Design of $\mathcal{H}_\infty$ Bounded Non-Fragile Controllers for Discrete-Time Systems

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December 2009

CDC 09 - Shanghai, China
Plan

- Introduction
- Main Results
- Examples
Introduction
Resilience of controllers [Keel et al. ’97]
  - Continuous-Time systems
    - Ricatti [Haddad, 97], [Yang et al, 01]
    - LMI [Jadbabaie et al. 97], [Peaucelle et al. 04]
  - Discrete-Time Systems?

Bounded controller design (NP-hard, [Blondel et al. 97])
  - Sporadic results, e.g. [Peaucelle et al. 08] in continuous-time
  - Discrete-time?
Goals

- Resilient Controllers (SF) Synthesis for DT systems
- Bounded Controllers Synthesis
- Efficient characterization of solutions (LMIs)
- Include performance optimization
• Discrete-time linear systems

\[
\begin{bmatrix}
  x(k+1) \\
  z(k)
\end{bmatrix}
= \begin{bmatrix}
  A & B & E \\
  C & D & F
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  u(k) \\
  w(k)
\end{bmatrix}
\]

state \( x \), control input \( u \), exogenous input \( w \), controlled output \( z \).

• Matrices supposed known

• Can be extended easily to the uncertain case
Control Laws

Find a control law of the form

\[ u(k) = Kx(k) \]

such that it

- stabilizes the system
- minimizes a performance criterium, e.g. $\mathcal{H}_\infty$.

Moreover, the controller must also satisfy

- A resilience (non-fragility) property
- A boundedness condition for the coefficients
Non-fragility property

- Self-robustness property of the controller
- Error on the controller implementation gain maintain closed-loop stability
- Model of the implementation error
- Two type of errors:
  - Additive error (rounding, uniform discrete valued space)
    \[ K_i = K_c + \delta K \]
  - Additive and multiplicative error (rounding+nonuniform discrete valued space)
    \[ K_i = K_c + \theta K_c + \Gamma \]

\( K_i \) implemented controller, \( K_c \) computed one, \( \delta K, \theta, \Gamma \) error terms
Additive Error

- Form of implemented gain

\[ K_i = K_c + \delta K \quad \delta K = U\Delta V \]

\( \Delta \) diagonal, \( ||\Delta||_2 \leq \alpha \)

- Coefficients of \( \delta K \) inside \([-\alpha, \alpha]\)

\( \delta K \)

\( +\alpha \)

\( -\alpha \)

possible values for \( \delta K \)
**Additive-Multiplicative Error**

- Form of implemented gain

\[ K_i = (1 + \theta)K_c + \Gamma \quad \Gamma = U\tilde{\Delta}V \]

\[ \theta \in [-\mu, \mu] \quad \|\tilde{\Delta}\|_2 \leq \tilde{\alpha} \]

- Illustration of possible error behavior:

\[ \delta K \]

**Linear approximation**

**Maximal error**

**Possible values for \( \delta K \)**

**\( \delta K \) total implementation error**
Bounded coefficients

- Form of implemented gain (with additive error)
  \[ K_i = M_1(K_0 + K_c)M_2 + \delta K \]
  previous \( K_c \)

  \( M_1, M_2 \) scaling terms, \( K_0 \) shifting term
- \( K_0 \) allows for looking for a controller centered around 0 such that

  \[
  \|K_c + M_1^{-1} \delta KM_2^{-1}\|_2 \leq \beta \sqrt{mn}
  \]

  \( m, n \) dimensions of input and state resp.
Main Results
Theorem

There exists a quadratically stabilizing resilient state-feedback if there exist a matrix $X = X^T \succ 0$, a diagonal matrix $Q \succ 0$ and a scalar $\gamma > 0$ such that the following LMI

$$
\begin{bmatrix}
-X & 0 & XV^T \\
* & -\gamma I & 0 \\
* & * & -Q \\
* & * & * \\
* & * & *
\end{bmatrix}
\begin{bmatrix}
\mathcal{M}_{14} & \mathcal{M}_{15} \\
F^T & E^T \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
-X + \alpha^2 B U Q U^T B^T \\
\alpha^2 D U Q U^T D^T
\end{bmatrix}
\prec 0
$$

holds where

$$
\mathcal{M}_{15} = [AX + BM_1 K_0 M_2 X + BM_1 Y]^T \\
\mathcal{M}_{14} = [CX + DM_1 K_0 M_2 X + DM_1 Y]^T
$$

In such a case, we have $K_c = Y (M_2 X)^{-1}$ and the closed-loop system satisfies $\|z\|_{\ell_2} \leq \gamma \|w\|_{\ell_2}$. 
Sketch of the proof

- Write the closed-loop system
- Substitute into the BRL
- Rewrite the BRL into the form

\[ \Psi + U^T \Delta V + V^T \Delta^T U \prec 0 \]

- Apply the Petersen’s lemma (or Scaled-bounded real lemma), congruence transformations, Schur complement and change of variables (standard)
Adding constraints on the controller coefficients (1)

- Idea: Add a condition to the previous design $\rightarrow$ add-on
- Nonlinear constraint on the controller (proved NP-hard, nonconvex) $\rightarrow$ no exact LMI formulation
- Relaxation necessary (Cone complementary algorithm or iterative LMI algorithm)
Adding constraints on the controller coefficients (2)

- Iterative LMI based result (no additional optimization cost)

**Theorem**

Find $N$, $Y$ and $X \succ 0$ of appropriate dimension such that

$$
\begin{bmatrix}
\Pi_{11} & Y & 0 & 0 \\
* & N^T M_2 X + X M_2^T N & X M_2^{-T} V^T & N^T \\
* & * & -H & 0 \\
* & * & * & -I \\
\end{bmatrix} \preceq 0
$$

$$
\Pi_{11} = -s^2 mn \beta^2 I + \alpha^2 M_1^{-1} U H U^T M_1^{-T}
$$

This will result in a gain $K_c$ satisfying

$$
||K_c + M_1^{-1} \delta K M_2^{-1}||_2 \leq s \sqrt{mn \beta}.
$$

- Iteration between $X$ and the slack-variable $N$
- Can be proved using the projection lemma.
Example
Example (1)

Let us consider the unstable system

\[
\begin{align*}
x(k + 1) &= Ax(k) + Bu(k) + Ew(k) \\
z(k) &= Cx(k) + Du(k) + Fw(k)
\end{align*}
\]

with matrices \( F = 0 \)

\[
A = \begin{bmatrix}
9.3547 & 0.5789 & 1.3889 & 2.7219 \\
9.1690 & 3.5287 & 2.0277 & 1.9881 \\
4.1027 & 8.1317 & 1.9872 & 0.1527 \\
8.9365 & 0.0986 & 6.0379 & 7.4679
\end{bmatrix}
\]

\[
C = 0.1 \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 0 \\
1 & 2 \\
-1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
E = 0.1 \begin{bmatrix}
1 & 0 \\
1 & 2 \\
1 & 0 \\
1 & 1
\end{bmatrix}
\]
Example (2)

- With no implementation error we get $\gamma^* = 14.78$
- System stabilizable for all $\alpha < 0.0032$ (need quite large precision)
- For a precision of $\alpha = 0.0020$, we find $\gamma_a = 132.9090$ (worst case)
- After rounding and verification, we get $\gamma_r = 88.4425$
Optimal Controller

\[ K^* = \begin{bmatrix} -18.6097 & -3.5441 & -5.8235 & -7.7459 \\ -2.3537 & -3.0544 & -0.8132 & -0.0420 \end{bmatrix} \]

Resilient Controller

\[ K_a = \begin{bmatrix} -53.4820 & -16.3820 & -20.9800 & -24.6260 \\ 42.0900 & 13.3060 & 18.5000 & 21.4660 \end{bmatrix} \]
Example (3)

- Location of eigenvalues of the closed-loop system for random implementation error lower than 0.0025.

Left: optimal controller, Right: Memory resilient controller

- Unstable behavior on the left
Conclusion and Future Works
Conclusion and Future Works

- Characterization of Resilient SF Controllers
- Two types of error
- LMI form (optimization)
- Additional nonlinear constraint for the boundedness of controllers (relaxation)
- Characterize more general class of errors
- Dynamic Output Feedback case
- Other formulations for boundedness of controllers (more relevant in continuous time)
Thank you for your attention