) - g(x_\tau(T^+)) = 0$ then it is straightforward to obtain

$$H(T^-) - H(T^+) = \lambda(T)^T (f(x(T^-)) - f(x(T^+))) \tag{34}$$

where

$$f(x(T^-)) - f(x(T^+)) = \begin{bmatrix} -\beta I(T) v \\ \beta I(T) v \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

Moreover as

$$g(x(T^+)) - g(x(T^-)) = \begin{bmatrix} 0 \\ 0 \\ -\mathrm{e}^{-\lambda\tau} I(T) v \\ -\tau\,\mathrm{e}^{-\lambda\tau} I(T) v \end{bmatrix} \tag{36}$$

then we get

$$\frac{dJ}{dT} = \lambda(T)^T \beta I(T) v \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \lambda(T+\tau)^T \mathrm{e}^{-\lambda\tau} I(T) v \begin{bmatrix} 0 \\ 0 \\ 1 \\ \tau \end{bmatrix} = 0 \tag{37}$$

and assuming that neither $I(T) = 0$ nor $v = 0$ we have

$$\beta\lambda(T)^T \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \mathrm{e}^{-\lambda\tau} \lambda(T+\tau)^T \begin{bmatrix} 0 \\ 0 \\ 1 \\ \tau \end{bmatrix} = 0 \tag{38}$$

□

5.2 Numerical Example

We consider the model identified in Section 4 (i.e. $\tau = 0.69, \beta = 0.00177, \gamma = 0.3$, and $\lambda = 0.75$). The results are summarized below with $u_0 = 100$ and $T_0 = 10$ as initial conditions:

c \ t_f	15	50
0.02	100	298
0.05	99.9	234
0.1	77.22	198
0.2	39.6	119.8
0.5	16	47.9
0.8	10	32.28
1	8	26.5
3	2.7	9.11

Table 1. Optimal values of u w.r.t. cost (i.e. c and t_f)

It is worth noting that for $t_f = 15$ we always find $T = 5.7$ while for $t_f = 50$ we get $T = 6.64$. For simulation purpose we choose an interesting case: $c = 0.02$ and $t_f = 15$ we obtain figure 4.

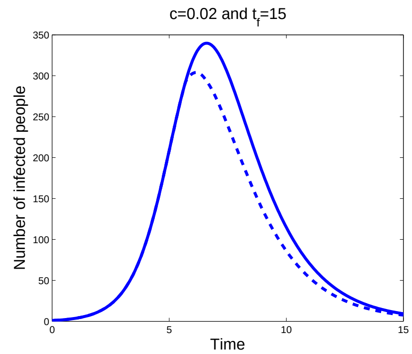

Fig. 4. Evolution of infected people without (plain) and with one pulse vaccination strategy (dashed) - criterium 1

If we compute the ratio of the integral term $\int_0^{t_f} I(t)dt$ with and without the vaccination strategy we obtain *ratio* $= 0.8577$ and this measures the reduction of the number of people who gets infected.

In the case $c = 0.02$ and $t_f = 50$ we obtain figure 5.

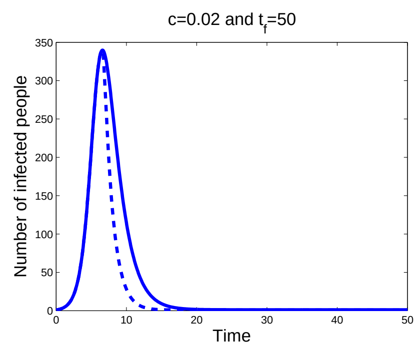

Fig. 5. Evolution of infected people without (plain) and with one pulse vaccination strategy (dashed) - criterium 2

In that case we obtain: *ratio* $= 0.6981$ and this shows that strategy 2 is better than the first one. Nevertheless, the second strategy is more expensive than the first one.

It is worth noting that, the multiple pulse vaccination strategy might lead to better result but it is not detailed here for sake of brevity. In this case, the new criterium to minimize should be

$$J_n := \sum_{i=1}^{N_p} c_i u_i^2 + \int_0^{t_f} I(t)dt \tag{39}$$

where $c_i > 0$ are chosen weighting parameters.

6. CONCLUSION

We have proposed a new epidemiological model. This new model considers the standard SIR model but includes a distribute delay modeling the rate at which infected people recover from the disease. Following the measurement of an real outbreak, we have identified the parameters and shows that it correctly describes the reality. On the other hand, we have developed an optimal pulse vaccination strategy minimizing a certain criterium measuring the cost of the campaign and the time spent by the population being sick. The interest of the approach is demonstrated through a realistic example.

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L Stability Analysis and Model-Based Control in EXTRAP-T2R with Time-Delay Compensation

Stability analysis and model-based control in EXTRAP-T2R with time-delay compensation

Erik Olofsson[1], Emmanuel Witrant[2], Corentin Briat[2], Silviu-Iulian Niculescu[3] and Per Brunsell[1]

Abstract— **In this paper, we investigate the stability problems and control issues that occur in a *reversed-field pinch* (RFP) device, EXTRAP-T2R (T2R), used for research in fusion plasma physics and general plasma (ionized gas) dynamics. The plant exhibits, among other things, magnetohydrodynamic *instabilities* known as *resistive-wall modes* (RWMs), growing on a time-scale set by a surrounding non-perfectly conducting shell. We propose a novel model that takes into account experimental constraints, such as the actuators dynamics and control latencies, which lead to a multivariable time-delay model of the system. The open-loop field-error characteristics are estimated and a stability analysis of the resulting closed-loop *delay differential equation* (DDE) emphasizes the importance of the delay effects. We then design an structurally constrained optimal PID controller by *direct eigenvalue optimization* (DEO) of this DDE. The presented results are substantially based on and compared with experimental data.**

I. INTRODUCTION

Control of magnetohydrodynamic (MHD) *instabilities* in toroidal devices for magnetic confinement is a crucial issue for thermonuclear fusion plasmas (high-temperature ionized gases) [1]. Indeed, advanced plasma confinement scenarios, as considered for the ITER experiment (a major step towards industrial fusion reactors) [2], motivate a better understanding of MHD phenomena and their regulation. The reversed-field pinch (RFP) device T2R, being the subject of this work, is particularly well suited for MHD studies in general (one of the main focuses of this facility) and more specifically for active control of MHD modes. Continuous research efforts have been done in this direction [3], [4], [5] based on physical approaches. We are now addressing the problem from a control-oriented point of view, highlighting impact of actuator dynamics to closed-loop stabilization.

T2R, sketched in Fig. 1(a), is a torus equipped with an equidistributed array of equally shaped 4×32 actuator saddle coils fully covering the surface outside a resistive wall (and vacuum container), and a corresponding set of 4×32 sensor saddle coils inside the wall (with 50% surface coverage). The coils inputs and outputs are subtracted pairwise in a top-down and inboard-outboard fashion, effectively implying 64 control and 64 measurement signals.

The MHD instabilities lead to non-symmetric electric currents within the plasma torus, causing perturbed magnetic fields outside of the plasma at the position of the surrounding wall. Complete stabilization would be achieved by an *ideally* conducting wall forcing the boundary magnetic field to vanish. In practice, eddy currents decay allow perturbed magnetic flux to penetrate the wall and hence the MHD instabilities to grow. To counteract this problem, the *intelligent-shell* (IS) concept [6] has been devised, to emulate the behavior of an ideally conducting wall by (decentralized) *feedback* control of external current-carrying coils. The RFP type of toroidal plasma confinement is particularly suited to study this method and stabilization of multiple independent MHD instabilities has recently been reported [5]. To emphasize the significance of IS feedback MHD-stabilization for T2R, note that the plasma is confined during $\sim 15-20\,\mathrm{ms}$ only *without* IS whereas a sustained plasma current is routinely achieved for over $90\mathrm{ms}$ *with* IS (limited by the experiment's power supply).

There is a strong motivation for developing this technique also for *Tokamak* fusion devices [7] (such as JET and ITER), the configuration mainly pursued today for magnetic confinement fusion research.

The aim of this paper is to introduce and analyze a new model for describing T2R dynamics, by explicitly taking into account the sensors/actuators configuration (aliasing and additional dynamics) and the control implementation (time-delays). To the best of the authors' knowledge, no systematic study of controller gain design for T2R IS operation explicitly including such experimental conditions has been made. We develop our description of the plant from a control viewpoint and employ a fixed-structure gain synthesis approach (presently instantiated for a classic PID) for T2R IS. Controller gains are directly optimized for a closed-loop delay differential equation (DDE) model. Experimental results illustrate the performance improvements in comparison with the explorative work [3], where PID gains scans and qualitative applicability of linear models were presented.

Paper organization as follows. First, a model describing the MHD unstable modes is introduced in Section II and the delay effects on the asymptotic stability of the corresponding model are analyzed in Section III. The design of a control law is presented in Section IV, and Section V is devoted to experimental results and highlights the performance improvements. Some concluding remarks end the paper.

II. MAGNETOHYDRODYNAMIC UNSTABLE MODES MODEL

The purpose of this section is threefold: first, to *outline* the unstable physics, second, to interface the corresponding model to a configuration of sensors and actuators and finally, to introduce an appropriate DDE (5) to be analyzed.

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A. Resistive-wall mode physics in the reversed-field pinch

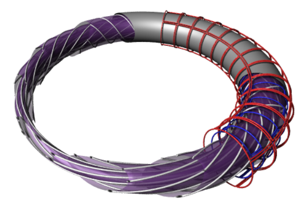
MHD theory [7], [8], [9] is the underlying physical level-of-detail employed here, a continuum description intended to capture behavior of conducting fluid matter, such as plasma gases and liquid metals. MHD effectively is a simultaneous application of Navier-Stokes' and Maxwell's equations. The system at hand is approximated by a periodic *cylinder*[1], with period $2\pi R$, R being the *major* toroidal radius, and thus reduced to the *minor* radial dimension r. The well-known MHD equations are: momentum $\rho\frac{d\mathbf{v}}{dt} = \mathbf{j}\times\mathbf{B} - \nabla p$, Ohm's law $\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j}$ together with Maxwell's, continuity and the adiabatic equation of state. For *ideal* MHD [8] resistivity $\eta \to 0$. A flowless $\mathbf{v} = \mathbf{0}$ and ideal equilibrium $\mathbf{j}_0\times\mathbf{B}_0 = \nabla p_0$, $\mathbf{E}_0 = \mathbf{0}$ is solved for using a standard current-profile and pressure parameterization [10], defining a magnetic structure in the plasma region $0 < r < r_a$, the *plasma column*. A vacuum layer isolates the plasma boundary $r = r_a$ from the conducting vessel wall at $r = r_w$. This wall is modeled *thin* [11]. Region $r_w < r < +\infty$ is air. An external source is positioned at $r = r_c > r_w$ (active coils outside the shell). Linear stability of perturbations around the nominal equilibrium is investigated by Fourier spectral decomposition $\mathbf{b}(r,t) = \sum_{mn}\mathbf{b}_{mn}(r)e^{j(t\omega+m\theta+n\phi)}$, yielding a discrete enumeration (m,n) of Fourier eigenmodes $\mathbf{b}_{mn}(r)$ with associated growth-rate $\gamma_{mn} = j\omega_{mn}$, after matching of boundary conditions. Eigenfunction first-order derivative discontinuity at $r = r_w$ determines modal growth-rate $\tau_w\gamma_{mn} = [\frac{rb_r'}{b_r}]_{r_w-}^{r_w+}$ (1(b)). These modes are the *resistive-wall modes* (RWMs), growing on the resistive time-scale set by the magnetic diffusion time τ_w. For the magnetic confinement configuration considered in this paper, the *reversed-field pinch* (RFP), named for its characteristic toroidal field reversal near the plasma boundary, it is customary to classify eigenmodes as *resonant/non-resonant* and *internal/external*. Internal modes share helicity with the equilibrium magnetic field inside the reversal surface, while external modes are reversed in this sense. *Ideal* resonant perturbations are zeroed for $0 < r < r_s < r_a$, r_s being the resonant position, as motivated in e.g. [11]. *Resistive* resonant modes are known as *tearing modes* (TMs), and are usually treated by inserting a thin resistive layer at r_s, and they typically seed *magnetic islands* governed by nonlinear dynamics [12]. Here, only ideal MHD modes are considered, modeled by

$$\tau_{mn}\dot{b}^r_{mn} - \tau_{mn}\gamma_{mn}b^r_{mn} = M_{mn}I_{mn} = b^{r,ext}_{mn} \qquad (1)$$

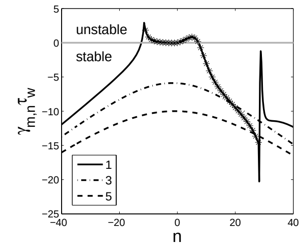
where b^r_{mn} is the radial Fourier component of the perturbed field, M_{mn}, I_{mn} respectively a geometric coefficient and a fourier harmonic for the external active coil current, while τ_{mn} is the mode-specific penetration time. A *range* n of unstable modes emerge for $m = 1$ (fig. 1(b)).

For a perfectly symmetric resistive wall, RWMs are uncoupled in ideal MHD regime, and growth rates are non-complex. Experimental support for this model (1) is reported in e.g. [10], [13], [4].

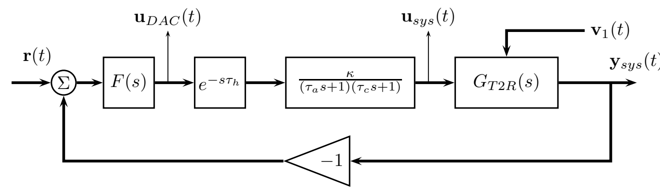
[1]Indeed, a good approximation for large aspect-ratio (R/r_a) devices, such as EXTRAP-T2R: $R = 1.24$m, $r_a = 0.183$m



(a) Cartoon of RFP magnetic equilibrium structure, vacuum vessel, sensor (blue) and actuator (red) saddle coils.



(b) Theoretical growth-rates $\tau_w\gamma_{mn}$. Integer-n non-resonant positions (RWMs) are marked (*) for $m = 1$.



(c) Closed-loop schematics. Taps indicate available measurements.

Fig. 1. RFP device 1(a) and RWM spectrum 1(b). All signal routings 1(c) are 64 parallel channels.

B. MIMO plant modeling by geometric coupling of SISO dynamics

From Faraday's and Biot-Savart's laws and assuming an ideal integrator on the sensor coil output voltage, the system dynamics write in the standard state-space form as

$$\left\{\begin{array}{rcl}\dot{\mathbf{x}} & = & A\mathbf{x} + B\mathbf{u} + N\mathbf{v}_1\\ \mathbf{z} & = & M\mathbf{x}\\ \mathbf{y} & = & C\mathbf{x} + \mathbf{v}_2\end{array}\right. \qquad (2)$$

where $\mathbf{x} \in \mathbb{R}^{2N_mN_n}$ is the vector of MHD-modes b^r_{mn}, $\mathbf{u} \in \mathbb{R}^{N_u}$ is the active coil currents, $\mathbf{z} \subset \mathbf{x}$ is the optional performance vector channel and $\mathbf{y}$ denotes *time-integrated* sensor voltages, corresponding to a measure of mode $\mathbf{x}$ (time-averaged radial magnetic field). A, B, C, M and N are matrices of appropriate dimensions, $\mathbf{v}_1$ is an exogenous signal, further detailed in section II-C.2, and $\mathbf{v}_2$ is a white noise signal. State matrix elements are obtained from

$$\begin{aligned}A_{mn,m'n'} &\sim \gamma_{mn}\delta_{mn,m'n'}\\ B_{mn,ij} &\sim \tau_{mn}^{-1}\int_\Omega e^{-\iota(m\theta+n\phi)}\left(\hat{\mathbf{r}}\cdot\oint_{l_{ij}}\frac{\mathbf{dl}_{ij}\times(\mathbf{r}-\mathbf{r}_{ij})}{|\mathbf{r}-\mathbf{r}_{ij}|^3}\right)d\Omega\\ C_{pq,mn} &\sim \int_\Omega e^{+\iota(m\theta+n\phi)}f_{pq}A_{pq}d\Omega\end{aligned} \qquad (3)$$

where mn, ij and pq enumerate Fourier modes, active coils and sensor coils, respectively, and f_{pq}, A_{pq} are sensor coils aperture and area functions. The integration set Ω is a full period of the toroidal surface $(\theta,\phi) \in [-\pi,\pi[\times[-\pi,\pi[$.

In the following, state matrices in (2)-(3) are instantiated for T2R geometry and routing. Note that the consideration of both intrinsic field-errors and peripheral dynamics is imperative for simulating open- and closed-loop *shots*[2] [14], [3].

1) Modes coupling and aliasing of spatial frequencies: The finite spatial arrays of sensors and actuators fundamentally affect the transition from a single-mode to a multiple-mode model due to *aliasing*. It generally renders the sensors and actuators imprecise, and even introduce a bias. Aliasing also has an important impact on the closed-loop control, as a zero on the output *could* in reality be a combination of non-zero modes $\sum_{mn} \mathbf{b}_{mn}$, deceptively summing to a small number. The traditional IS regulator [6] consequently drives the output to zero but does so happily ignorant of individual mode amplitudes. This is a fairly recent appreciation of the need for further development of control systems for MHD experiments [15], [14]. Indeed, IS operation typically excites higher mode numbers, which are, supposedly, stable and mainly transient.

TABLE I
CHARACTERISTIC TIMES FOR CURRENT SETUP OF T2R.

Symbol	Value/order	Description/comment
τ_w	$\sim 10\,\text{ms}$	Resistive wall time
τ_{mn}	$\lesssim \frac{1}{2}\tau_w$	Actual model mode time
τ_{MHD}	$\sim 1\mu$s	Internal MHD activity/fluctuations
τ_d	$100\,\mu$s	Digital sampling time, controller cycle
τ_h	$\sim 100\,\mu$s	Control latency, dead time
τ_{CPU}	$< 100\,\mu$s	Algorithm-dependent part of τ_h
τ_a	$8\,\mu$s	Active amplifier first-order time
τ_c	$1\,\text{ms}$	Active coil L/R-time
$\tau_{A\&D}$	$\sim 1\mu$s	ADC/DAC settle, ns/μs respectively

2) Actuators dynamics, latencies and PID control: Consideration of the actuators dynamics and control latency is essential for a realistic description of the control problem, as detailed in [3]. Table I suggests[3] that we can consider a (lumped) active amplifier and an active coil model together with a dead-time τ_h in series with RWM dynamics. Using a first-order description, the system input $\mathbf{u}_{sys}(t)$ is inferred from the digital control signal $\mathbf{u}_{DAC}(t)$ through a relationship

$$\mathbf{u}_{sys}(t) \approx \frac{1}{\tau_c s + 1} \frac{\kappa}{\tau_a s + 1} \mathbf{u}_{DAC}(t - \tau_h) \tag{4}$$

Introducing the system (A_ξ, B_ξ, C_ξ) to describe the previous dynamics, the resulting state-space matrices $(\bar{A}, \bar{B}, \bar{C})$ are obtained as

$$\bar{A} = \begin{pmatrix} A & BC_\xi \\ 0 & A_\xi \end{pmatrix},\ \bar{B} = \begin{pmatrix} 0 \\ B_\xi \end{pmatrix},\ \bar{C} = \begin{pmatrix} C & 0 \end{pmatrix}$$

The closed-loop dynamics, using a PID controller, is obtained as follows. The state considered is $\tilde{\mathbf{x}} = (\mathbf{x}^T\ \mathbf{q}^T)^T$, which includes the integrator state $\mathbf{q}(t) = \int_{-\infty}^{t} \mathbf{e}(\tau) d\tau$, where $e(t) = y(t)$ is the error (the reference is zero). Modeling the derivative action by finite time-difference renders

[2]One single experiment is known as a *shot*. Open- and closed-loop here specifically refers to RWMs.

[3]Neglecting $\tau_{A\&D}$ and quantization.

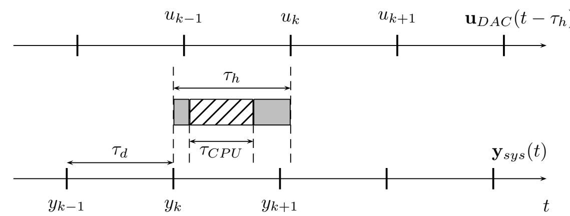


Fig. 2. Delay from control system. Note τ_h, dependent of control algorithm, possibly greater than τ_d but obviously $\tau_{CPU} < \tau_d$ for a working system. Sample frequency $f_s = 1/\tau_d$. Input y_k sampled from sensor coils, output u_k is the DAC-output subsequently fed to the active coil amplifiers as augmented system input.

the controller

$$\mathbf{u}_{DAC}(t) = K_p\mathbf{e}(t) + K_i\mathbf{q}(t) + \tau_d^{-1} K_d(\mathbf{e}(t) - \mathbf{e}(t-\tau_d))$$

The closed-loop dynamics is consequently obtained as

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}(t) &= \mathcal{A}_0\tilde{\mathbf{x}}(t) + \mathcal{A}_1(\theta)\tilde{\mathbf{x}}(t-\tau_h) \\ &\quad + \mathcal{A}_2(\theta)\tilde{\mathbf{x}}(t-\tau_h-\tau_d) + \mathcal{E}\mathbf{v}_1(t)\end{aligned} \tag{5}$$

where the control parameters $\theta = (K_p, K_i, K_d)$ enter affinely and

$$\mathcal{A}_0 = \begin{pmatrix} \bar{A} & 0 \\ \bar{C} & 0 \end{pmatrix}, \mathcal{A}_1(\theta) = \begin{pmatrix} \bar{B}(K_p + K_d/\tau_d)\bar{C} & \bar{B}K_i \\ 0 & 0 \end{pmatrix}$$
$$\mathcal{A}_2(\theta) = \begin{pmatrix} \bar{B}(-K_d/\tau_d)\bar{C} & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{E} = \begin{pmatrix} \bar{N} \\ 0 \end{pmatrix} \tag{6}$$

Note that the gain matrices have a diagonal form $K_\eta = k_\eta I$, where k_η is a scalar and $\eta \in \{p, i, d\}$ for the IS scheme.

C. Open-loop error estimation and parameter identification

Here, see fig. 1(c), an error field $\mathbf{v}_1$ estimate is obtained from experimental data via *model-based filtering* of open-loop (in the sense of RWM-control) shots, while actuators $G_{act}(s)$ are found by straighforward parametric identification. The controller cycle time τ_d is set at a nominal value $\tau_d = 100\,\mu$s.

1) Actuators dynamics identification: In order to identify the actuator dynamics (4), we consider the transfer channels i: $u_{DAC}^{ij}(t) \mapsto u_{sys}^{ij}(t)$, for each experiment j. The amplifiers' time constants τ_a^{ij} are fixed and we identify the remaining parameters $\rho^{ij} \doteq \{\tau_c^{ij}, \tau_h^{ij}, \kappa^{ij}\}$. Following the approach presented in the previous section (MIMO model built from a set of SISO dynamics), we determine the optimal averaged model $\rho^* = \langle\langle \rho^{ij*} \rangle_i\rangle_j$. This model averages the optimal parameters $\rho^{ij*} = \arg\min_{\rho^{ij}} J^{ij}$ that minimize the error functional

$$J^2(\rho^{ij}) = \frac{1}{T}\int_0^T \left(u_{sys}^{ij}(\tau) - u_{sim}^{ij}(\tau, \rho^{ij})\right)^2 d\tau$$

where u_{sys} is the experimental data and u_{sim} is the model output, for each transfer channel.

A real-time PRBS (*pseudorandom binary sequence*) generator [16] was implemented producing $64\times$ parallel SISO identification inputs. This generator spent $\sim 5\,\mu$s per cycle of τ_{CPU} and can thus be considered to yield an identification of *minimum* latency. The optimal set of parameters ρ^{ij*} is obtained by minimizing $J(\rho^{ij})$ with a *Quasi-Newton* method initialized from a nominal guess $\rho_0^{ij} = (1\text{ms}, 100\mu\text{s}, 4\text{A/V})$. The overall average model ρ^* was found to be $(\tau_c^*, \tau_h^*, \kappa^*) = (0.989\text{ms}, 77.7\mu\text{s}, 3.96\text{A/V})$ by residual minimization. A finite difference gradient approximations and a scaling of the decision variables to the order of unity led to a rapid convergence (1-14 iterations $\forall i, j$) of the numerical scheme.

The identification data set shows channel-by-channel variations and all the computations involving the *full* MIMO model (2) consequently use the individual channel averages $\langle \rho^{i,j} \rangle_j$, except for the time-delay, which is set identical for all channels. A worst-case τ_h (using maximum τ_{CPU}) exceeding 200 μs is consistent with recorded data for particular channels.

2) Error-field estimation and filtering: To estimate the error-field time-evolution, a set of open-loop shots are analyzed in the scope of model (2). A standard *Kalman Filter* (KF, e.g. [17]) is formed from (2) by adding *placeholder* states that represent the error-field $\tau_s \dot{\mathbf{x}}_s + \mathbf{x}_s = 0$. More precisely, the KF estimates the state vector for

$$\begin{cases} \dot{\tilde{\mathbf{x}}} &= \begin{pmatrix} A & N \\ 0 & -\tau_s^{-1} I \end{pmatrix} \tilde{\mathbf{x}} + \begin{pmatrix} B \\ 0 \end{pmatrix} \mathbf{u} + \mathbf{v}_1' \\ \mathbf{y} &= \begin{pmatrix} C & 0 \end{pmatrix} \tilde{\mathbf{x}} + \mathbf{v}_2 \end{cases} \quad (7)$$

where $\tilde{\mathbf{x}}(t) \doteq (\mathbf{x}(t)^T \, \mathbf{x}_s(t)^T)^T$, and $\mathbf{v}_1'$ and $\mathbf{v}_2$ are white noise. The filter takes $(\mathbf{u}(t)^T \, \mathbf{y}(t)^T)^T$ as inputs and outputs the state-estimate $\hat{\tilde{\mathbf{x}}} = (\hat{\mathbf{x}}^T(t) \, \hat{\mathbf{x}}_s^T(t))^T$. Note that the estimated error field $\hat{\mathbf{x}}_s(t)$ has a specific physical interpretation as it corresponds to a driving term for *inter alia* RWM-instabilities. The KF is tuned for very fast error-state $\hat{\mathbf{x}}_s$ due to the fact that an error in the growth-rate $\gamma_{mn}^{true} = \gamma_{mn}^{nominal} + \gamma_{mn}^{(1)}(t)$ affects in principle $\mathbf{v}_1$. This is expressed by the relation

$$\begin{aligned} \tau_{mn} \dot{b}_{mn}^r &= \tau_{mn} \left\{ \gamma_{mn} + \gamma_{mn}^{(1)}(t) \right\} b_{mn}^r + b_{mn}^{r,err} + b_{mn}^{r,act} \\ &= \tau_{mn} \gamma_{mn} b_{mn}^r + b_{mn,eff}^{r,err} + b_{mn}^{r,act} \end{aligned}$$

which implies that the *effective* error $b_{mn,eff}^{r,err} \equiv \tau_{mn} \gamma_{mn}^{(1)}(t) b_{mn}^r + b_{mn}^{r,err}$, associated with $\mathbf{v}_1$ in the model considered, depends on the mode amplitude itself. The discretized augmented model (7) is used for offline smoothing with the well-known *Rauch-Tung-Striebel* [17] forward-backward algorithm.

III. STABILITY ANALYSIS AND DELAY EFFECTS

Consider the (asymptotic) stability of the DDE-class (5). The corresponding characteristic equation of (5) reads as (for $n = 2$)

$$\det \Delta(s) = \det \left(sI - \mathcal{A}_0 - \sum_{i=1}^{n} \mathcal{A}_i e^{-s\tau_i} \right) = 0 \quad (8)$$

It is well-known that (8) has an *infinite* number of roots $s = \lambda_j$ and that (5) has a point spectrum. Furthermore, since the set $\{\lambda_j : \det \Delta(\lambda_j) = 0, \, \mathrm{Re}(\lambda_j) > a\}$ with a real is *finite* (see, e.g., [18] and the references therein), it follows that the stability problem is reduced to analyze the location of the *rightmost* characteristic roots with respect to the imaginary axis (see, for instance, [19] for numerical computations).

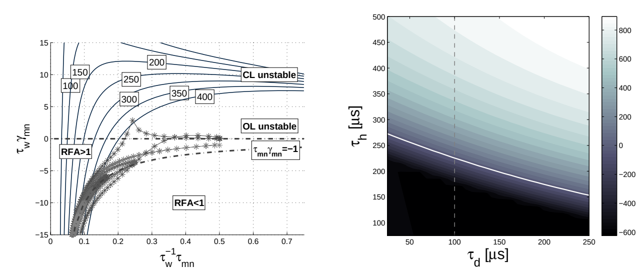
Finally, the continuity properties of the spectral abscissa with respect to the system parameters (including the delays) allows a better understanding of the effects induced by the parameters' change on the stability of the system. Without entering into details, such properties will be exploited in the sequel. For the sake of brevity, we will discuss some of the properties of our delay system without giving a complete characterization of the stability regions in the corresponding parameter space. Such an issue will be addressed in a different work.

A. Mode-control and perfect decoupling; SISO dynamics

Consider here a fictitious situation where *perfect* actuators and sensors are available (in a no-aliasing sense; infinite array of vanishing-size coils). Ideally, we could then, according to (1), measure and affect each Fourier mode (m, n) independently, achieving perfect decoupling and effectively reducing the dynamics to a SISO system with actuator delay:

$$G_{mn}(s) = \frac{1}{\tau_{mn} s - \tau_{mn} \gamma_{mn}} \frac{1}{\tau_c s + 1} \frac{1}{\tau_a s + 1} e^{-s\tau_h} \quad (9)$$

readily converted to a closed-loop description (5) with $\mathcal{A}_i \in \mathbb{R}^{4 \times 4}$. A static mode-control (MC) decoupling controller would typically be computed by taking SVD pseudoinverses of (3), and it can be demonstrated that doing this produce aliased side-bands on the inverse approximations [14]. IS operation, considered here is "far-from-perfect" mode-control, but the underlying SISO dynamics (1) is fundamentally important, and is considered a benchmark case. Fig. 3(a)



(a) Stability in RWM parameter space.

(b) SISO-set stability in (τ_h, τ_d)-space.

Fig. 3. Stability and time-delay impact on RWM dynamics.

shows stability contours $(\max_j \mathrm{Re}(\lambda_j) = 0)$ for $\tau_h = \{100 - 400\mu\text{s}\}$ in RWM parameter space (τ, γ), for fixed PID gains[4] $\theta_{old} = (-10.4, -1040, -0.0026)$.

In fig. 3(a) *Resonant-field amplification* (RFA) regions [13] are indicated, an effect related to the error-field, as modeled in (7).

[4]When *quoting numerical* gains: k_η correspond to dimensionless loop-gains (*negative*), related to (*positive*) experiment settings $k_\eta = \beta K_\eta$ [3], with a nominal conversion factor $\beta = -6.5 \times 10^{-2}$.

B. Spectrum dependence on τ_h; MIMO and SISO cases

Consider now the dependence of the spectral abscissa with respect to the parameter τ_h. In this sense, we fix the gains $(K_p, K_i, K_d) = (146, 57000, 0.085)$ and the delay $\tau_d = 100\mu$s for IS operation on full MIMO model (5), and we will concentrate on the effects induced by varying the delay τ_h. Computing rightmost closed-loop roots, critical crossing of the imaginary axis occurs at $\tau_h \approx 201\mu$s. This compared to the SISO-set analog 3(b) where instability occurs at $\tau_h \approx 225\mu$s. The set of modes was in both cases $\mathcal{K} = \{1,3\}\times\{-24,\ldots,+23\}$. In conclusion, plant geometry lowers latency stability margin.

IV. MODEL-BASED CONTROL AND DELAY COMPENSATION

Our aim is to select PID gains for DDE (5) to ensure stability and minimize the closed-loop spectral abscissa. The PID in the actual experiment control system (IS) is regarded as fixed, imposing a structural constraint on the optimization problem. A fixed-order/fixed-structure controller synthesis approach is utilized to find gains for T2R IS operation. The method, as instantiated in this work, concerns model (5), i.e. it handles time-delays explicitly, which has a significant practical benefit: developing control algorithms with varying computational complexity (varying τ_{CPU}) implies varying τ_h, which can be accounted for.

It is recognized that other widely spread iterative tuning techniques such as [20] also could be applied for this particular problem. This is subject for the sequel.

A. Direct eigenvalue optimization (DEO)

The asymptotic damping maximization of (5) is formulated as minimizing the spectral abscissa of the characteristic equation [18] with

$$\theta^* = \arg\min_{\theta}\max_{\lambda}\{\mathrm{Re}\,(\lambda):\ \det\Delta(\lambda,\theta)=0\}$$

This problem is generally both nonconvex and nonsmooth, which motivates a hybrid SISO/MIMO method. The general MIMO problem size (5) is typically large; e.g. a set $(m,n)\in\mathcal{K}=\mathcal{M}\times\mathcal{N}=\{1,3\}\times\{-16,\ldots,+15\}$ results in $\mathcal{A}_i\in\mathbb{R}^{384\times384}$. However, for IS, each coil measures a linear combination of fundamental dynamics (1) over $\mathcal{K}$, but does not discriminate between modes. This relates to the previously discussed hypothesis that the MIMO model can be approximated by a set of SISO systems. The MIMO optimization problem is then approximated to the problem of minimizing the maximum SISO spectral abscissas over $\mathcal{K}$ with

$$\tilde{\theta}^* = \arg\min_{\theta}\max_{k\in\mathcal{K}}\max_{\lambda}\{\mathrm{Re}\,(\lambda):\ \det\Delta_k(\lambda,\theta)=0\} \quad (10)$$

where Δ_k denotes the characteristic matrix ($\in\mathbb{R}^{4\times4}$) for (5) for a single mode $k=(m,n)$. We employ the recently developed *gradient-sampling* (GS) method [21], a *robustified* steepest-descent method suitable for nonsmooth optimization, to solve (10) using finite difference gradient approximations.

We investigate two different parameterizations of the controller, implicitly assigning the closed-loop performance and control-input norm, respectively by:

a) varying k_p and searching for the optimal $\tilde{\theta}^* = (k_i^*, k_d^*)$ for a nominal τ_h, and

b) varying τ_h and determining the full optimal PID $\tilde{\theta}^* = (k_p^*, k_i^*, k_d^*)$.

The mode-sets are considered for $a)-b)$ were $\mathcal{K}_1=\{1,3\}\times\{-16,\ldots,+15\}$ and $\mathcal{K}_2=\{1,3\}\times\{-24,\ldots,+23\}$ respectively. In *a)* $\tau_h = 77.7\mu$s corresponding to the *minimum* actuator latency.

B. Optimization results

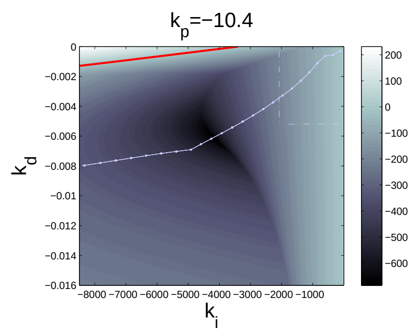


Fig. 4. Closed-loop spectrum optimization.

Gain design strategy a), where the optimal (k_i^*, k_d^*) are obtained with preset values of k_p and τ_h is illustrated in Fig. 4, which depicts the rightmost values of the closed-loop spectrum for a fixed k_p in (k_i, k_d)-space (the optimum corresponding to the darkest region). The bold dotted line corresponds to the evolution of (k_i^*, k_d^*) when k_p varies (going left when the magnitude of k_p is increased). The red line in the upper left corner is the stability boundary and the rectangle in upper-right corner is the region of uniformly randomized initializations for the GS method.

Comparable numerical values were obtained for gain design strategy b). A few optimal settings are seen in table II. For b) the minimum objective value is increased (spectral abscissa traveling rightwards) as the time-delay increases, as expected.

The optimization algorithm was numerically robust on problems a), b). All runs converged, normally within $10-30$ iterations, from randomized starting controllers. Multiple runs were taken for each controller, yielding identical results (within reasonable numerical accuracy).

V. EXPERIMENTAL RESULTS

The new control approach presented in the previous sections motivated new series of experiments on T2R: shot numbers $\#20743-\#20755$ and $\#20824-\#20838$. Experimental plasma equilibrium conditions were set with a toroidal plasma current $I_p \approx 85\,\mathrm{kA}$, a shot length $\tau_p =\sim 50-70\,\mathrm{ms}$ and *reversal* and *pinch* parameter values (typically used to characterize RFP equilibria [8]) $(F,\Theta)\approx(-0.27, 1.72)$.

TABLE II
T2R EXPERIMENTAL RESULTS.
$[J_y] = (\mathrm{mT})^2 \times 10^{-3}$, $[J_u] = (\mathrm{A})^2 \times 10^3$.

Shot#	K_p	K_i	K_d	J_y	J_u	Remark
20743	150	16000	0.05	1.04	1.66	old gain 1
20744	160	16000	0.04	1.14	1.80	old gain 2
20746	106	37500	0.061	0.581	2.12	series *a)*
20747	126	47500	0.073	0.808	1.94	*a)*
20827	150	16000	0.05	1.12	1.60	old gain 1
20833	119.6	46800	0.065	0.680	1.77	*b)*
20835	106.8	39860	0.058	0.645	1.64	*b)*

We only consider strict IS performance in terms of plant output, and introduce a suitable scalar measure to compare experimental (and simulated) performance.

A. Generic measure of experimental performance

The overall controller performance is summarized with the general quadratic measure

$$J_x(\tilde{\theta}) \equiv \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mathbf{x}^T(\tau, \tilde{\theta}) Q_x \mathbf{x}(\tau, \tilde{\theta}) d\tau \qquad (11)$$

where $\mathbf{x} = \mathbf{y}_{sys}$ or $\mathbf{u}_{sys}$, and $\tilde{\theta}$ the controller setting. We do not consider any specific channel weighting ($Q_y = Q_u = I \in \mathbb{R}^{64\times64}$) and the integral is approximated by trapezoidal summation of non-filtered sampled data. The nature of T2R shots [3] suggests a split of the timespan $[t_0, t_1]$ into two parts, corresponding to the *transient* (first $10\,\mathrm{ms}$) and *steady-state* behaviors (between 10 and $\sim 50\,\mathrm{ms}$).

B. Performance improvements

The performance improvements are summarized in Table II for the *steady-state* interval $10-45$ms, using cost function (11). The optimized controllers a) and b) clearly reveal a significant 44% $(1 - 0.581/1.04)$ reduction of average field energy at the sensors during steady-state period. This is at the expense of a higher input power, increased by 28% $(2.12/1.66 - 1)$. Furthermore, simulations with the MIMO model, as driven by the identified $\mathbf{v}_1$ of section II-C.2, reveal that the *old PID coefficients are significantly suboptimal* in both full model (2) and experiment compared to the new PID coefficients.

VI. CONCLUSIONS

A new model for MHD instabilities in T2R, explicitly including important geometrical and engineering aspects was presented. Direct closed-loop PID gain optimization for the corresponding DDE model was shown to provide useful results for experimental IS feedback in a RFP fusion research device. Simulations and experiments for the T2R device have shown some qualitative agreement, further indicating the applicability of the model to real experimental conditions. In short, results strongly encourage future work, theoretically and experimentally, in both physical modeling and multivariable control.

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